GOLDEN RULE and PHASE SPACE FACTORS

Fermi's Golden Rule

Consider a perturbation theory in quantum mechanics, $\hat{H} = \hat{H}_0 + \hat{V}$, where we use the eigenstates of the un-perturbed Hamiltonian $\hat{H}_0$ as a basis while the perturbation $\hat{V}$ causes transitions between these basis states. In particular, consider the transition from some initial state $|i\rangle$ into a *continuum* of some similar final states $|f\rangle$. *Fermi's Golden Rule* gives us the *rate* of such transitions to the first order in the perturbation $\hat{V}$:

$$\Gamma \stackrel{\text{def}}{=} \frac{d\,\text{probability}}{d\,\text{time}} = \frac{2\pi\rho(f)}{\hbar} \times \left|\langle f|\,\hat{V}\,|i\rangle\right|^2 \tag{1}$$

where

$$\rho(f) \stackrel{\text{def}}{=} \frac{dN_{\text{final states}}}{dE_{\text{final}}} \tag{2}$$

is the *final state density*. Equivalently,

$$\Gamma = \int dN_{\text{final}} \left|\langle f|\,\hat{V}\,|i\rangle\right|^2 \times \frac{2\pi}{\hbar}\delta(E_f - E_i), \tag{3}$$

where the energies E_i and E_f are measured by the un-perturbed Hamiltonian $\hat{H}_0$.

To derive the Fermi's Golden Rule, suppose the quantum system in question at time $t_0 = 0$ is in some eigenstate $|i\rangle$ of the un-perturbed Hamiltonian. The probability of finding this state at some later time $t > 0$ in a different eigenstate $|f\rangle$ is

$$P(i \to f) = \left|\langle f|\,\hat{U}_I(t,0)\,|i\rangle\right|^2, \tag{4}$$

where to the first order in the perturbation $\hat{V}$

$$\hat{U}_I(t,0) \approx 1 - \frac{i}{\hbar}\int_0^t dt'\,\hat{V}_I(t') = 1 - \frac{i}{\hbar}\int_0^t dt'\,\exp(+it'\hat{H}_0/\hbar)\hat{V}_S\exp(-it'\hat{H}_0/\hbar), \tag{5}$$

cf. my notes on the Dyson series. Consequently,

$$\begin{aligned}
\langle f|\,\hat{U}_I(t,0)\,|i\rangle \;&\approx\; -\frac{i}{\hbar}\int\limits_0^t dt'\,\langle f|\,e^{+it\hat{H}_0/\hbar}\hat{V}_S e^{-it\hat{H}_0/\hbar}\,|i\rangle \\
&=\; -\frac{i}{\hbar}\int\limits_0^t dt'\;\exp(+it'E_f/\hbar)\,\langle f|\,\hat{V}_S\,|i\rangle\,\exp(-it'E_i/\hbar) \\
&=\; -\frac{i}{\hbar}\,\langle f|\,\hat{V}_S|i\rangle\times\int\limits_0^t dt'\;\exp(it'(E_f-E_i)/\hbar) \\
&=\; \langle f|\,\hat{V}_S|i\rangle\times\frac{1-\exp(it(E_f-E_i)/\hbar)}{E_f-E_i}\,,
\end{aligned}\tag{6}$$

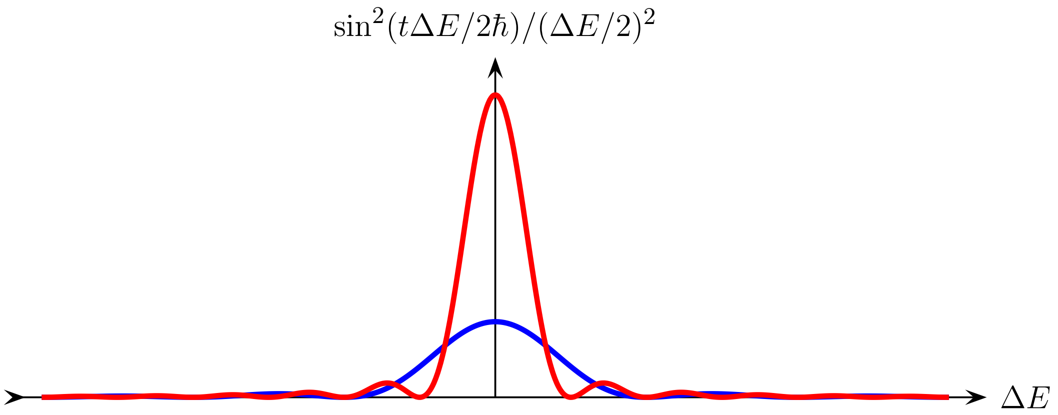
and therefore

$$P(i\to f)\;=\;\left|\langle f|\,\hat{V}_S|i\rangle\right|^2\times\frac{\sin^2(t(E_f-E_i)/2\hbar)}{((E_f-E_i)/2)^2}\,.\tag{7}$$

