Problem **1**(a):

The oscillating magnetic field of the incident wave looks approximately uniform from the sphere's point of view, but it cannot penetrate the sphere itself due to the skin effect. Or rather, it penetrates only to the skin depth, and for a perfectly good conductor skin depth $\to 0$. Thus, the incident magnetic field is screened from the inside of the sphere by the surface currents, just like the incident electric field is screened by the surface charges.

To find the surface currents and hence the net magnetic dipole moment, let's compare the sphere at hand to a uniformly magnetized spherical permanent magnet and the bound currents on its surface. We saw in my notes on polarization and magnetization (pages 15–16) that the magnetic field inside a spherical magnet is uniform

$$\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}. \tag{S.1}$$

By the superposition principle, if we add an external uniform field $\mathbf{H}_{\text{ext}}$ without changing the magnetization, we would get

$$\mathbf{B}_{\text{inside}} = \mu_0\left(\tfrac{2}{3}\mathbf{M} + \mathbf{H}_{\text{ext}}\right), \tag{S.2}$$

so for $\mathbf{M} = -\frac{3}{2}\mathbf{H}_{\text{ext}}$ the net magnetic field inside the sphere would vanish. For the same magnetization, the net magnetic moment of the sphere would be

$$\mathbf{m} = \left(\text{Volume} = \frac{4\pi r^3}{3}\right)\mathbf{M} = -2\pi a^3\mathbf{H}_{\text{ext}}. \tag{S.3}$$

For the conducting sphere at hand, the situation is physically different but mathematically similar: There is no magnetization or bound currents, instead there are conduction currents on the surface of the sphere, but their net effect on the magnetic field inside the sphere is exactly the same — they precisely cancel the external field $\mathbf{H}_{\text{ext}} = \mathbf{H}_{\text{inc}}$. Consequently, the conduction currents on the sphere's surface — or rather the amplitudes of these currents — are precisely the same as the bound currents on the surface or a permanent magnet ball which happens to have zero magnetic field inside it in a similar $\mathbf{H}_{\text{ext}}$. Therefore, the magnetic dipole moments are precisely similar in both cases, or rather the amplitude of the conducting sphere's dipole moment is related to the incident magnetic field's amplitude by the same formula (S.3) as the magnetic moment of the spherical magnet, thus

$$\mathbf{m}[\text{conducting sphere}] \;=\; -2\pi a^3 \mathbf{H}_{\text{inc}}\,. \tag{1}$$

Quod erat demonstrandum.

<u>Problem **1**(b)</u>:

For the incident wave of wavelength $\lambda \gg a$, we may treat the incident electric field as approximately uniform external electric field $\mathbf{E}_{\text{ext}}$. Also, the response of the conducting sphere to this external field is much faster than $1/\omega$, so we may use the electrostatics techniques to find the induced electric dipole moment at any given time. Thus, as explained in any undergraduate textbook — for example, *Introduction to Electrodynamics* by David Griffith, example **3.8**, or in my notes on separation of variables for 352K class, pages 35–36, — there are induced charges on the sphere's surface

$$\rho(r, \mathbf{n}) \;=\; 3\epsilon_0 E_{\text{ext}} \cos\theta\, \delta(r-a) \tag{S.4}$$

and hence net dipole moment

$$\mathbf{p} \;=\; \int d^3\mathbf{x}\, \rho(\mathbf{x})\, \mathbf{x} \;=\; 4\pi\epsilon_0 a^3 \mathbf{E}_{\text{ext}}\,. \tag{S.5}$$

In the context of the incident EM wave, this dipole moment oscillates with amplitude

$$\mathbf{p}[\text{conducting sphere}] \;=\; +4\pi a^3 \epsilon_0\, \mathbf{E}_{\text{inc}}\,. \tag{2}$$

Next, to relate the electric and the magnetic dipole moments of the conducting sphere to each other, we note that the electric and the magnetic amplitudes of the incident wave are related to each other as

$$\mathbf{H}_{\text{inc}} = \frac{1}{Z_0}\,\mathbf{n}_0 \times \mathbf{E}_{\text{inc}} \tag{S.6}$$

where $\mathbf{n}_0$ is the unit vector in the direction of the incident wave. Consequently,

$$\begin{aligned}\frac{\mathbf{m}}{c} &= -\frac{2\pi a^3}{c}\,\mathbf{H}_{\text{inc}} = -\frac{2\pi a^3}{Z_0 c}\,\mathbf{n}_0 \times \mathbf{E}_{\text{inc}} = -2\pi a^3 \epsilon_0\,\mathbf{n}_0 \times \mathbf{E}_{\text{inc}} = -\frac{1}{2}\,\mathbf{n}_0 \times \left(4\pi a^3 \epsilon_0\,\mathbf{E}_{\text{inc}}\right) \\ &= -\frac{1}{2}\,\mathbf{n}_0 \times \mathbf{p}.\end{aligned} \tag{3}$$

Quod erat demonstrandum.

Problem **1**(c):

In the radiation zone far away from the oscillating multipoles, the EM fields are

$$\mathbf{E}_{\text{sc}} = ikZ_0(\mathbf{n} \times (\mathbf{n} \times \mathbf{f}))\,\frac{e^{ikr-i\omega t}}{r}, \quad \mathbf{H}_{\text{sc}} = -ik(\mathbf{n} \times \mathbf{f})\,\frac{e^{ikr-i\omega t}}{r} \tag{S.7}$$

for

$$\mathbf{f}(\mathbf{n}) = \frac{1}{4\pi}\int d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\,\exp(-ik\mathbf{n}\cdot\mathbf{y}). \tag{S.8}$$

Specifically, for an electric dipole of amplitude $\mathbf{p}$

$$\mathbf{f}_{\text{ED}} = \frac{i\omega}{4\pi}\,\mathbf{p} \tag{S.9}$$

while for a magnetic dipole

$$\mathbf{f}_{\text{MD}}(\mathbf{n}) = -\frac{i\omega}{4\pi c}\,\mathbf{n} \times \mathbf{m} \tag{S.10}$$

