Fermions of the ElectroWeak Theory

The Quarks, The Leptons, and their Masses.

This is my second set of notes on the Glashow–Weinberg–Salam theory of weak and electromagnetic interactions. The first set was about the bosonic fields of the theory — the gauge fields of the $SU(2)\times U(1)$ gauge theory and the Higgs fields that give mass to the $W^\pm_\mu$ and Z^0_μ vector particles. This set is about the fermionic fields — the quarks and the leptons.

From the fermionic point of view, the electroweak gauge symmetry $SU(2)_W\times U(1)_Y$ is *chiral* — the left-handed and the right-handed fermions form different types of multiplets — and consequently, the weak interactions do not respect the parity or the charge-conjugation symmetries. Specifically, all the left-handed quarks and leptons form doublets of the $SU(2)_W$ while the all right-handed quarks and leptons are singlets, so the charged weak currents are purely left-handed,

$$J^\mu_\pm \;=\; \tfrac12\bigl(V^\mu\,-\,A^\mu\bigr) \;=\; \overline{\Psi}\gamma^\mu\frac{1-\gamma^5}{2}\Psi \;=\; \psi^\dagger_L\bar\sigma^\mu\psi_L \;\;\text{without a } \psi_R \text{ term.} \tag{1}$$

The left-handed and the right-handed fermions also have different $U(1)$ hypercharges, which is needed to give them similar electric charges $Q=Y+T^3$. For example, the LH up and down quarks — which form an $SU(2)_W$ doublet — have $Y=+\frac16$, while the RH quarks are $SU(2)$ singlets and have $Y_u=+\frac23$ and $Y_d=-\frac13$. Consequently, their electric charges come out to be

$$\begin{aligned}
\left.\begin{aligned}
Q(u,L) &= Y(u,L)+T^3(u,L) = +\tfrac16+\tfrac12 = +\tfrac23\\
Q(u,R) &= Y(u,R)+T^3(u,R) = +\tfrac23+0 = +\tfrac23
\end{aligned}\right\}&\quad\text{same,}\\
\left.\begin{aligned}
Q(d,L) &= Y(d,L)+T^3(d,L) = +\tfrac16-\tfrac12 = -\tfrac13\\
Q(d,R) &= Y(d,R)+T^3(d,R) = -\tfrac13+0 = -\tfrac13
\end{aligned}\right\}&\quad\text{same.}
\end{aligned}\tag{2}$$

Similarly, the other four flavors — strange, charm, bottom, and top — of the left-handed quarks form $SU(2)_W$ singlets of hypercharge $Y=+\frac16$, while the right-handed quarks of the same flavors are $SU(2)_W$ singlets of hypercharges $Y_s=Y_b=-\frac13$ and $Y_c=Y_t=+\frac23$. Consequently, once the $SU(2)_W\times U(1)_Y$ is Higgsed down to the $U(1)_{\rm EM}$, all the quarks end up with vector-like electric charges

$$\begin{aligned} Q(s,L) &= Q(s,R) = -\tfrac{1}{3},\\ Q(c,L) &= Q(c,R) = +\tfrac{2}{3},\\ Q(b,L) &= Q(b,R) = -\tfrac{1}{3},\\ Q(t,L) &= Q(t,R) = +\tfrac{2}{3}. \end{aligned} \tag{3}$$

Likewise, the left-handed leptons form 3 $SU(2)_W$ doublets $(\nu_e, e^-)_L$, $(\nu_\mu, \mu^-)_L$, $(\nu_\tau, \tau^-)_L$ of hypercharge $Y = -\frac{1}{2}$, while the right handed leptons are singlets: the charged RH leptons have hypercharges $Y(e^-_R) = Y(\mu^-_R) = Y(\tau^-_R) = -1$, while the RH neutrinos — if they exist at all — have zero hypercharges, $Y(\nu_e, R) = Y_(\nu_\mu, R) = Y(\nu_\tau, R) = 0$. Again, the LH and the RH fermions of the same species end up with similar electric charges,

$$\begin{aligned} &\left.\begin{aligned} Q(e^-,L) &= Q(\mu^-,L) = Q(\tau^-,L) = Y + T^3 = -\tfrac{1}{2} - \tfrac{1}{2} = -1\\ Q(e^-,R) &= Q(\mu^-,R) = Q(\tau^-,R) = Y + T^3 = -1 + 0 = -1 \end{aligned}\right\} \text{same,}\\ &\left.\begin{aligned} Q(\nu_e,L) &= Q(\nu_\mu,L) = Q(\nu_\tau,L) = Y + T^3 = -\tfrac{1}{2} + \tfrac{1}{2} = 0\\ Q(\nu_e,R) &= Q(\nu_\mu,R) = Q(\nu_\tau,R) = Y + T^3 = 0 + 0 = 0 \end{aligned}\right\} \text{same.} \end{aligned} \tag{4}$$