In general, this probability is very small for any particular final state $|f\rangle$ *unless* $E_f\approx E_i$. However, for $|f\rangle$ belonging to the continuous spectrum of $\hat{H}_0$, we are interested not in the transition to a specific final state but rather into any one of the similar final states $f\in F$,

$$P_{\text{net}}\big(i\to(\text{any f}\in\text{F})\big)\;=\;\int\limits_{\text{f}\in\text{F}}\text{dN}_\text{f}\,\text{P(i}\to\text{f)}\;=\;\int\limits_{\text{f}\in\text{F}}\text{dN}_\text{f}\,\left|\langle\text{f}|\,\hat{\text{V}}_\text{S}|\text{i}\rangle\right|^2\times\frac{\sin^2(\text{t}(\text{E}_\text{f}-\text{E}_\text{i})/2\hbar)}{((\text{E}_\text{f}-\text{E}_\text{i})/2)^2}\,.\tag{8}$$

Under the integral, the energy-dependent factor is strongly peaked for $E_f\approx E_i$, and this peak becomes taller and narrower with the increasing time t. Indeed, let's plot this factor as a function of E_f-E_i for two different values of t: t_1 (blue) and $t_2=2t_1$ (red),



(9)

In the long time limit $t\to\infty$, this energy-dependent factor becomes an infinitely narrow but infinitely high peak, so we may approximate it by a δ-function, or rather

$$\frac{\sin^2(t(E_f-E_i)/2\hbar)}{((E_f-E_i)/2\hbar)^2} \;\to\; \frac{2\pi t}{\hbar}\times\delta(E_f-E_i) \tag{10}$$

where the coefficient stems from

$$\int\limits_{-\infty}^{+\infty} d(\Delta E)\,\frac{\sin^2(t\Delta E/2\hbar)}{(\Delta E/2\hbar)^2} \;=\; \frac{2\pi t}{\hbar}\,. \tag{11}$$

Consequently, in the long time limit eq. (8) for the transition probability becomes

$$P_{\rm net}\big(i\to(\mathrm{any\,f}\in\mathrm{F})\big) \;=\; \frac{2\pi\mathrm{t}}{\hbar}\int\limits_{\mathrm{f}\in\mathrm{F}}\mathrm{dN_f}\,\left|\langle\mathrm{f}|\,\hat{\mathrm{V}}_{\mathrm{S}}|\mathrm{i}\rangle\right|^2\times\delta(\mathrm{E_f}-\mathrm{E_i}), \tag{12}$$

and the overall factor of t in this formula means a *uniform transition rate*

$$\Gamma(i\to f\in F) \;\stackrel{\rm def}{=}\; \frac{dP(i\to f\in F)}{dt} \;=\; \int\limits_{f\in F} dN_f\,\left|\langle f|\,\hat{V}|i\rangle\right|^2\times\frac{2\pi}{\hbar}\delta(E_f-E_i), \tag{3}$$

exactly as in the Fermi's Golden Rule.

Alas, the uniformity-in-time of the transition rate (11) is an artefact of the first-order perturbation theory, which also leads to $P(|i\rangle\text{ stays }|i\rangle)\approx 1$. In reality, we have a uniform ratio

$$\Gamma(i\to f\in F) \;\stackrel{\rm def}{=}\; \frac{1}{P(i\to i)(t)}\times\frac{d}{dt}P(i\to f\in F)(t), \tag{13}$$

where

$$P(|i\rangle\text{ stays }|i\rangle) \;=\; \exp\big(-t\times\Gamma_{\rm tot}(i\to\text{anything})\big) \tag{14}$$

while to the first order in $\hat{V}$

$$\Gamma(i\to f\in F) \;=\; \int\limits_{f\in F} dN_f\,\left|\langle f|\,\hat{V}|i\rangle\right|^2\times\frac{2\pi}{\hbar}\delta(E_f-E_i), \tag{15}$$