For the case at hand, both electric and magnetic oscillating dipoles are present and have comparable magnitudes, or rather $m/c \sim p$. Therefore

$$\mathbf{f}_{\text{net}} \approx \mathbf{f}_{\text{ED}} + \mathbf{f}_{\text{MD}}(\mathbf{n}) = \frac{i\omega}{4\pi}\left(\mathbf{p} - \mathbf{n} \times \frac{\mathbf{m}}{c}\right). \tag{S.11}$$

In light of eqs. (3) and (2), this formula evaluates to

$$\mathbf{f} = \frac{i\omega}{4\pi}\left(\mathbf{p} + \frac{1}{2}\,\mathbf{n} \times (\mathbf{n}_0 \times \mathbf{p})\right) = i\omega a^3 \epsilon_0 E_0\left(\mathbf{e}_0 + \frac{1}{2}\,\mathbf{n} \times (\mathbf{n}_0 \times \mathbf{e}_0)\right). \tag{S.12}$$

Problem **1**(d):

The polarized partial cross section obtains from $\mathbf{f}(\mathbf{n})$ according to

$$\frac{d\sigma}{d\Omega} = \frac{k^2 Z_0^2}{E_0^2}\left|\mathbf{e}^*\cdot(\mathbf{n}\times(\mathbf{n}\times\mathbf{f}))\right|^2 = \frac{k^2 Z_0^2}{E_0^2}\left|\mathbf{e}^*\cdot\mathbf{f}\right|^2. \tag{S.13}$$

In particular, for $\mathbf{f}(\mathbf{n})$ as in eq. (S.12),

$$\frac{d\sigma}{d\Omega} = (kZ_0\,\omega a^3\epsilon_0)^2\left|\mathbf{e}^*\cdot\mathbf{e}_0 + \frac{1}{2}\mathbf{e}^*\cdot(\mathbf{n}\times(\mathbf{n}_0\times\mathbf{e}_0))\right|^2. \tag{S.14}$$

To simplify this formula, note that in the first factor

$$kZ_0\times\omega a^3\epsilon_0 = k^2a^3\times cZ_0\epsilon_0 = k^2a^3 \quad\Longrightarrow\quad (kZ_0\,\omega a^3\epsilon_0)^2 = k^4a^6, \tag{S.15}$$

while inside $|\cdots|^2$

$$\mathbf{e}^*\cdot(\mathbf{n}\times(\mathbf{n}_0\times\mathbf{e}_0)) = (\mathbf{n}_0\times\mathbf{e}_0)\cdot(\mathbf{e}^*\times\mathbf{n}) = -(\mathbf{n}\times\mathbf{e}^*)\cdot(\mathbf{n}_0\times\mathbf{e}_0). \tag{S.16}$$

Consequently,

$$\frac{d\sigma}{d\Omega} = k^4a^6\left|\mathbf{e}^*\cdot\mathbf{e}_0 - \tfrac{1}{2}(\mathbf{n}\times\mathbf{e}^*)\cdot(\mathbf{n}_0\times\mathbf{e}_0)\right|^2. \tag{S.17}$$

exactly as in eq. (4).

Next, let's specialize to linear polarizations $\perp$ or $\|$ to the scattering plane. Note that if $\mathbf{e}_0 \parallel$ the scattering plane then $(\mathbf{n}_0\times\mathbf{e}_0)$ $\perp$the plane and vice verse; likewise if $\mathbf{e} \parallel$ the plane then $(\mathbf{n}\times\mathbf{e}) \perp$ the plane and vice verse. Consequently, IF $\mathbf{e}_0 \perp$ the plane while $\mathbf{e} \parallel$ the plane OR IF $\mathbf{e}_0 \parallel$ the plane while $\mathbf{e} \perp$ the plane THEN both $\mathbf{e}^*\cdot\mathbf{e}_0 = 0$ and $(\mathbf{n}\times\mathbf{e}^*)\cdot(\mathbf{n}_0\times\mathbf{e}_0) = 0$, and hence $d\sigma/d\Omega = 0$, exactly as in the two middle eqs. (5). In other words, if the incident wave happens to be polarized $\perp$ to the scattering plane, then the scattering wave is also polarized $\perp$ to the scattering plane, and likewise if the incident wave happens to be polarized $\|$ to the scattering plane, then the scattering wave is also polarized $\|$ to the scattering plane.

Now, suppose both waves are polarized $\perp$ to the plane of scattering. In the coordinate system where z axis points along the incident wave direction $\mathbf{n}_0$ while the xz plane is the scattering plane,

$$\mathbf{n}_0 \;=\; (0,0,1), \quad \mathbf{n} \;=\; (\sin\theta, 0, \cos\theta), \tag{S.18}$$

we have

$$\mathbf{e}_0 \;=\; (0,1,0), \quad \mathbf{e} \;=\; (0,1,0), \tag{S.19}$$

hence

$$\mathbf{n}_0 \times \mathbf{e}_0 \;=\; (-1,0,0), \quad \mathbf{n} \times \mathbf{e} \;=\; (-\cos\theta, 0, +\sin\theta),$$

and therefore

$$\mathbf{e}^* \cdot \mathbf{e}_0 \;=\; +1, \quad (\mathbf{n} \times \mathbf{e}^*) \cdot (\mathbf{n}_0 \times \mathbf{e}_0) \;=\; +\cos\theta. \tag{S.20}$$

Plugging this geometry into eq. (4), we get

$$\frac{d\sigma^{\perp}}{d\Omega} \;=\; k^4 a^6 \, (1 \;-\; \tfrac{1}{2}\cos\theta)^2, \tag{S.21}$$

in perfect agreement with the first eq. (5).