In light of different $SU(2)_W \times U(1)_W$ quantum numbers for the LH and RH fermions, the electroweak Lagrangian cannot have any fermion mass terms $\psi^\dagger_L \psi_R$ or $\psi^\dagger_R \psi_L$. Instead, the physical quark and lepton masses arise from the Yukawa couplings of the quarks to the Higgs scalars H_i, as we shall see in a moment.

Yukawa Couplings, Scalar VEVs, and Fermion Masses.

Many field theories have chiral symmetries — global or local, discrete or continuous, does not matter — which forbid the fermion mass terms $\psi^\dagger_L \psi_R$ or $\psi^\dagger_R \psi_L$ in the theory's Lagrangian. However, if the symmetry is spontaneously broken by non-zero vacuum expectation values (VEVs) of some scalar fields, and if these scalar fields have Yukawa couplings to the fermions, then the fermions may end up with non-zero physical masses.

To see how it works, consider a theory of one massless Dirac field Ψ and one real scalar field ϕ with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{8}(\phi^2 - v^2)^2 + i\overline{\Psi}\partial\!\!\!/\Psi - g\phi \times \overline{\Psi}\Psi. \tag{5}$$

This theory has a discrete $\mathbf{Z}_2$ symmetry

$$\phi(x) \;\to\; -\phi(x), \quad \Psi(x) \;\to\; +\gamma^5\Psi(x), \quad \overline{\Psi}(x) \;\to\; -\overline{\Psi}(x)\gamma^5 \tag{6}$$

which acts on the fermions in a chiral manner and thus forbids the fermion mass term in the Lagrangian (5). However, this symmetry is spontaneously broken by the scalar's VEV $\langle\phi\rangle = \pm v \neq 0$, and in terms of the shifted scalar field

$$\sigma(x) \;=\; \phi(x) \;-\; \langle\phi\rangle \tag{7}$$

the Lagrangian (5) becomes

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}(\partial_\mu\sigma)^2 \;-\; \frac{\lambda v^2}{2}\,\sigma^2 \;\mp\; \frac{\lambda v}{2}\,\sigma^3 \;-\; \frac{\lambda}{8}\,\sigma^4 \\ & +\; i\overline{\Psi}\partial\!\!\!/\Psi \;-\; g\,\langle\phi\rangle \times \overline{\Psi}\Psi \;-\; g\sigma \times \overline{\Psi}\Psi.\end{aligned} \tag{8}$$

As far as the Dirac field is concerned, the red term on the second line here is the *emergent mass term* $m = g\,\langle\phi\rangle$, while the blue term is the Yukawa coupling to the physical scalar field $\sigma(x)$.

In general, the Yukawa couplings of fermions to scalars (or pseudoscalars) have form

$$\text{true scalar:} \quad g_s\phi \times \overline{\Psi}\Psi \;=\; g_s\phi \times \left(\psi_L^\dagger\psi_R \,+\, \psi_R^\dagger\psi_L\right), \tag{9}$$

$$\text{pseudoscalar:} \quad ig_p\phi \times \overline{\Psi}\gamma^5\Psi \;=\; g_p\phi \times \left(i\psi_L^\dagger\psi_R \,-\, i\psi_R^\dagger\psi_L\right), \tag{10}$$

or for a complex scalar field Φ without parity symmetry

$$\mathcal{L} \;\supset\; g\Phi \times \psi_L^\dagger\psi_R \;+\; g^*\Phi^* \times \psi_R^\dagger\psi_L \tag{11}$$

with a complex coupling constant $g = g_s + ig_p$. Theories with multiple fermionic and scalar fields may have different Yukawa couplings for different scalar and fermionic species, as long as they are invariant under all the required symmetries. In particular, if a symmetry allows the Yukawa coupling (11) but forbids the Lagrangian mass terms $\psi_L^\dagger\psi_R$ and $\psi_R^\dagger\psi_L$, then it must act non-trivially on the scalar fields Φ. Consequently, if these scalar fields happen to acquire non-zero VEVs, then such VEVs will spontaneously break the symmetry in question, and that's what allows the fermions to get an emergent mass $m = g\,\langle\Phi\rangle \neq 0$.