To illustrate the Fermi's Golden Rule (15), consider an atom in an excited state emitting a photon while the atom itself drops to a lower energy state. In this example, the initial and the final states are eigenstates of the free Hamiltonian

$$\hat{H}_0 \;=\; \hat{H}(\text{atom}) \;+\; \hat{H}(\text{free photons}) \tag{16}$$

while the interactions between the quantum EM fields and the atom's electrons comprise the perturbation $\hat{V}$. For a moment, let's fix the specific initial and final states of the atom as well as the photon's polarization λ. However, the final states still form a continuous family parametrized by the photon's momentum $\mathbf{p}_\gamma$. In the large-box normalization, the number of such final states is

$$dN_{\text{final}} \;=\; \left(\frac{L}{2\pi}\right)^3 d^3\mathbf{k}_\gamma \;=\; \frac{L^3}{(2\pi)^3}\times k_\gamma^2\,dk_\gamma\,d^2\Omega_\gamma \tag{17}$$

where $d^2\Omega_\gamma$ is the infinitesimal solid angle into which the photon is emitted. At the same time,

$$E_{\text{final}}^{\text{net}} \;-\; E_{\text{initial}}^{\text{net}} \;=\; \hbar c k_\gamma \;+\; E_{\text{final}}^{\text{atom}} \;-\; E_{\text{initial}}^{\text{atom}} \;=\; \hbar c k_\gamma \;-\; \Delta E^{\text{atom}}, \tag{18}$$

hence (to the first order of the perturbation $\hat{V}$)

$$\Gamma \;=\; \frac{1}{(2\pi)^2\hbar}\int\! d^2\Omega_\gamma \int\! dk_\gamma\, k_\gamma^2\times L^3 \left|\left\langle \text{atom}_f+\gamma\right|\hat{V}\left|\text{atom}_i\right\rangle\right|^2\times\delta(\hbar c k_\gamma-\Delta E^{\text{atom}}). \tag{19}$$

In this formula, the L^3 factor in the density of states factor cancels against the (square of the) $L^{-3/2}$ factor in the matrix element due to the photon's wave function in the large-box normalization. Specifically, in the electric dipole approximation to the interaction between the EM fields and the atom

$$\hat{V} \;\approx\; -\hat{\mathbf{E}}(\mathbf{x}_{\text{atom}})\cdot\hat{\mathbf{d}} \tag{20}$$

where $\hat{\mathbf{d}}$ is the atom's electric dipole moment, we have

$$\begin{aligned}\left\langle \text{atom}_f+\gamma\right|\hat{V}\left|\text{atom}_i\right\rangle \approx&\; -\left\langle\gamma(\mathbf{k},\lambda)\right|\hat{\mathbf{E}}(\mathbf{x}_{\text{atom}}\left|\text{vac}\right\rangle\cdot\left\langle f\right|\hat{\mathbf{d}}\left|i\right\rangle_{\text{atom}}\\ =&\; -iL^{-3/2}\sqrt{2\pi\hbar\omega_{\mathbf{k}}}\,e^{-i\mathbf{k}\cdot\mathbf{x}_{\text{atom}}}\,\mathbf{e}^*_{\mathbf{k},\lambda}\cdot\left\langle f\right|\hat{\mathbf{d}}\left|i\right\rangle_{\text{atom}}\end{aligned} \tag{21}$$

where $\mathbf{e}_{\mathbf{k},\lambda}$ is the photon's polarization vector. Plugging this matrix element into the Golden

Rule equation (19), we arrive at

$$\begin{aligned}\Gamma &\approx \int d^2\Omega_\gamma \int dk_\gamma\, k_\gamma^2 \times \frac{\omega = ck_\gamma}{2\pi} \left| \mathbf{e}^*_{\mathbf{k},\lambda} \cdot \langle f | \hat{\mathbf{d}} | i \rangle_{\text{atom}} \right|^2 \times \delta(\hbar c k_\gamma - \Delta E^{\text{atom}}) \\ &\quad \langle\!\langle \text{ integrating over the } k_\gamma \rangle\!\rangle \\ &= \frac{(\Delta E^{\text{atom}})^3}{2\pi\hbar^4 c^3} \int d^2\Omega_\gamma \left| \mathbf{e}^*_{\mathbf{k},\lambda} \cdot \langle f | \hat{\mathbf{d}} | i \rangle_{\text{atom}} \right|^2 .\end{aligned} \tag{22}$$

At this point, we may drop the $\int d\Omega$ integral and get the partial rate of photon emission in a particular direction,

$$\frac{d\Gamma}{d\Omega_\gamma} = \frac{(\Delta E^{\text{atom}})^3}{2\pi\hbar^4 c^3} \times \left| \mathbf{e}^*_{\mathbf{k},\lambda} \cdot \langle f | \hat{\mathbf{d}} | i \rangle_{\text{atom}} \right|^2 . \tag{23}$$

For the completely specified initial and final states of the atom — including their angular momentum quantum numbers j and m_j — the partial emission rates (23) dependent of the photon's direction, so they are worth calculating and comparing to the experiment.