Finally, suppose both incident and scattered waves are polarized $\|$ to the scattering plane. In this case, in the coordinate system (S.18) we have

$$\mathbf{e}_0 \;=\; (1,0,0), \quad \mathbf{e} \;=\; (\cos\theta, 0, -\sin\theta), \tag{S.22}$$

hence

$$\mathbf{n}_0 \times \mathbf{e}_0 \;=\; (0,1,0), \quad \mathbf{n} \times \mathbf{e} \;=\; (0,1,0), \tag{S.23}$$

and therefore

$$\mathbf{e}^* \cdot \mathbf{e}_0 \;=\; +\cos\theta, \quad (\mathbf{n} \times \mathbf{e}^*) \cdot (\mathbf{n}_0 \times \mathbf{e}_0) \;=\; +1. \tag{S.24}$$

This time, plugging this geometry into eq. (4) yields

$$\frac{d\sigma^{\|}}{d\Omega} \;=\; k^4 a^6 \, (\tfrac{1}{2} \;-\; \cos\theta)^2, \tag{S.25}$$

in perfect agreement with the last eq. (5).

Quod erat demonstrandum.

Problem **1**(e):

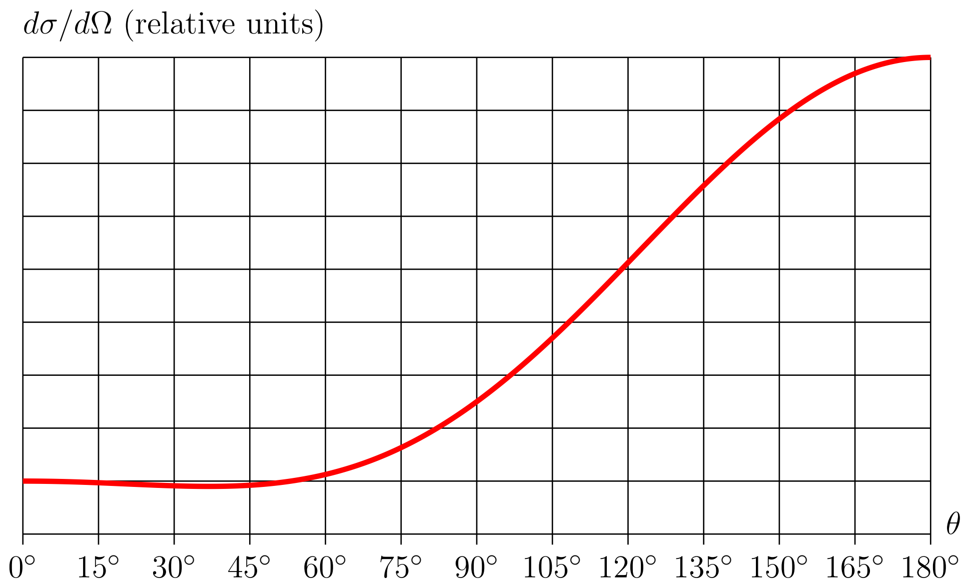
Saying that the incident wave is un-polarized means that in any polarization basis half of the net power belongs to one polarization and half to the other. In particular, half of the net incident energy flux belongs to the linear polarization $\parallel$ to the scattering plane and the other half to the polarization $\perp$ to the scattering plane. Consequently, the partial cross-section in which the scattered wave's polarization is not detected is simply

$$\frac{d\sigma^{\text{unpolarized}}}{d\Omega} = \frac{1}{2}\frac{d\sigma(\perp\to\text{any})}{d\Omega} + \frac{1}{2}\frac{d\sigma(\|\to\text{any})}{d\Omega} = \frac{1}{2}\frac{d\sigma(\perp\to\perp)}{d\Omega} + \frac{1}{2}\frac{d\sigma(\|\to\|)}{d\Omega}. \tag{S.26}$$

Specifically, for the polarized partial cross-sections as in the first and the last eqs. (5),

$$\begin{aligned}\frac{d\sigma^{\text{unpolarized}}}{d\Omega} &= \frac{k^4a^6}{2}\times\Bigl((\tfrac{1}{2}-\cos\theta)^2 + (1-\tfrac{1}{2}\cos\theta)^2\Bigr)\\ &= \frac{k^4a^6}{2}\times\Bigl((\tfrac{1}{4}-\cos\theta+\cos^2\theta) + (1-\cos\theta+\tfrac{1}{4}\cos^2\theta)\Bigr)\\ &= \frac{k^4a^6}{2}\times\Bigl(\tfrac{5}{4} - 2\cos\theta + \tfrac{5}{4}\cos^2\theta\Bigr).\end{aligned} \tag{S.27}$$

Note that this partial cross-section does not have a forward-backward symmetry. Instead, the scattering into the backward hemisphere ($\theta > 90°$ so that $\cos\theta < 0$) is significantly stronger than the scattering into the forward hemisphere ($\theta < 90°$ so that $\cos\theta > 0$). Graphically,