Now let's see how this mass-generating mechanism works in the electroweak theory. In this theory, all the left-handed quarks and leptons are $SU(2)$ doublets while all the right-handed quarks and leptons are $SU(2)$ singlets, so all the bilinears $\psi_L^\dagger\psi_R$ and $\psi_R^\dagger\psi_L$ transform as $SU(2)$ doublets. This immediately forbids the Lagrangian mass terms but allow the Yukawa couplings of those bilinears to the Higgs doublet H^i or its conjugate H_i^*, depending on the $U(1)_Y$ hypercharges of the fields involved. For example, consider the up and down quark flavors. The bilinear $\psi_L^\dagger(u,d)\psi_R(u)$ had net hypercharge

$$Y\bigl(\psi_L^\dagger(u,d)\psi_R(u)\bigr) \;=\; -Y\bigl(\psi_L^{u,d}\bigr) \;+\; Y\bigl(\psi_R^u\bigr) \;=\; -\tfrac{1}{6} \;+\; \tfrac{2}{3} \;=\; +\tfrac{1}{2}, \tag{12}$$

so it may couple to the H^* field which has $Y(H^*) = -\frac{1}{2}$. Likewise, the bilinear $\psi_L^\dagger(u,d)\psi_R(d)$ had net hypercharge

$$Y\bigl(\psi_L^\dagger(u,d)\psi_R(d)\bigr) \;=\; -Y\bigl(\psi_L^{u,d}\bigr) \;+\; Y\bigl(\psi_R^d\bigr) \;=\; -\tfrac{1}{6} \;-\; \tfrac{1}{3} \;=\; -\tfrac{1}{2}, \tag{13}$$

so it may couple to the H field which has $Y(H) = +\frac{1}{2}$. Altogether, we get

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} \;=\;& g_u\epsilon^{ij}H_i^*\times\psi_{L,j}^\dagger\psi_R^u \;-\; g_dH^i\times\psi_{L,i}^\dagger\psi_R^d \\ &+\; g_u\epsilon_{ij}H^i\times\psi_R^{u\dagger}\psi_L^j \;-\; g_dH_i^*\times\psi_R^{d\dagger}\psi_L^i\end{aligned} \tag{14}$$

where $i,j=1,2$ are the $SU(2)$ doublet indices (and the color indices are not written down). Note that every term in eq. (14) is $SU(2)$ invariant and has zero net Y charge.

But *when the Higgs doublet of the GWS theory develops a non-zero Vacuum Expectation Value, it breaks the chiral symmetry of the fermions and gives them emergent masses.* Indeed, *in the unitary gauge*

$$\langle H\rangle \;=\; \frac{v}{\sqrt{2}}\times\begin{pmatrix}0\\1\end{pmatrix}, \qquad v\approx 247\text{ GeV}, \quad H(x) \;=\; \frac{v+h(x)}{\sqrt{2}}\times\begin{pmatrix}0\\1\end{pmatrix}, \tag{15}$$

so the Yukawa terms (14) become

$$\mathcal{L}_{\text{Yukawa}} \;=\; -\frac{g_u(v+h)}{\sqrt{2}}\Bigl(\psi_R^{u\dagger}\psi_L^1 \;+\; \psi_L^{1\dagger}\psi_R^u\Bigr) - \frac{g_d(v+h)}{\sqrt{2}}\Bigl(\psi_R^{d\dagger}\psi_L^2 \;+\; \psi_L^{2\dagger}\psi_R^d\Bigr). \tag{16}$$

These terms contain both the Yukawa couplings to the physical Higgs field h and the *emergent* *mass terms*

$$\mathcal{L}_{\text{mass}} \;=\; -m_u\Big(\psi_R^{u\dagger}\psi_L^1 \,+\, \psi_L^{1\dagger}\psi_R^u\Big) \;-\; m_d\Big(\psi_R^{d\dagger}\psi_L^2 \,+\, \psi_L^{2\dagger}\psi_R^d\Big) \tag{17}$$

where

$$m_u \;=\; g_u \times \frac{v}{\sqrt{2}}\,, \qquad m_d \;=\; g_d \times \frac{v}{\sqrt{2}}\,. \tag{18}$$