Alternatively, we may not only integrate over the photon's directions as in eq. (22) but also sum over some discrete quantum numbers which we are not going to detect experimentally, for example the photon's polarization λ and the m_j of the atom's final state. This gives us a more inclusive transition rate

$$\begin{aligned}\Gamma &= \frac{(\Delta E^{\text{atom}})^3}{2\pi\hbar^4 c^3} \times \int d^2\Omega_\gamma \sum_\lambda \sum_{m_j(f)} \left| \mathbf{e}^*_{\mathbf{k},\lambda} \cdot \langle f | \hat{\mathbf{d}} | i \rangle_{\text{atom}} \right|^2 \\ &= \frac{4(\Delta E^{\text{atom}})^3}{3\hbar^4 c^3} \times \sum_{m_j(f)} \left| \langle f | \hat{\mathbf{d}} | i \rangle_{\text{atom}} \right|^2 .\end{aligned} \tag{24}$$

Beyond the First Order

Beyond the first order of the perturbation theory, Fermi's Golden Rule for the transition rates — or rather ratios of transitions rates to the probability of the initial state's survival

$$\Gamma(i\to f\in F) \stackrel{\rm def}{=} \frac{1}{P(i\to i)(t)}\times\frac{d}{dt}P(i\to f\in F)(t), \tag{25}$$

— becomes

$$\Gamma(i\to f\in F) = \int\limits_{f\in F} dN_f \left|\langle f|\,\hat{T}(E)\,|i\rangle\right|^2 \times \frac{2\pi}{\hbar}\delta(E_f-E_i) \tag{26}$$

where $E=E_f=E_i$ and

$$\hat{T}(E) = \hat{V} + \hat{V}\frac{1}{E+i\epsilon-\hat{H}_0}\hat{V} + \hat{V}\frac{1}{E+i\epsilon-\hat{H}_0}\hat{V}\frac{1}{E+i\epsilon-\hat{H}_0}\hat{V} + \cdots. \tag{27}$$

The modified transition rate (26) and the Lippmann–Schwinger series (27) follow from the higher-order terms in the Dyson series for the evolution operator $\hat{U}_i(t,0)$, similarly to the first order term giving rise to the Fermi's Golden rule. However, the formal derivation of eqs. (26) and (27) is rather complicated, so let me skip it from these notes, especially since this subject is not directly germane to the Quantum Field Theory.

Also, let me state without proof the relation between the $\hat{T}(E)$ operator and the S–matrix elements. For any 2 eigenstates $|i\rangle$ and $\langle f|$ of the un-perturbed Hamiltonian $\hat{H}_0$,

$$\langle f|\,\hat{S}\,|i\rangle = \langle f|i\rangle + 2\pi i\delta(E_f-E_i)\times\langle f|\,\hat{T}(E)\,|i\rangle\,. \tag{28}$$

Also, for the potential scattering of two non-relativistic particles — for which

$$\hat{H}_0 = \frac{\hat{\mathbf{P}}^2_{\rm reduced}}{2M_{\rm reduced}} \quad\text{and}\quad \hat{V} = V(\hat{\mathbf{x}}_{\rm relative}), \tag{29}$$

— the scattering amplitude is related to the $\hat{T}(E)$ operator as

$$f(\mathbf{p}\to\mathbf{p}') = -\frac{M_{\rm red}}{2\pi\hbar^2}\,\langle\mathbf{p}'|\,\hat{T}(E)\,|\mathbf{p}\rangle\,. \tag{30}$$