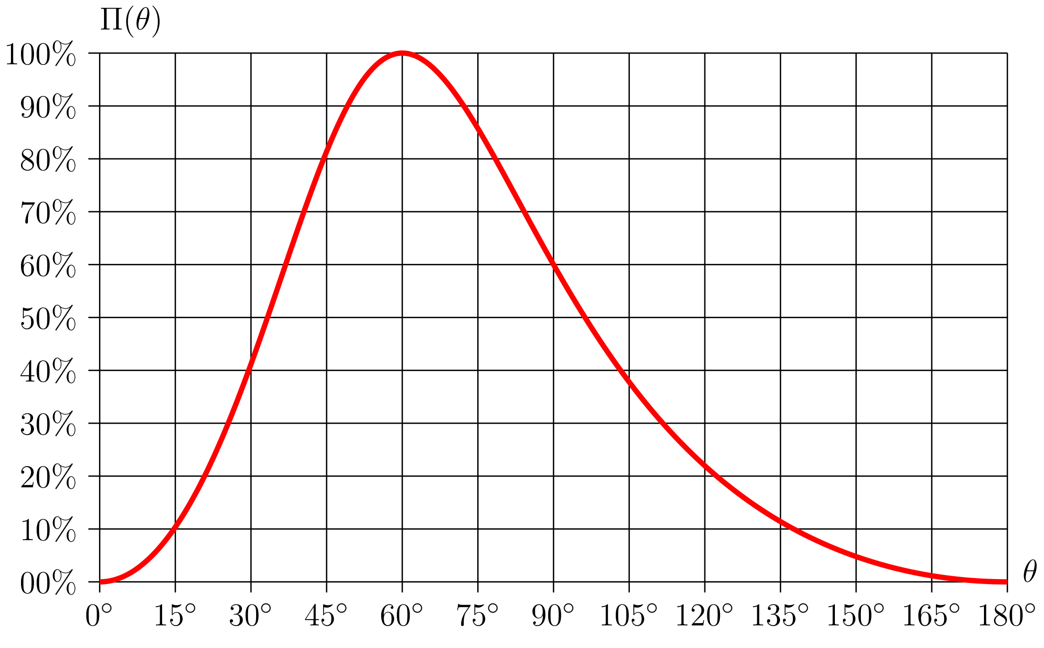
Now consider the partial polarization of the scattered wave. Although the incident wave has equal powers of the two polarizations, they scatter with different strengths due to un-equal polarized cross-sections. Consequently, the degree to which the scattered wave is polarized is

$$\Pi(\theta) \;=\; \frac{dP^{\perp} \,-\, dP^{\|}}{dP^{\perp} \,+\, dP^{\|}} \;=\; \frac{d\sigma^{\perp} \,-\, d\sigma^{\|}}{d\sigma^{\perp} \,+\, d\sigma^{\|}}\,. \tag{S.28}$$

For the polarized cross-sections as in eqs. (5), this formula yields

$$\begin{aligned}
\Pi(\theta) \;&=\; \frac{(1-\frac{1}{2}\cos\theta)^2 \,-\, (\frac{1}{2}-\cos\theta)^2}{(1-\frac{1}{2}\cos\theta)^2 \,+\, (\frac{1}{2}-\cos\theta)^2} \\
&=\; \frac{\frac{3}{4} \,-\, \frac{3}{4}\cos^2\theta}{\frac{5}{4} \,-\, 2\cos\theta \,+\, \frac{5}{4}\cos^2\theta} \\
&=\; \frac{3\sin^2\theta}{5-8\cos\theta+5\cos^2\theta}\,.
\end{aligned} \tag{S.29}$$

Graphically,



Note: at $\theta = 60^\circ$ the scattered wave is 100% polarized $\perp$ to the scattering plane — the $\|$ polarization does not scatter in this direction.

Problem **1**(f):

Finally, let's calculate the net un-polarized cross-sections for scattering into the forward and the backward hemispheres. Integrating the partial unpolarized cross-section (S.27), we have

$$\begin{aligned}
\sigma_{\text{forward}} &= \int_0^{\pi/2} d\theta\, 2\pi \sin\theta\, \frac{d\sigma}{d\Omega} \\
&= \pi k^4 a^6 \times \int_0^{+1} d\cos\theta \big(\tfrac{5}{4} - 2\cos\theta + \tfrac{5}{4}\cos^2\theta\big) \\
&= \pi k^4 a^6 \times \left(\frac{5}{4} - \frac{2}{2} + \frac{5/4}{3} = \frac{2}{3}\right), \qquad \text{(S.30)} \\
\sigma_{\text{backward}} &= \int_{\pi/2}^{\pi} d\theta\, 2\pi \sin\theta\, \frac{d\sigma}{d\Omega} \\
&= \pi k^4 a^6 \times \int_{-1}^{0} d\cos\theta \big(\tfrac{5}{4} - 2\cos\theta + \tfrac{5}{4}\cos^2\theta\big) \\
&= \pi k^4 a^6 \times \left(\frac{5}{4} + \frac{2}{2} + \frac{5/4}{3} = \frac{8}{3}\right). \qquad \text{(S.31)}
\end{aligned}$$

Consequently, the total cross-section (for scattering in all possible directions) is

$$\sigma_{\text{net}} = \sigma_{\text{forward}} + \sigma_{\text{backward}} = \left(\frac{2}{3} + \frac{8}{3} = \frac{10}{3}\right) \times \pi k^4 a^6, \qquad \text{(S.32)}$$

while the forward-backward asymmetry is

$$A \stackrel{\text{def}}{=} \frac{\sigma_{\text{forward}} - \sigma_{\text{backward}}}{\sigma_{\text{forward}} + \sigma_{\text{backward}}} = \frac{\frac{2}{3} - \frac{8}{3}}{\frac{2}{3} + \frac{8}{3}} = -60\%. \qquad \text{(S.33)}$$

Problem **2**(a):

In a spherical cavity, we can find the standing-wave solution of the Maxwell equations by separating variables in spherical coordinates (r,θ,ϕ). This separation of variables works *exactly* as in my notes on the spherical waves, and we end up with the *transverse magnetic waves* with

$$\begin{aligned} \mathbf{x}\cdot\mathbf{H}(\mathbf{x}) &= 0, \\ \mathbf{x}\cdot\mathbf{E}(\mathbf{x}) &= \frac{E_{\ell,m}}{k} * f_\ell(kr) * Y_{\ell,m}(\theta,\phi), \end{aligned} \tag{Notes.56}$$

and the *transverse electric waves* with

$$\begin{aligned} \mathbf{x}\cdot\mathbf{E}(\mathbf{x}) &= 0, \\ \mathbf{x}\cdot\mathbf{H}(\mathbf{x}) &= \frac{H_{\ell,m}}{k} * f_\ell(kr) * Y_{\ell,m}(\theta,\phi), \end{aligned} \tag{Notes.57}$$

where for both kinds of waves, the radial profiles $f_\ell(kr)$ obey the spherical Bessel equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + k^2\right) f_\ell(kr) = 0. \tag{S.34}$$