Note: *in the unitary gauge* (15), the mass terms (17) connect the upper member ψ_L^1 of the LH quark doublet to the RH up quark and the lower member ψ_L^2 to the RH down quark, so we may identify the ψ_L^1 as the LH up quark and the ψ_L^2 as the LH down quark. But this identification depends on the Higgs VEV being as in eq. (15), so in the non-unitary gauges we would need Higgs-dependent identifications which of the LH quarks is up and which is down:

$$\psi_L^u \;=\; -\frac{\sqrt{2}}{v}\,\epsilon_{ij}H^i\psi_L^j\,, \qquad \psi_L^d \;=\; +\frac{\sqrt{2}}{v}\,H_i^*\psi_L^i\,. \tag{19}$$

To avoid this complication, we shall stick to the unitary gauge throughout these notes, so the emergent quark mass terms are simply (17). These are Dirac mass terms in the Weyl fermion notations; in Dirac fermion notations, they become

$$\mathcal{L}_{\text{mass}} \;=\; -m_u\overline{\Psi}_u\Psi^u \;-\; m_d\overline{\Psi}_d\Psi^d \tag{20}$$

for

$$\Psi^u \;=\; \begin{pmatrix} \psi_L^1 \\ \psi_R^u \end{pmatrix}, \qquad \Psi^d \;=\; \begin{pmatrix} \psi_L^2 \\ \psi_R^d \end{pmatrix}. \tag{21}$$

The other 4 quark flavors — charm, strange, top, and bottom — have similar quantum numbers to the up and down quarks, so they do not have any Lagrangian mass terms but they do have Yukawa couplings similar to (14),

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} \supset\; & g_c\epsilon^{ij}H_i^* \times \psi_{L,j}^\dagger(c,s)\psi_R(c) \;-\; g_sH^i \times \psi_{L,i}^\dagger(c,s)\psi_R(s) \\ & +\; g_t\epsilon^{ij}H_i^* \times \psi_{L,j}^\dagger(t,b)\psi_R(t) \;-\; g_bH^i \times \psi_{L,i}^\dagger(t,b)\psi_R(b) \\ & +\; \text{Hermitian conjugates.}\end{aligned} \tag{22}$$

And when the Higgs scalar develops a non-zero VEV and breaks the chiral $SU(2)\times U(1)$ symmetry down to the non-chiral $U(1)_{\text{EM}}$, all these quarks get emergent masses just like the up and the down quarks:

$$\mathcal{L}_{\text{Yukawa}} \supset \mathcal{L}_{\text{mass}} = -m_s\overline{\Psi}_s\Psi^s - m_c\overline{\Psi}_c\Psi^c - m_b\overline{\Psi}_b\Psi^b - m_t\overline{\Psi}_t\Psi^t, \tag{23}$$

where

$$\Psi^s = \begin{pmatrix} \psi^2_L(c,s) \\ \psi_R(s) \end{pmatrix}, \quad \Psi^c = \begin{pmatrix} \psi^1_L(c,s) \\ \psi_R(c) \end{pmatrix}, \quad \Psi^b = \begin{pmatrix} \psi^2_L(t,b) \\ \psi_R(b) \end{pmatrix}, \quad \Psi^t = \begin{pmatrix} \psi^1_L(t,b) \\ \psi_R(t) \end{pmatrix}, \tag{24}$$

and

$$m_s = g_s \times \frac{v}{\sqrt{2}}, \quad m_c = g_c \times \frac{v}{\sqrt{2}}, \quad m_b = g_b \times \frac{v}{\sqrt{2}}, \quad m_t = g_t \times \frac{v}{\sqrt{2}}. \tag{25}$$

However, while the charge $= +\frac{2}{3}$ quarks u, c, t have exactly similar electroweak quantum numbers, they have very different values of the Yukawa couplings, $g_u \ll g_c \ll g_t$, and hence very different physical masses, $m_u \ll m_c \ll m_t$. Likewise, the charge $= -\frac{1}{3}$ quarks d, s, t have exactly similar electroweak quantum numbers but different Yukawa couplings, $g_d \ll g_s \ll g_b$, and hence different physical masses, $m_d \ll m_s \ll m_b$. Experimentally

$$m_u \approx 2.15 \text{ MeV} \ll m_c \approx 1.28 \text{ GeV} \ll m_t \approx 173 \text{ GeV}, \tag{26}$$

$$m_d \approx 4.7 \text{ MeV} \ll m_s \approx 94 \text{ MeV} \ll m_b \approx 4.2 \text{ GeV}, \tag{27}$$

but we do not have a good explanation of this hierarchical pattern. In the Standard Model, the Yukawa couplings are arbitrary parameters to be determined experimentally. Beyond the Standard Model, there have been all kinds of speculative explanations over the last 40+ years, but none of them can be supported by any experimental evidence whatsoever.