Phase Space Factors

Let us apply the Fermi's Golden Rule — or rather the modified rule (26) — to the decay of an unstable particle into n daughter particles. For simplicity, let's ignore all the particles' spins and focus only on their momenta, thus $|\text{initial}\rangle = |\mathbf{p}_0\rangle$ and $\langle\text{final}| = \langle\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_n|$. Naively, in the bog-box normalization, the density of such n-particle final states is

$$dN_f = \prod_{i=1}^{n} d\#(\mathbf{p}_i) = \prod_{i=1}^{n} \frac{L^3\, d^3\mathbf{p}'_i}{(2\pi\hbar)^3}\,; \tag{31}$$

however, due to net momentum conservation

$$\left\langle \mathbf{p}'_1, \ldots, \mathbf{p}'_n \right| \hat{T} \left| \mathbf{p}_0 \right\rangle = \delta_{\mathbf{p}'_1+\cdots+\mathbf{p}'_n,\mathbf{p}_0} \times \text{analytic} \left\langle \mathbf{p}'_1, \ldots, \mathbf{p}'_n \right| \widetilde{T} \left| \mathbf{p}_0 \right\rangle, \tag{32}$$

and since the square of the Kronecker $\delta_{\mathbf{p}'_{\text{net}},\mathbf{p}_0}$ is the same as the $\delta_{\mathbf{p}'_{\text{net}},\mathbf{p}_0}$ itself, the Golden Rule integral (26) becomes

$$\begin{aligned}
\Gamma &= \int d\#(\mathbf{p}'_1) \cdots \int d\#(\mathbf{p}'_n) \left| \left\langle \mathbf{p}'_1, \ldots, \mathbf{p}'_n \right| \widetilde{T} \left| \mathbf{p}_0 \right\rangle \right|^2 \times \delta_{\mathbf{p}'_1+\cdots+\mathbf{p}'_n,\mathbf{p}_0} \times \frac{2\pi}{\hbar} \delta(E'_1 + \cdots + E'_n - E_0) \\
&= \int \frac{L^3 d^3\mathbf{p}'_1}{(2\pi\hbar)^3} \cdots \int \frac{L^3 d^3\mathbf{p}'_n}{(2\pi\hbar)^3} \left| \left\langle \mathbf{p}'_1, \ldots, \mathbf{p}'_n \right| \widetilde{T} \left| \mathbf{p}_0 \right\rangle \right|^2 \times \\
&\qquad \times \left(\frac{2\pi\hbar}{L} \right)^3 \delta^{(3)}(\mathbf{p}'_1 + \cdots + \mathbf{p}'_n - \mathbf{p}_0) \times \frac{2\pi}{\hbar} \delta(E'_1 + \cdots + E'_n - E_0).
\end{aligned} \tag{33}$$

Now let's change the normalization convention from the big-box to the infinite space. This removes the powers of L^3 from all the momentum integrals, δ-functions, and also the matrix elements in our formulae; in particular, eq. (32) becomes

$$\left\langle \mathbf{p}'_1, \ldots, \mathbf{p}'_n \right| \hat{T} \left| \mathbf{p}_0 \right\rangle = (2\pi\hbar)^3 \delta^{(3)}(\mathbf{p}'_1 + \cdots + \mathbf{p}'_n - \mathbf{p}_0) \times \left\langle \mathbf{p}'_1, \ldots, \mathbf{p}'_n \right| \widetilde{T} \left| \mathbf{p}_0 \right\rangle, \tag{34}$$

while eq. (33) becomes

$$\begin{aligned}
\Gamma &= \int \frac{d^3\mathbf{p}'_1}{(2\pi)^3} \cdots \int \frac{d^3\mathbf{p}'_n}{(2\pi)^3} \left| \left\langle \mathbf{p}'_1, \ldots, \mathbf{p}'_n \right| \widetilde{T} \left| \mathbf{p}_0 \right\rangle \right|^2 \times \\
&\qquad \times (2\pi)^3 \delta^{(3)}(\mathbf{p}'_1 + \cdots + \mathbf{p}'_n - \mathbf{p}_0) \times (2\pi)\delta(E'_1 + \cdots + E'_n - E_0)
\end{aligned} \tag{35}$$

(where I have also changed the units to $\hbar = c = 1$). Furthermore, in a relativistic theory we may combine the δ-functions for the momentum conservation and the energy conservation into a single 4D δ-function

$$(2\pi)^3\delta^{(3)}(\mathbf{p}'_1+\cdots+\mathbf{p}'_n-\mathbf{p}_0)\times(2\pi)\delta(E'_1+\cdots+E'_n-E_0) \;=\; (2\pi)^4\delta^{(4)}(p'_1+\cdots+p'_n-p_0). \quad (36)$$

Also, we should use the relativistic normalization of the particle states, which changes the transition matrix element to

$$\begin{aligned}\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n|\,\hat{\mathcal{M}}\,|\mathbf{p}_0\rangle \;&\equiv\; \langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n|\,\widetilde{T}\,|\mathbf{p}_0\rangle_{\text{rel}}\\ &=\; \sqrt{2E_0}\prod_{i=1}^{n}\sqrt{2E'_i}\times\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n|\,\widetilde{T}\,|\mathbf{p}_0\rangle_{\text{nonrel}}\,.\end{aligned} \quad (37)$$