However, these radial profiles $f_\ell(kr)$ obey different boundary conditions than the profiles $g_\ell(kr)$ of the divergent spherical waves: Instead of

$$\begin{aligned} g_\ell(kr) &\longrightarrow \frac{e^{+ikr}}{kr} \quad \text{for} \quad kr \to \infty \\ \text{but never mind if} \quad g_\ell(kr) &\longrightarrow \infty \quad \text{for} \quad kr \to 0, \end{aligned} \tag{S.35}$$

we now need

$$g_\ell(kr) \longrightarrow \text{finite} \quad \text{for} \quad kr \to 0, \tag{S.36}$$

plus another condition at $r = R$ we shall deal with in part (b). Consequently, instead of $g_\ell(kr)$ being the spherical Hankel functions — or rather

$$g_\ell(kr) = i^{\ell+1} h_\ell(kr) = i^{\ell+1} j_\ell(kr) + i^{\ell+2} n_\ell(kr), \tag{S.37}$$

— for the spherical cavity at hand, the radial profiles are the *regular* spherical Bessel functions, $f_\ell(kr) = j_\ell(kr)$.

Besides these radial profiles, the EM fields $\mathbf{E}(r,\theta,\phi)$ and $\mathbf{H}(r,\theta,\phi)$ of the TM and TE wave modes work out exactly as in my notes on the spherical waves. In particular, separating the EM fields into their radial and transverse (or lateral) components, we arrive at the close cousins of eqs. (152) through (160) on pages 24–25 of my notes:

for a $\mathrm{TM}_{\ell,m}$ wave:

$$H_r = 0, \tag{TM.1}$$

$$\mathbf{H}_t = -\frac{E_{\ell,m}}{\ell(\ell+1)Z_0} * j_\ell(kr) * \hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n}), \tag{TM.2}$$

$$E_r = E_{\ell,m} * \frac{j_\ell(kr)}{kr} * Y_{\ell,m}(\mathbf{n}), \tag{TM.3}$$

$$\mathbf{E}_t = -i\frac{E_{\ell,m}}{\ell(\ell+1)} * \left(\frac{1}{kr} + \frac{\partial}{\partial(kr)}\right) j_\ell(kr) * \mathbf{n}\times\hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n}); \tag{TM.4}$$

for a $\mathrm{TE}_{\ell,m}$ wave:

$$E_r = 0, \tag{TE.1}$$

$$\mathbf{E}_t = +\frac{Z_0 H_{\ell,m}}{\ell(\ell+1)} * j_\ell(kr) * \hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n}), \tag{TE.2}$$

$$H_r = H_{\ell,m} * \frac{j_\ell(kr)}{kr} * Y_{\ell,m}(\mathbf{n}), \tag{TE.3}$$

$$\mathbf{H}_t = -i\frac{H_{\ell,m}}{\ell(\ell+1)} * \left(\frac{1}{kr} + \frac{\partial}{\partial(kr)}\right) j_\ell(kr) * \mathbf{n}\times\hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n}). \tag{TE.4}$$

Problem **2**(b):

For a perfectly conducting wall surrounding the spherical cavity in question, the boundary conditions at $r=R$ amount to

$$\mathbf{E}_t = 0 \quad\text{and}\quad H_r = 0 \quad \text{for } r=R \text{ and any } (\theta,\phi). \tag{S.38}$$

For the TE waves obeying eqs. (TE.1–4), both $\mathbf{E}_t$ and H_r are proportional to the $j_\ell(kr)$, so both boundary conditions (S.38) are satisfied if and only if $j_\ell(kR)=0$, *cf.* eq. (7.a). For a cavity of a given radius this means $kR =$ one of the $x_{\ell,n}$, hence the resonant frequency

$$\omega_n(\mathrm{TE}_\ell) = ck = \frac{c}{R}\times x_{\ell,n} \tag{8.a}$$

(assuming an empty cavity).

As to the TM waves obeying eqs. (TM.1-4), the radial magnetic field is zero everywhere, so the only non-trivial boundary condition is $\mathbf{E}_t = 0$ at the cavity's surface. In light of eq. (TM.4), this means

$$\frac{j_\ell(kR)}{kR} + j'_\ell(kR) = 0 \tag{S.39}$$

or equivalently

$$j_\ell(kR) + (kR)j'_\ell(kR) = \left|\frac{d}{dy}\big(yj_\ell(y)\big)\right|_{y=kR} = 0. \tag{7.b}$$

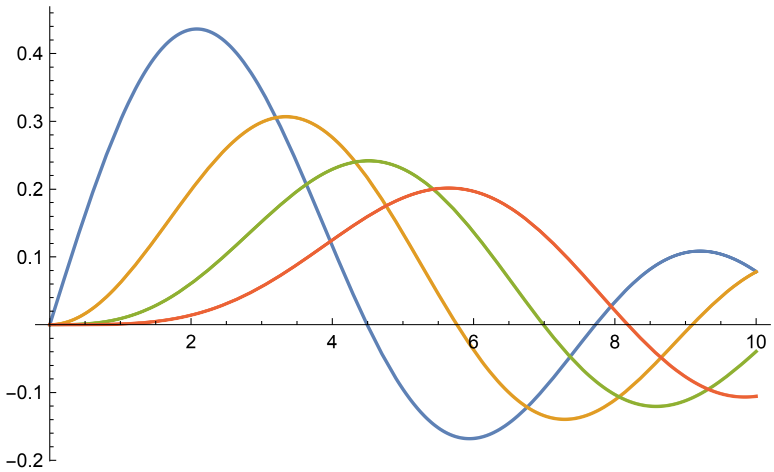
This calls for $kR =$ one of the $y_{\ell,n}$ and hence resonant frequencies

$$\omega_n(\mathrm{TM}_\ell) = ck = \frac{c}{R}\times y_{\ell,n} \tag{8.b}$$

PS: In eqs. (8) for the resonant frequencies, I refer to these frequencies as $\omega_n(\mathrm{TE}_\ell)$ and $\omega_n(\mathrm{TM}_\ell)$ rather than $\omega_n(\mathrm{TE}_{\ell,m})$ and $\omega_n(\mathrm{TM}_{\ell,m})$ because none of these frequencies depends on m. So for each of the resonant frequencies (8) there are $2\ell+1$ exactly degenerate modes.