Besides the quarks, there are 3 species of charged leptons — the electron e^-, the muon μ^-, and the tau τ^-. Similar to the quarks, the LH and the RH leptons have different $SU(2)_W \times U(1)_Y$ quantum numbers, so they cannot have any mass terms in the electroweak Lagrangian. Instead, their masses emerge from the Higgs VEV and the Yukawa couplings

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} = &-g_e H^*_i \times \psi^\dagger_R(e)\psi^i_L(\nu_e, e) - g_e H^i \times \psi^\dagger_{L,i}(\nu_e, e)\psi_R(e) \\ &- g_\mu H^*_i \times \psi^\dagger_R(\mu)\psi^i_L(\nu_\mu, \mu) - g_\mu H^i \times \psi^\dagger_{L,i}(\nu_\mu, \mu)\psi_R(\mu) \\ &- g_\tau H^*_i \times \psi^\dagger_R(\tau)\psi^i_L(\nu_\tau, \tau) - g_\tau H^i \times \psi^\dagger_{L,i}(\nu_\tau, \tau)\psi_R(\tau). \end{aligned} \tag{28}$$

Since all the LH leptons here are $SU(2)$ doublets with $Y = -\frac{1}{2}$, all the RH charged leptons are singlets with $Y = -1$, and Higgs H^i is a doublet with $Y = +\frac{1}{2}$, all these Yukawa terms are $SU(2)_W \times U(1)_Y$ invariant, so they are perfectly legitimate Lagrangian terms. And when the Higgs gets its VEV, these Yukawa terms yield emergent mass terms

$$\mathcal{L}_{\text{Yukawa}} \supset \mathcal{L}_{\text{mass}} = -m_e \overline{\Psi}_e \Psi^e - m_\mu \overline{\Psi}_\mu \Psi^\mu - m_\tau \overline{\Psi}_\tau \Psi^\tau, \tag{29}$$

where

$$\Psi^e = \begin{pmatrix} \psi_L^2(\nu_e, e) \\ \psi_R(e) \end{pmatrix}, \quad \Psi^\mu = \begin{pmatrix} \psi_L^2(\nu_\mu, \mu) \\ \psi_R(\mu) \end{pmatrix}, \quad \Psi^\tau = \begin{pmatrix} \psi_L^2(\nu_\tau, \tau) \\ \psi_R(\tau) \end{pmatrix}, \tag{30}$$

(in the unitary gauge), and

$$m_e = g_e \times \frac{v}{\sqrt{2}}, \qquad m_\mu = g_\mu \times \frac{v}{\sqrt{2}}, \qquad m_\tau = g_\tau \times \frac{v}{\sqrt{2}}. \tag{31}$$

Experimentally, these masses are

$$m_e = 0.511 \text{ MeV} \ll m_\mu = 106 \text{ MeV} \ll m_\tau = 1777 \text{ MeV}. \tag{32}$$

Similar to the quarks, these masses form a hierarchy due to very different Yukawa couplings $g_e \ll g_\mu \ll g_\tau$, but nobody knows why these couplings are so different.

As to the neutrino masses, the original Glashow–Weinberg–Salam theory considered them to be massless LH Weyl fields without any RH counterparts. But once the neutrino oscillations were experimentally discovered, the GWS theory was extended to allow tiny but non-zero neutrino masses. In fact, there are two different theories of the neutrino masses: In one theory, the neutrinos are left-handed Weyl spinors with emergent Majorana masses, while the RH neutrino fields do not exist. In the other theory, the neutrinos have emergent Dirac masses, so the RH neutrino fields do exists, but they are $SU(2)$ singlets with $Y = 0$ and therefore do not have any weak interactions. Since they also do not have strong or EM interactions, this makes the RH neutrinos completely invisible to the experiment — and that's why we do not know if they exist or not. For the moment, let me focus on the simplest version without the $\psi_R(\nu)$; I'll come back to the other theory I'll in the separate set of notes on neutrino masses.

