Fermionic Functional Integrals

Fermions and Grassmann Numbers

In the classical limit $\hbar \to 0$, the commutators between bosonic fields vanish, so the classical bosonic fields can be though as ordinary real-number valued (or complex-number valued) functions of x. But for the fermionic fields, it's the *anti*commutators which vanish, so *the classical fermionic fields anticommute with each other*, $\Psi_\alpha(x)\Psi_\beta(y) = -\Psi_\beta(y)\Psi_\alpha(x)$. Consequently, their values are anticommuting Grassmann numbers rather than the ordinary real or complex numbers.

So let me start with a quick review of *Grassmann algebras*. An algebra allows addition and multiplication by real or complex numbers which obey the usual vector-space rules, and it also allows multiplications of any two elements. In a Grassmann algebra, the product is associative but not always commutative; instead the algebra is $\mathbf{Z}_2$ graded into even and odd elements: the even Grassmann numbers act as bosons and the odd Grassmann numbers act as fermions,

$$B_1 B_2 \;=\; +B_2 B_1\,, \quad F_1 F_2 \;=\; -F_2 F_1\,, \quad FB \;=\; +BF. \tag{1}$$

A simple way to construct a Grassmann algebra is to start with N anticommuting fermionic generators $\theta_1, \ldots, \theta_N$, and let the algebra span all linear combinations of all independent products of the generators. For finite N such algebras have finite dimensions 2^N as vector spaces. Indeed, the generators anticommute with each other and with themselves, hence they all square to zero, (any $\theta_i)^2 = 0$, and in any non-zero product of θ's no generator may appear more then once. Consequently, any non-zero generator product has form

$$\text{product} \;=\; \pm(1\,\text{or}\,\theta_1)\times(1\,\text{or}\,\theta_2)\times\cdots(1\,\text{or}\,\theta_n), \tag{2}$$

and there are 2^n distinct products of this kind. For example, for $N=3$ there are 8 distinct products: 4 bosons (even products)

$$1, \quad \theta_1\theta_2\,, \quad \theta_1\theta_3\,, \quad \theta_2\theta_3\,, \tag{3}$$

and 4 fermions (odd products)

$$\theta_1\,, \quad \theta_2\,, \quad \theta_3\,, \quad \theta_1\theta_2\theta_3\,. \tag{4}$$

However, a classical fermionic field $\Psi_\alpha(x)$ acts as an infinite family of independent Grassmann numbers, thus and *independent generator* $\Psi_\alpha(x)$ for each component α and each spacetime point x. Hence, the Grassmann algebra generated by this fermionic field is infinite dimensional.

Next, consider functions of Grassmann numbers. A function of an ordinary real or complex number x can be expanded into a power series,

$$f(x) \;=\; f_0 \;+\; f_1 \times x \;+\; f_2 \times x^2 \;+\; f_3 \times x^3 \;+\; \cdots. \tag{5}$$

However, a similar expansion for a function of an odd GN θ stops after the linear term,

$$f(\theta) \;=\; f_0 \;+\; f_1 \times \theta \;+\; \text{nothing else because } \theta^2 = 0. \tag{6}$$

Thus, a function of an odd GN is a linear polynomial with constant coefficients f_0 and f_1. Likewise, a function of 2 independent odd GN θ_1 and θ_2 amounts to a quadratic polynomial with 4 independent terms

$$f(\theta_1, \theta_2) \;=\; f_{00} \;+\; f_{10} \times \theta_1 \;+\; f_{01} \times \theta_2 \;+\; f_{11} \times \theta_1\theta_2\,. \tag{7}$$

for some constants $f_{00}, f_{10}, f_{01}, f_{11}$. More generally, a function of N independent odd GN is a degree-N polynomial with 2^N independent terms.