Problem **2**(c):

Zeroes $x_{\ell,n}$ of the spherical Bessel functions $j_\ell(x)$ increase with both ℓ and n, so the first 4 zeroes must belong to the $j_\ell(x)$ with $\ell \leq 4$. Plotting the $j_\ell(x)$ for $\ell = 1, 2, 3, 4$ using Mathematica, I get

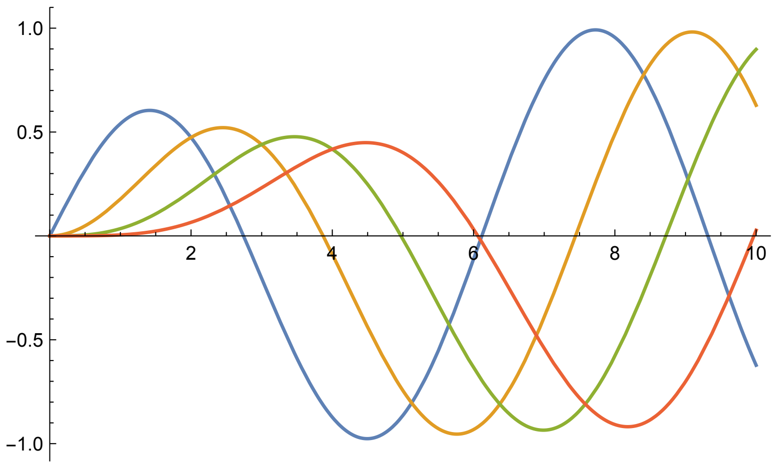


According to this plot, the first four zeroes are $x_{1,1} \approx 4.5$, $x_{2,1} \approx 5.8$, $x_{3,1} \approx 7.0$, and

$x_{1,2} \approx 7.7$. Physically, this means the first 4 TE-wave resonances are

$$\omega_1(\mathrm{TE}_1) \;\approx\; 4.5\,\frac{c}{R}\,, \qquad \omega_1(\mathrm{TE}_2) \;\approx\; 5.8\,\frac{c}{R}\,, \qquad \omega_1(\mathrm{TE}_3) \;\approx\; 7.0\,\frac{c}{R}\,, \qquad \omega_2(\mathrm{TE}_1) \;\approx\; 7.7\,\frac{c}{R}\,. \tag{S.40}$$

Similarly, zeroes $y_{\ell,n}$ of the derivatives (7) increase with both ℓ and n, so the first 4 zeroes must belong to the $f_\ell(y)$ with $\ell \leq 4$. Plotting these four functions using Mathematica, I get



According to this plot, the first four zeroes are $y_{1,1} \approx 2.7$, $y_{2,1} \approx 3.9$, $y_{3,1} \approx 5.0$, and $y_{4,1} \approx 6.1$. Physically, this means the first 4 TM-wave resonances are

$$\omega_1(\mathrm{TM}_1) \;\approx 2.7\,\frac{c}{R}\,, \qquad \omega_1(\mathrm{TM}_2) \;\approx 3.9\,\frac{c}{R}\,, \qquad \omega_1(\mathrm{TM}_3) \;\approx 5.0\,\frac{c}{R}\,, \qquad \omega_1(\mathrm{TM}_4) \;\approx 6.1\,\frac{c}{R}\,. \tag{S.41}$$

Finally, comparing the lists (S.40) and (S.41), and selecting the 4 lowest frequencies regardless of the type of the wave, we end up with

$$\begin{aligned}
\Omega_1 &= \omega_1(\mathrm{TM}_{\ell=1}) \;\approx\; 2.7437\,\frac{c}{R}\,,\\
\Omega_2 &= \omega_1(\mathrm{TM}_{\ell=2}) \;\approx\; 3.8702\,\frac{c}{R}\,,\\
\Omega_3 &= \omega_1(\mathrm{TE}_{\ell=1}) \;\approx\; 4.4934\,\frac{c}{R}\,,\\
\Omega_4 &= \omega_1(\mathrm{TM}_{\ell=3}) \;\approx\; 4.9734\,\frac{c}{R}\,.
\end{aligned} \tag{S.42}$$

The extra precision of the $y_{1,1}$, $y_{2,1}$, $x_{1,1}$, and $y_{3,1}$ factors here obtains by zooming up the plots of the f_1, f_2, j_1, and f_3 functions to the progressively narrower and narrower intervals of kr.

Problem **2**(d):

As explained in my notes on waveguides and resonant cavities, the quality factor of a vacuum-filled microwave cavity obtains as

$$Q \;=\; \frac{Z_0}{R_s}\times\hat{G} \tag{S.43}$$

where $Z_0 \approx 377\,\Omega$ is the wave impedance of the vacuum, $R_s = 1/\sigma\delta$ is the surface resistivity of the cavity wall(s), and

$$\hat{G} \;=\; \frac{\omega}{c}\times\frac{\iiint|\mathbf{H}|^2\,d^3\mathrm{volume}}{\iint|\mathbf{H}|^2\,d^2\mathrm{area}} \tag{S.44}$$

is the dimensionless geometry factor of the cavity. The area integral in eq. (S.44) is over the entire inner surface of the cavity, while the volume integral is over the whole cavity's volume.