Consequently, in eq. (35) we may replace the non-relativistic $\left|\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n|\,\widetilde{T}\,|\mathbf{p}_0\rangle\right|^2$ factor with the relativistic $|\mathcal{M}|^2$ divided by a $2E$ factor for each initial or final particle, thus

$$\Gamma \;=\; \frac{1}{2E_0}\int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\,\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_0\rangle\right|^2\times(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_0). \quad (38)$$

In other words, an unstable particle (0) decays into n final-state particles $(1')+\cdots+(n')$ at the rate

$$\Gamma \;=\; \int d\mathcal{P}_{\text{decay}}\times\left|\langle 1',2',\ldots,n'|\,\mathcal{M}\,|0\rangle\right|^2 \quad (39)$$

where $\mathcal{M}(0\to 1'+\cdots+n')\equiv\langle 1',\ldots,n'|\,\hat{\mathcal{M}}\,|0\rangle$ is the relativistic decay amplitude calculated according to the Feynman rules, and $d\mathcal{P}$ is the *phase space factor*

$$d\mathcal{P}_{\text{decay}} \;=\; \frac{1}{2E_0}\times\prod_{i=1}^{n}\frac{d^3\mathbf{p}'_i}{(2\pi)^3 2E'_i}\times(2\pi)^4\delta^{(4)}(E'_1+\cdots+E'_n-E_0). \quad (40)$$

Likewise, the transition rate for a generic 2-particle to n-particle scattering process is given by

$$\begin{aligned}\Gamma \;=\; \frac{1}{2E_1\times 2E_2}\int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\,&\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_1,p_2\rangle\right|^2\times\\ &\times(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_1-p_2).\end{aligned} \quad (41)$$

In terms of the scattering cross-section σ, the rate (41) is $\Gamma=\sigma\times$flux of initial particles. In the large-box normalization the flux is $L^{-3}|\mathbf{v}_1-\mathbf{v}_2|$, so in the continuum normalization it's simply the relative speed $|\mathbf{v}_1 - \mathbf{v}_2|$. Consequently, the total scattering cross-section is given by

$$\sigma_{\rm tot} \;=\; \frac{1}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|}\int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\,\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_1,p_2\rangle\right|^2\times \\ \times\,(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_1-p_2), \tag{42}$$

or in other words

$$\sigma_{\rm tot} \;=\; \int d\mathcal{P}_{\rm scattering}\times\left|\langle 1',2',\ldots,n'|\,\mathcal{M}\,|1,2\rangle\right|^2 \tag{43}$$

$$\text{for}\quad d\mathcal{P}_{\rm scattering} \;=\; \frac{1}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|}\times\prod_{i=1}^{n}\frac{d^3\mathbf{p}'_i}{(2\pi)^3 2E'_i}\times(2\pi)^4\delta^{(4)}(E'_1+\cdots+E'_n-E_0). \tag{44}$$

A note on Lorentz invariance of decay rates or cross-sections. The matrix elements $\langle\text{final}|\,\mathcal{M}\,|\text{initial}\rangle$ are Lorentz invariant, and so are all the integrals over the final-particles' momenta and the δ-functions. The only non-invariant factor in the decay-rate formula (38) is the pre-integral $1/E_0$, hence the decay rate of a moving particle is

$$\Gamma(\text{moving}) \;=\; \Gamma(\text{rest frame})\times\frac{M}{E} \tag{45}$$

where M/E is precisely the time dilation factor in the moving frame.

As to the scattering cross-section, it should be invariant under Lorentz boosts along the initial axis of scattering, thus the same cross-section in any frame where $\mathbf{p}_1 \parallel \mathbf{p}_2$. This includes the *lab frame* where one of the two particles is initially at rest, the *center-of-mass frame* where $\mathbf{p}_1+\mathbf{p}_2=0$, and any other frame where the two particles collide head-on. And indeed, in any frame where both $\mathbf{p}_1$ and $\mathbf{p}_2$ are parallel to the z axis, the pre-integral factor in eq. (42) for the cross-section becomes

$$\frac{1}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|} \;=\; \frac{1}{4|E_1\mathbf{p}_2-E_2\mathbf{p}_1|} \;=\; \frac{1}{4|\epsilon_{xy\mu\nu}p_1^\mu p_2^\nu|}\,, \tag{46}$$

which is manifestly invariant under the $SO^+(1,1)$ group of Lorentz boosts along the z axis.