Now let's integrate over odd Grassmann numbers. Integration should be linear, so

$$\int d\theta \left(f(\theta) \;=\; f_0 + f_1\theta \right) \tag{8}$$

should be a linear combination of the coefficients f_0 and f_1. Felix Berezin came up with the definition

$$\int d\theta \left(f(\theta) \;=\; f_0 + f_1\theta\right) \;\stackrel{\text{def}}{=}\; f_1 \tag{9}$$

which is invariant under shifting the integration variable θ by a constant odd GN, $f(\theta) \to f(\theta + \eta)$; indeed,

$$f(\theta+\eta) \;=\; f_0 \;+\; f_1(\theta+\eta) \;=\; \left(f_0 + f_1\eta\right) \;+\; f_1\theta, \tag{10}$$

hence

$$\int d\theta\, f(\eta+\eta) \;=\; f_1 \;=\int d\theta\, f(\theta). \tag{11}$$

Berezin integral (9) has a straightforward generalization to integrals over several odd GN, for example

$$\begin{aligned}\int d\theta_2 \int d\theta_1 \left(f(\theta_1,\theta_2) \;=\; f_{00} + f_{10}\theta_1 + f_{01}\theta_2 + f_{11}\theta_1\theta_2\right)\\ =\int d\theta_2 \left(f_{10} + f_{11}\theta_2\right) \;=\; f_{11}\,.\end{aligned} \tag{12}$$

More generally,

$$\begin{aligned}\int d^N\theta\, f(\theta_1,\ldots,\theta_N) \;\stackrel{\text{def}}{=}\; \int d\theta_N \cdots \int d\theta_1\, f(\theta_1,\ldots,\theta_N)\\ =\;\text{coefficient of the senior } \theta_1\theta_2\cdots\theta_N \text{ term in the expansion of } f.\end{aligned} \tag{13}$$

Grassmann algebras with complex coefficients usually define complex conjugation $g \to \overline{g}$ of the Grassmann numbers themselves. From the Physics point of view, the Grassmann numbers are classical limits of quantum operators, so their conjugation should behave similar to the Hermitian conjugation in QM. In particular, for any two GN g_1 and g_2, odd or even,

$$\overline{g_1 \times g_2} \;=\; +\overline{g}_2 \times \overline{g}_1\,. \tag{14}$$

Also, complex Grassmann algebras have independent generators θ_i and $\overline{\theta}_i$, thus

$$\theta \times \bar\theta \;=\; -\bar\theta \times \theta \;\neq\; 0. \tag{15}$$

Consequently,

$$f(\theta,\bar\theta) \;=\; f_{00} \;+\; f_{10}\theta \;+\; f_{01}\bar\theta \;+\; f_{11}\theta\bar\theta\,, \tag{16}$$

$$\int d\bar{\theta} \int d\theta\, f(\theta, \bar{\theta}) \;=\; f_{11} \;=\; \text{coefficient of } \theta\bar{\theta}, \tag{17}$$

and likewise for functions of several complex GN and their conjugates. Note however that conjugation reverses the order in which the GN are multiplied, hence

$$\int d^N\theta \;=\; \int d\theta_N \cdots \int d\theta_1 \quad \text{but} \quad \int d^N\bar{\theta} \;=\; \int d\bar{\theta}_1 \cdots \int d\bar{\theta}_N\,, \tag{18}$$

and therefore

$$\int d^N\bar{\theta} \int d^N\theta\, f(\theta_1, \ldots, \theta_N; \bar{\theta}_1, \ldots, \bar{\theta}_N) \;=\; \text{coefficient of } \theta_1 \cdots \theta_N \times \bar{\theta}_N \cdots \bar{\theta}_1 \text{ in } f. \tag{19}$$

Gaussian Integrals over Odd Grassmann Numbers

Of particular interest to QFT are the Gaussian integrals over the odd Grassmann numbers such as

$$\int d^N\bar{\theta} \int d^N\theta\, \exp\Bigl(-\Theta^\dagger A\Theta\Bigr), \qquad \int d^N\bar{\theta} \int d^N\theta\, \exp\Bigl(-\Theta^\dagger A\Theta\Bigr) \times \bar{\theta}_k\theta_\ell, \qquad \textit{etc.}, \quad \textit{etc.}, \tag{20}$$

where

$$\Theta^\dagger A\Theta \;=\; \bar{\theta}_i A_{ij}\theta_j \;=\; \sum_{i,j=1}^{N} \bar{\theta}_i A_{ij}\theta_j\,, \tag{21}$$

for some $N \times N$ bosonic matrix A_{ij}.