Weak Currents

Altogether, the fermionic fields of the electroweak theory and their couplings to the bosonic gauge and Higgs fields can be summarized by the Lagrangian

$$\mathcal{L}_F \;=\; \sum_{\substack{\text{LH quarks}\\ \text{\& leptons}}} i\psi_L^\dagger \bar{\sigma}^\mu D_\mu \psi_L \;+\; \sum_{\substack{\text{RH quarks}\\ \text{\& leptons}}} i\psi_R^\dagger \sigma^\mu D_\mu \psi_R \;+\; \mathcal{L}_{\text{Yukawa}}\,. \qquad (33)$$

In the first section of these notes I was focused on the Yukawa couplings that give rise to the fermion masses when the Higgs field gets its VEV, but now let's turn our attention to the interactions of quarks and leptons with the electroweak $SU(2)\times U(1)$ gauge fields. In the Lagrangian (33), the gauge interactions are hidden inside the covariant derivatives D_μ, so let's spell them out. Since these notes are about on the electroweak interactions rather than strong interactions, let me skip suppress the quarks' color indices and ignore their couplings to the gluon fields, thus:

- The left-handed quarks form $SU(2)$ doublets

$$\psi_L^i \;=\; \begin{pmatrix} u \\ d \end{pmatrix}_L \quad\text{or}\quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad\text{or}\quad \begin{pmatrix} t \\ b \end{pmatrix}_L \qquad (34)$$

  of hypercharge $Y=+\frac{1}{6}$, so for the LH quark fields

$$D_\mu\psi_L^i \;=\; \partial_\mu\psi_L^i \;+\; \frac{ig_2}{2}\,W_\mu^a(\tau^a)^i{}_j\psi_L^j \;+\; \frac{ig_1}{6}\,B_\mu\psi_L^i\,.$$

- The left-handed leptons also form $SU(2)$ doublets

$$\psi_L^i \;=\; \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad\text{or}\quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad\text{or}\quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \qquad (35)$$

  but of hypercharge $Y=-\frac{1}{2}$, so for the LH lepton fields

$$D_\mu\psi_L^i \;=\; \partial_\mu\psi_L^i \;+\; \frac{ig_2}{2}\,W_\mu^a(\tau^a)^i{}_j\psi_L^j \;-\; \frac{ig_1}{2}\,B_\mu\psi_L^i\,.$$

- The right handed quarks are $SU(2)$ singlets of hypercharges $Y = +\frac{2}{3}$ or $Y = -\frac{1}{3}$, thus

$$\begin{aligned} \text{for } \psi_R &= u_R \text{ or } c_R \text{ or } t_R\,, & D_\mu\psi_R &= \partial_\mu\psi_R + \frac{2ig_1}{3}\,B_\mu\psi_R\,, \\ \text{for } \psi_R &= d_R \text{ or } s_R \text{ or } b_R\,, & D_\mu\psi_R &= \partial_\mu\psi_R - \frac{ig_1}{3}\,B_\mu\psi_R\,. \end{aligned} \tag{36}$$

- The right-handed charged leptons are $SU(2)$ singlets of hypercharge $Y = -1$, thus

$$\text{for } \psi_R = e^-_R \text{ or } \mu^-_R \text{ or } \tau^-_R\,, \quad D_\mu\psi_R = \partial_\mu\psi_R - ig_1\,B_\mu\psi_R\,. \tag{37}$$

— Finally, if the right-handed neutrino fields exist at all, they are $SU(2)$ singlets and have zero hypercharge, thus

$$\text{for } \psi_R = \nu^e_R \text{ or } \nu^\mu_R \text{ or } \nu^\tau_R\,, \quad D_\mu\psi_R = \partial_\mu\psi_R + 0. \tag{38}$$