Let's simplify eq. (42) for a 2 particle $\to$ 2 particle scattering process in the center-of-mass frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. In this frame,

$$E_1 E_2 \, |\mathbf{v}_1 - \mathbf{v}_2| \; = \; |E_2 \mathbf{p}_1 - E_1 \mathbf{p}_2| \; = \; |\mathbf{p}| \times (E_1 + E_2 = E_{\text{tot}}), \tag{47}$$

hence the phase-space factor

$$\mathcal{P}_{\text{scattering}} \; = \; \frac{1}{4|\mathbf{p}|E_{\text{tot}}} \times \mathcal{P}_{\text{int}} \tag{48}$$

for

$$\begin{aligned}
\mathcal{P}_{\text{int}} &= \int\!\frac{d^3\mathbf{p}_1'}{(2\pi)^3\, 2E_1'} \int\!\frac{d^3\mathbf{p}_2'}{(2\pi)^3\, 2E_2'}\, (2\pi)^4 \delta^{(3)}(\mathbf{p}_1' + \mathbf{p}_2') \delta(E_1' + E_2' - E_{\text{net}}) \\
&= \int\!\frac{d^3\mathbf{p}_1'}{(2\pi)^3 \times 2E_1' \times 2E_2'}\, (2\pi) \delta(E_1'(\mathbf{p}_1') + E_2'(-\mathbf{p}_1') - E_{\text{net}}) \\
&= \int\! d^2\Omega_{\mathbf{p}'} \times \int_0^\infty\! dp'\, \frac{p'^2}{16\pi^2 E_1' E_2'} \times \delta(E_1' + E_2' - E_{\text{tot}}) \\
&= \int\! d^2\Omega_{\mathbf{p}'} \left[\frac{p'^2}{16\pi^2 E_1' E_2'} \Big/ \frac{d(E_1' + E_2')}{dp'} \right]_{E_1' + E_2' = E_{\text{tot}}}^{\text{when}} .
\end{aligned} \tag{49}$$

On the last 3 lines here $E_1' = E_1'(\mathbf{p}_1') = \sqrt{p'^2 + m_1'^2}$ while $E_2' = E_2'(\mathbf{p}_2' = -\mathbf{p}_1') = \sqrt{p'^2 + m_2'^2}$. Consequently,

$$\frac{dE_1'}{dp'} \; = \; \frac{p'}{E_1'}\,, \quad \frac{dE_2'}{dp'} \; = \; \frac{p'}{E_2'}\,, \tag{50}$$

$$\frac{d(E_1' + E_2')}{dp'} \; = \; \frac{p'}{E_1'} + \frac{p'}{E_2'} \; = \; \frac{p'}{E_1' E_2'} \times (E_2' + E_1' \; = \; E_{\text{tot}}), \tag{51}$$

$$\left[\frac{p'^2}{16\pi^2 E_1' E_2'} \Big/ \frac{d(E_1' + E_2')}{dp'} \right]_{E_1' + E_2' = E_{\text{tot}}}^{\text{when}} \; = \; \frac{p'}{16\pi^2 E_{\text{tot}}}\,, \tag{52}$$

and therefore

$$\begin{aligned}
\mathcal{P}_{\text{scattering}} &= \frac{1}{4|\mathbf{p}|E_{\text{tot}}} \times \frac{p'}{16\pi^2 E_{\text{tot}}} \times \int\! d^2\Omega_{\mathbf{p}'} \\
&= \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{tot}}^2} \times \int\! d^2\Omega_{\mathbf{p}'}\,.
\end{aligned} \tag{53}$$

Note: since the scattering amplitude $\mathcal{M}$ may depend on the directions of the scattered particles, we should multiply the phase space factor by the $|\mathcal{M}|^2$ *before* integrating over those directions. This means that we should not evaluate the angular integral in eq. (53) but rather re-interpret that formula as

$$d\mathcal{P}_{\text{scattering}} = \frac{(p'/p)_{\text{cm}}}{64\pi^2 E_{\text{cm}}^2} \times d\Omega_{\text{cm}} \tag{54}$$

where

$$E_{\text{cm}}^2 = E_{\text{tot}}^2 \quad \text{in the center-of-mass frame,} \tag{55}$$

$$d\Omega_{\text{cm}} = d^2\Omega_{\mathbf{p}_1'} = d^2\Omega_{\mathbf{p}_2'} \quad \text{in the center-of-mass frame,} \tag{56}$$