Let's calculate these integrals for a TM wave. According to eqs. (TM.1–2),

$$|\mathbf{H}(r,\theta,\phi)|^2 \;=\; |H_0|^2\,\bigl(j_\ell(kr)\bigr)^2\,|\hat{\mathbf{L}}Y_{\ell,m}(\theta,\phi)|^2 \tag{S.45}$$

where

$$H_0 \;=\; \frac{E_{\ell,m}}{\ell(\ell+1)Z_0}\,, \tag{S.46}$$

hence the surface integral over the sphere of radius R amounts to

$$I_S \;\stackrel{\rm def}{=}\; \iint|\mathbf{H}|^2\,d^2\mathrm{area} \;=\; |H_0|^2\,\bigl(j_\ell(kR)\bigr)^2\times R^2\iint d^2\Omega(\mathbf{n})\,|\hat{\mathbf{L}}Y_{\ell,m}(\mathbf{n})|^2. \tag{S.47}$$

The remaining angular integral on the RHS here evaluates to $\ell(\ell+1)$, indeed

$$\begin{aligned}
\iint d^2\Omega\,|\hat{\mathbf{L}}Y_{\ell,m}|^2 \;&=\; \iint d^2\Omega\,(\hat{\mathbf{L}}^*Y^*_{\ell,m})\cdot(\hat{\mathbf{L}}Y_{\ell,m})\\
\langle\!\langle\,\text{integrating by parts}\,\rangle\!\rangle \qquad &=\; -\iint d^2\Omega\,Y^*_{\ell,m}\,(\hat{\mathbf{L}}^*\cdot\hat{\mathbf{L}})Y_{\ell,m}\\
\langle\!\langle\,\text{using } \hat{\mathbf{L}} = -i\mathbf{x}\times\nabla,\text{ hence } \hat{\mathbf{L}}^* = -\hat{\mathbf{L}}\,\rangle\!\rangle \qquad &=\; +\iint d^2\Omega\,Y^*_{\ell,m}\hat{\mathbf{L}}^2Y_{\ell,m}\\
&=\; \ell(\ell+1)\iint d^2\Omega\,|Y_{\ell,m}|^2\\
&=\; \ell(\ell+1)\times 1.
\end{aligned} \tag{S.48}$$

Thus altogether, the surface area integral evaluates to

$$I_S \;=\; \ell(\ell+1)R^2\,|H_0|^2\times\bigl(j_\ell(kR)\bigr)^2. \tag{S.49}$$

As to the volume integral, in spherical coordinates it factorizes into a product of a radial integral and the directional integral:

$$I_V \;\stackrel{\rm def}{=}\; \iiint|\mathbf{H}|^2\,d^3\text{volume} \;=\; |H_0|^2\times\int\limits_0^R dr\,r^2\,\bigl(j_\ell(kr)\bigr)^2\times\iint d^2\Omega(\mathbf{n})\,|\hat{\mathbf{L}}Y_{\ell,m}|^2. \tag{S.50}$$

Note that the directional integral here is exactly as in eq. (S.48), which leaves us with just the radial integral,

$$I_V \;=\; \ell(\ell+1)\,|H_0|^2\times\int\limits_0^R dr\,r^2\,\bigl(j_\ell(kr)\bigr)^2 \;=\; \ell(\ell+1)\,|H_0|^2\times\frac{1}{k^3}\int\limits_0^{kR} dx\,x^2\,\bigl(j_\ell(x)\bigr)^2. \tag{S.51}$$

Moreover, for a resonant TM wave, $Y=kR$ must be one of the zeroes $y_{\ell,n}$ of the derivative (9), so the integral on the RHS here obtains from the spherical Bessel function identity (13):

$$\int\limits_0^{Y=kR} dx\,x^2\,\bigl(j_\ell(x)\bigr)^2 \;=\; \tfrac{1}{2}(kR)\bigl((kR)^2-\ell(\ell+1)\bigr)\times\bigl(j_\ell(kR)\bigr)^2, \tag{S.52}$$

and therefore

$$I_V \;=\; \ell(\ell+1)\,|H_0|^2\times\frac{R}{2}\left(R^2\,-\,\frac{\ell(\ell+1)}{k^2}\right)\times\bigl(j_\ell(kR)\bigr)^2. \tag{S.53}$$

Taking the ratio of this volume integral to the surface integral (S.49), we arrive at

$$\frac{I_V}{I_S} \;=\; \frac{R}{2}\left(1\,-\,\frac{\ell(\ell+1)}{(kR)}\right), \tag{S.54}$$

hence

$$\hat{G} \;=\; k\times\frac{I_V}{I_S} \;=\; \frac{kR}{2}\left(1\,-\,\frac{\ell(\ell+1)}{(kR)^2}\right) \;=\; \frac{1}{2}\left(kR\,-\,\frac{\ell(\ell+1)}{(kR)}\right) \;=\; \frac{1}{2}\left(y_{\ell,n}\,-\,\frac{\ell(\ell+1)}{y_{\ell,n}}\right) \tag{S.55}$$

and therefore cavity's quality factor

$$Q = \frac{Z_0}{2R_s} \times \left(kR - \frac{\ell(\ell+1)}{(kR)} \right) \tag{S.56}$$