Now let's plug these covariant derivatives into the fermionic Lagrangian (33), extract the terms containing the $SU(2) \times U(1)$ gauge fields, and organize the fermionic fields interacting with those gauge fields into the currents according to

$$\mathcal{L} \supset -g_2 W^a_\mu J^\mu_{Ta} - g_1 B_\mu J^\mu_Y\,, \tag{39}$$

cf. eq. (21) from my notes on the bosonic sector on the electroweak theory. Since the right-handed quarks and leptons are $SU(2)$ singlets, the $SU(2)$ currents turn out to be purely left-handed,

$$J^\mu_{Ta} = \sum_{(u,d),(c,s),(t,b)}^{\text{LH quarks}} \psi^\dagger_{L,i} \left(\frac{\tau^a}{2}\right)^i_j \bar\sigma^\mu \psi^j_L + \sum_{(\nu_e,e),(\nu_\mu,\mu),(\nu_\tau,\tau)}^{\text{LH leptons}} \psi^\dagger_{L,i} \left(\frac{\tau^a}{2}\right)^i_j \bar\sigma^\mu \psi^j_L\,. \tag{40}$$

However, the $U(1)$ current has both left-handed and right handed contributions,

$$\begin{aligned} J^\mu_Y = {} & \sum_{u,c,t \text{ quarks}} \left(\tfrac{1}{6}\psi^\dagger_L\bar\sigma^\mu\psi_L + \tfrac{2}{3}\psi^\dagger_R\sigma^\mu\psi_R\right) + \sum_{d,s,b \text{ quarks}} \left(\tfrac{1}{6}\psi^\dagger_L\bar\sigma^\mu\psi_L - \tfrac{1}{3}\psi^\dagger_R\sigma^\mu\psi_R\right) \\ & + \sum_{e,\mu,\tau \text{ leptons}} \left(-\tfrac{1}{2}\psi^\dagger_L\bar\sigma^\mu\psi_L - \psi^\dagger_R\sigma^\mu\psi_R\right) + \sum_{\text{neutrinos}} \left(-\tfrac{1}{2}\psi^\dagger_L\bar\sigma^\mu\psi_L + 0\right). \end{aligned} \tag{41}$$

In the my notes on the bosonic sector I had re-organized these 4 gauge currents into currents which couple to the specific electroweak gauge field, namely the electric current

$$J^\mu_{\text{EM}} \;=\; J^\mu_{T3} \;+\; J^\mu_Y \tag{42}$$

which couples to the EM field A_μ, the *charged weak currents*

$$J^{+\mu} \;=\; J^\mu_{T1} \;-\; iJ^\mu_{T2} \quad \text{and} \quad J^{-\mu} \;=\; J^\mu_{T1} \;+\; iJ^\mu_{T2} \tag{43}$$

which couple to the charged $W^\pm_\mu$ massive vector fields, and the *neutral weak current*

$$J^\mu_Z \;=\; J^\mu_{T3} \;-\; \sin^2\theta J^\mu_{\text{EM}} \tag{44}$$

which couples to the neutral massive vector field Z^0_μ. Now let's spell out all these currents in terms of the fermionic fields. For the charged currents, reorganizing the weak isospin currents (40) into the $J^{\pm\mu}$ amounts to combining the isospin Pauli matrices τ^a in the same way as the currents (43),

$$\tau^+ \;\equiv\; \tau^1 \;-\; i\tau^2 \;=\; \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad \tau^- \;\equiv\; \tau^1 \;+\; i\tau^2 \;=\; \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{45}$$

Consequently, in eqs. (40) we have

$$\psi^\dagger_{L,i} \left(\frac{\tau^+}{2}\right)^i_j \bar\sigma^\mu \psi^j_L \;=\; \psi^\dagger_{L,2}\bar\sigma^\mu\psi^1_L\,, \qquad \psi^\dagger_{L,i} \left(\frac{\tau^-}{2}\right)^i_j \bar\sigma^\mu \psi^j_L \;=\; \psi^\dagger_{L,1}\bar\sigma^\mu\psi^2_L\,, \tag{46}$$

and therefore

$$\begin{aligned} J^{+\mu} \;&=\; \psi^\dagger_L(d)\bar\sigma^\mu\psi_L(u) \;+\; \psi^\dagger_L(s)\bar\sigma^\mu\psi_L(c) \;+\; \psi^\dagger_L(b)\bar\sigma^\mu\psi_L(t) \\ &\qquad +\; \psi^\dagger_L(e)\bar\sigma^\mu\psi_L(\nu_e) \;+\; \psi^\dagger_L(\mu)\bar\sigma^\mu\psi_L(\nu_\mu) \;+\; \psi^\dagger_L(\tau)\bar\sigma^\mu\psi_L(\nu_\tau), \\ J^{-\mu} \;&=\; \psi^\dagger_L(u)\bar\sigma^\mu\psi_L(d) \;+\; \psi^\dagger_L(c)\bar\sigma^\mu\psi_L(s) \;+\; \psi^\dagger_L(t)\bar\sigma^\mu\psi_L(b) \\ &\qquad +\; \psi^\dagger_L(\nu_e)\bar\sigma^\mu\psi_L(e) \;+\; \psi^\dagger_L(\nu_\mu)\bar\sigma^\mu\psi_L(\mu) \;+\; \psi^\dagger_L(\nu_\tau)\bar\sigma^\mu\psi_L(\tau). \end{aligned} \tag{47}$$