$$(p'/p)_{\text{cm}} = \frac{|\mathbf{p}_1'| = |\mathbf{p}_2'|}{|\mathbf{p}_1| = |\mathbf{p}_2|} \quad \text{in the center-of-mass frame;} \tag{57}$$

$$\text{for an elastic scattering} \quad (p'/p)_{\text{cm}} = 1. \tag{58}$$

In light of eq. (54), in the center-of-mass frame

$$\begin{aligned} d\sigma(1+2\to 1'+2') &= \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2 \times d\mathcal{P}_{\text{scattering}} \\ &= \frac{(p'/p)_{\text{cm}}}{64\pi^2 E_{\text{cm}}^2} \times \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2 \times d\Omega_{\text{cm}} \end{aligned} \tag{59}$$

and hence the *partial cross-section* for scattering in a particular direction is

$$\frac{d\sigma(1+2\to 1'+2')}{d\Omega_{\text{cm}}} = \frac{(p'/p)_{\text{cm}}}{64\pi^2 E_{\text{cm}}^2} \times \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2 . \tag{60}$$

Finally, the net cross-section — into specific final particle species but emitted in any direction — obtains as an integral

$$\sigma_{\text{net}}(1+2\to 1'+2') = \frac{(p'/p)_{\text{cm}}}{64\pi^2 E_{\text{cm}}^2} \times \int d^2\Omega_{\text{cm}} \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2 . \tag{61}$$

Note: the net cross-sections have same values in all frames where the initial momenta are collinear, so you may use eq. (61) in any such frame, provided you translate the total energy and the (p'/p) ratio to the center-of-mass frame according to

$$E_{\text{cm}}^2 = (E_1+E_2)^2 - (\mathbf{p}_1+\mathbf{p}_2)^2 = (p_1+p_2)^2 = \text{Mandelstam's } s, \tag{62}$$

$$(p'/p)^2_{\rm cm} \;=\; \frac{s^2 - 2s(m_1'^2 + m_2'^2) + (m_1'^2 - m_2'^2)^2}{s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}\,. \tag{63}$$

But the infinitesimal solid angles $d\Omega$ are not invariant under Lorentz boosts along the scattering axis, so eq. (60) for the partial cross-section applies only in the center-of mass frame. In any other collinear frame, we would have

$$\frac{d\sigma}{d\Omega} \;=\; \frac{d\sigma}{d\Omega_{\rm cm}} \times \frac{d\Omega_{\rm cm}}{d\Omega} \tag{64}$$

with a non-trivial frame-dependent factor $d\Omega_{\rm cm}/d\Omega$.

Finally, let me write down the phase-space factor for a 2-body decay (1 particle $\to$ 2 particles) in the rest frame of the initial particle. The under-the-integral factors for such a decay are the same as in eq. (49) for a $2 \to 2$ scattering, but the pre-integral factor is $1/2M_0$ instead of $1/4pE_{\rm cm}$, thus instead of eq. (54) we get

$$d\mathcal{P}_{\rm decay} \;=\; \frac{1}{2M_0} \times \frac{p'}{16\pi^2(E_{\rm tot} = M_0)} \times d\Omega_{\rm cm} \;=\; \frac{p'}{32\pi^2 M_0^2} \times d\Omega_{\rm cm}\,. \tag{65}$$

Consequently, the partial decay rate (for the final particles flying along a particular axis) is

$$\frac{d\Gamma(0 \to 1' + 2')}{d\Omega_{\rm cm}} \;=\; \frac{p'}{32\pi^2 M^2} \times \left|\langle p_1' + p_2'\right| \mathcal{M} \left|p_0\rangle\right|^2, \tag{66}$$

and the net decay rate — into specific particle species but flying in any directions — is

$$\Gamma(0 \to 1' + 2') \;=\; \frac{p'}{32\pi^2 M^2} \times \int\! d^2\Omega_{\rm cm}\, \left|\langle p_1' + p_2'\right| \mathcal{M} \left|p_0\rangle\right|^2. \tag{67}$$

Postscript: In these notes, I have treated all particles as scalars and ignored their spin states. For scattering of decay processes involving particles with non-zero spins, we should distinguish between the polarized cross-sections or decay rates — in which we know the spin states of all the initial and final particles, — and the un-polarized cross-sections or decay rates — in which we sum over the final particles' spin states and average over the initial particles' spin states. I shall explain this issue later in class when we get to QED.