— which is exactly as in eq. (11) for

$$kR = \frac{\omega R}{c} = y_{\ell,n}\,. \tag{S.57}$$

Now consider the TE waves (TE). For a resonant TE wave mode, we must have $kR = X =$ one of the zeroes $x_{\ell,n}$ of the spherical Bessel function $j_\ell(x)$, hence at the cavity's surface $r = R$, the radial magnetic field (TE.3) vanishes, while the transverse magnetic field (TE.4) becomes simply

$$\mathbf{H}_t(r = R, \theta, \phi) = -iH_0\, j'_\ell(kR) * \mathbf{n} \times \hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi) \tag{S.58}$$

where this time

$$H_0 = \frac{H_{\ell,m}}{\ell(\ell+1)}\,. \tag{S.59}$$

Consequently, at the cavity's surface

$$\begin{aligned} |\mathbf{H}(r = R, \theta, \phi)|^2 &= |\mathbf{H}_t(r = R, \theta, \phi)|^2 \\ &= |H_0|^2 * \big(j'_\ell(kR)\big)^2 * \Big|\mathbf{n} \times \hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi)\Big|^2 \\ &= |H_0|^2 * \big(j'_\ell(kR)\big)^2 * \Big|\hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi)\Big|^2 \\ &\qquad \langle\!\langle \text{ because } \mathbf{n} \cdot \hat{\mathbf{L}} Y_{\ell,m} = 0 \,\rangle\!\rangle, \end{aligned} \tag{S.60}$$

and the area integral of this $|\mathbf{H}|^2$ amounts to

$$I_S \stackrel{\text{def}}{=} \iint |\mathbf{H}|^2 \, d^2\text{area} = |H_0|^2 * \big(j'_\ell(kR)\big)^2 * R^2 \ell(\ell+1). \tag{S.61}$$

As to the volume integral of the $|\mathbf{H}|^2$, we may relate it to the volume integral of $|\mathbf{E}|^2$ since for any harmonic oscillator the electric and the magnetic average energies must be equal, thus

$$I_V \stackrel{\text{def}}{=} \iiint |\mathbf{H}|^2\, d^3\text{volume} \;=\; \frac{1}{Z_0^2} \iiint |\mathbf{E}|^2\, d^3\text{volume}\,. \tag{S.62}$$

For the electric field (TE.1–2) of a TE wave,

$$|\mathbf{E}(r,\theta,\phi)|^2 \;=\; Z_0^2 |H_0|^2 * \big(j_\ell(kr)\big)^2 * \Big|\hat{\mathbf{L}} Y_{\ell,m}(\theta,\phi)\Big|^2, \tag{S.63}$$

so the volume integral (S.62) factorizes into the radial integral and the directional integral

$$\iiint |\mathbf{E}|^2\, d^3\text{volume} \;=\; Z_0^2 |H_0|^2 * \int_0^R dr\, r^2 \big(j_\ell(kr)\big)^2 * \iint d^2\Omega(\mathbf{n})\, |\hat{\mathbf{L}} Y_{\ell,m}(\mathbf{n})|^2. \tag{S.64}$$

The directional integral here is exactly as in eq. (S.48), so it evaluates to $\ell(\ell+1)$, thus

$$I_V \;=\; |H_0|^2\, \ell(\ell+1) \times \int_0^R dr\, r^2 \big(j_\ell(kr)\big)^2 \;=\; |H_0|^2\, \ell(\ell+1) \times \frac{1}{k^3} \int_0^{kR} dx\, x^2 \big(j_\ell(x)\big)^2. \tag{S.65}$$

Furthermore, the upper limit of the last integral here is $kR = X$, one of the zeroes $x_{\ell,n}$ of the $j_\ell(x)$ function. Hence, thanks to the identity (12), the integral evaluates to

$$\int_0^{X=kR} dx\, x^2 \big(j_\ell(x)\big)^2 \;=\; \frac{X^3}{2} \times \big(j'_\ell(X)\big)^2 \;=\; \frac{(kR)^3}{2} \times \big(j'_\ell(kR)\big)^2, \tag{S.66}$$

so altogether

$$I_V \;=\; \frac{|E_0|^2}{Z_0^2}\, \ell(\ell+1) \times \frac{R^3}{2} \times \big(j'_\ell(kR)\big)^2. \tag{S.67}$$

Taking the ration of this volume integral to the surface integral (S.61), we get

$$\frac{I_V}{I_S} \;=\; \frac{R}{2} \tag{S.68}$$

hence geometry factor

$$\hat{G} \;=\; (k = \omega/c) \times \frac{I_V}{I_S} \;=\; \frac{kR}{2} \;=\; \frac{X = x_{\ell,n}}{2} \tag{S.69}$$

and therefore the quality factor

$$Q \;=\; \frac{Z_0}{2R_s} \times \Big(kR \;=\; (\omega R/c) \;=\; x_{\ell,n}\Big), \tag{S.70}$$

exactly as in eq. (10).

Quod erat demonstrandum.

PS: In terms of the skin depth δ of the metal surrounding the cavity,

$$\frac{Z_0}{2R_s} \times \frac{\omega R}{c} \;=\; \frac{R}{\delta} \tag{S.71}$$

(assuming the metal is non-magnetic, $\mu = 1$), so eqs. (11) and (12) for the quality factor become

$$Q \;=\; \frac{R}{\delta} \tag{S.72}$$

for all the TE waves, and

$$Q \;=\; \frac{R}{\delta} \times \left(1 \;-\; \frac{\ell(\ell+1)}{y_{\ell,n}^2 = (\omega R/c)^2}\right) \tag{S.73}$$

for the TM waves. Indeed, using

$$R_s \;=\; \frac{1}{\sigma\delta}\,, \qquad \delta^2 \;=\; \frac{2}{\sigma\omega\mu_0}\,, \tag{S.74}$$

we find

$$\frac{Z_0}{2R_s} \times \frac{\omega R}{c} \Big/ \frac{R}{\delta} \;=\; \frac{Z_0\sigma\delta}{2} \times \frac{\omega\delta}{c} \;=\; \frac{Z_0\sigma\omega}{2c} \times \left(\delta^2 \;=\; \frac{2}{\sigma\omega\mu_0}\right) \;=\; \frac{Z_0}{c\mu_0} \;=\; 1 \tag{S.75}$$

and hence eq. (S.71).