In terms of Dirac fermions for the quarks and leptons,

$$\psi^\dagger_L\bar\sigma^\mu\psi_L \;=\; \overline{\Psi}\gamma^\mu\frac{1-\gamma^5}{2}\Psi, \tag{48}$$

hence

$$J^{+\mu} = \overline{\Psi}^d\gamma^\mu\frac{1-\gamma^5}{2}\Psi^u + \overline{\Psi}^s\gamma^\mu\frac{1-\gamma^5}{2}\Psi^c + \overline{\Psi}^b\gamma^\mu\frac{1-\gamma^5}{2}\Psi^t + \overline{\Psi}^e\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_e} + \overline{\Psi}^\mu\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_\mu} + \overline{\Psi}^\tau\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_\tau},$$

$$J^{-\mu} = \overline{\Psi}^u\gamma^\mu\frac{1-\gamma^5}{2}\Psi^d + \overline{\Psi}^c\gamma^\mu\frac{1-\gamma^5}{2}\Psi^s + \overline{\Psi}^t\gamma^\mu\frac{1-\gamma^5}{2}\Psi^b + \overline{\Psi}^{\nu_e}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^e + \overline{\Psi}^{\nu_\mu}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\mu + \overline{\Psi}^{\nu_\tau}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\tau. \tag{49}$$

As promised, these charged weak currents are purely left-handed, so they completely violate the parity and the charge-conjugation symmetries. But please note that this left-handedness is in terms of chirality of the fermionic fields rather than helicities of the fermionic particles. In terms of helicities, *the quarks and the leptons participating in charged-current weak interactions are polarized left, but the antiquarks and the antileptons are polarized right*; the degree of polarization is $\beta = v/c$, which approaches 100% for the ultra-relativistic particles.

On the other hand, the electric current is left-right symmetric,

$$J^\mu_{\rm EM} = \tfrac{2}{3}\sum_{q=u,c,t}^{\rm quarks}\overline{\Psi}^q\gamma^\mu\Psi^q - \tfrac{1}{3}\sum_{q=d,s,b}^{\rm quarks}\overline{\Psi}^q\gamma^\mu\Psi^q - \sum_{\ell=e,\mu\tau}^{\rm leptons}\overline{\Psi}^\ell\gamma^\mu\Psi^\ell. \tag{50}$$

Finally, the neutral weak current has both left-handed and right-handed components but it is not left-right symmetric. In terms of Dirac spinor fields,

$$\begin{aligned} J^\mu_Z &= J^\mu_{T3}[\text{left-handed}] - \sin^2\theta\times J^\mu_{\rm EM}[\text{left-right symmetric}] \\ &= \sum_{q=u,c,t}^{\rm quarks}\overline{\Psi}^q\gamma^\mu\left(+\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta\right)\Psi^q + \sum_{q=d,s,b}^{\rm quarks}\overline{\Psi}^q\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta\right)\Psi^q \\ &\quad + \sum_{\ell=e,\mu,\tau}^{\rm leptons}\overline{\Psi}^\ell\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \sin^2\theta\right)\Psi^\ell + \sum_{\nu=\nu_e,\nu_\mu,\nu_\tau}^{\rm neutrinos}\overline{\Psi}^\nu\gamma^\mu\left(+\frac{1-\gamma^5}{4} - 0\right)\Psi^\nu. \end{aligned} \tag{51}$$

PS: In these notes, I have ignored the Cabibbo–Kobayashi–Maskawa (CKM) mixing of the quark flavors. I shall address that issue in a next set of notes. For the moment, let me simply say that in the Standard Model the neutral weak current J^μ_Z is exactly as in eq. (51) despite the flavor mixing. However, the charged weak currents $J^\pm_\mu$ become more complicated than in eqs. (49).