Theorem:

$$\int d^N\bar{\theta} \int d^N\theta\, \exp\Bigl(-\Theta^\dagger A\Theta\Bigr) \;=\; \det(A). \tag{22}$$

Before proving this theorem for general N, let's see how it works for $N = 1$ and $N = 2$. For

$N=1$, A is just a number (real or complex), $\Theta^\dagger A\Theta$ is simply $\bar\theta A\theta$, and

$$\exp\bigl(-\bar\theta A\theta\bigr)\ =\ 1\ -\ \bar\theta A\theta\ +\ \text{nothing else}\ =\ 1\ +\ A\times\theta\bar\theta,$$

hence

$$\int\!d\bar\theta\int\!d\theta\,\exp\bigl(-\bar\theta A\theta\bigr)\ =\int\!d\bar\theta\int\!d\theta\,(1\,+\,A\times\theta\bar\theta)\ =\ A. \tag{23}$$

Next, for $N=2$

$$\exp\bigl(-\Theta^\dagger A\Theta\bigr)\ =\ 1\ -\ \bigl(\Theta^\dagger A\Theta\bigr)+\ \tfrac12\bigl(\Theta^\dagger A\Theta\bigr)^2 \tag{24}$$

where the highest component is

$$\tfrac12\bigl(\Theta^\dagger A\Theta\bigr)^2\ =\ \tfrac12 A_{ij}A_{k\ell}\times\bar\theta_i\theta_j\bar\theta_k\theta_\ell\ =\ -\tfrac12 A_{ij}A_{k\ell}\times\theta_j\theta_\ell\times\bar\theta_i\bar\theta_k\,. \tag{25}$$

Moreover, since there are only two independent θ's at play and they anticommute with each other,

$$\theta_j\theta_\ell\ =\ \epsilon_{j\ell}\theta_1\theta_2 \tag{26}$$

where $\epsilon_{j\ell}$ is the 2D Levi–Civita tensor, and likewise

$$\bar\theta_i\bar\theta_k\ =\ \epsilon_{ik}\bar\theta_1\bar\theta_2\ =\ -\epsilon_{ik}\bar\theta_2\bar\theta_1\,. \tag{27}$$

Consequently,

$$\Bigl[\exp\bigl(-\Theta^\dagger A\Theta\bigr)\Bigr]^{\text{highest}}_{\text{component}}\ =\ +\tfrac12 A_{ij}A_{k\ell}\times\epsilon_{j\ell}\epsilon_{ik}\times\theta_1\theta_2\bar\theta_2\bar\theta_1 \tag{28}$$

and therefore

$$\int\!d^2\bar\theta\int\!d^2\theta\,\exp\Bigl(-\Theta^\dagger A\Theta\Bigr)\ =\ +\tfrac12 A_{ij}A_{k\ell}\times\epsilon_{j\ell}\epsilon_{ik}\,. \tag{29}$$

Finally, on the RHS here

$$\begin{aligned}+\tfrac12 A_{ij}A_{k\ell}\times\epsilon_{j\ell}\epsilon_{ik}\ &=\ \tfrac12 A_{11}A_{22}\ -\ \tfrac12 A_{12}A_{21}\ -\ \tfrac12 A_{21}A_{12}\ +\ \tfrac12 A_{22}A_{11}\\ &=\ A_{11}A_{22}\ -\ A_{12}A_{21}\ =\ \det(A),\end{aligned} \tag{30}$$

which verifies the theorem for $N=2$.

In the same way, for $N \geq 3$, the highest component of $\exp\bigl(-\Theta^\dagger A\Theta\bigr)$ is

$$\begin{aligned}\frac{1}{N!}\bigl(-\Theta^\dagger A\Theta\bigr)^N &= \frac{(-1)^N}{N!}\,A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\times\bar\theta_{i_1}\theta_{j_1}\bar\theta_{i_2}\theta_{j_2}\cdots\bar\theta_{i_N}\theta_{j_N}\\ &= \frac{(-1)^N}{N!}\,A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\times(-1)^{N(N+1)/2}\,\theta_{j_1}\cdots\theta_{j_N}\times\bar\theta_{i_1}\cdots\bar\theta_{i_N}\,.\end{aligned}\tag{31}$$

Furthermore, since there are only N θ's at play

$$\theta_{j_1}\cdots\theta_{j_N} = \epsilon_{j_1,\ldots,j_N}\times\theta_1\cdots\theta_N \tag{32}$$

where $\epsilon_{j_1,\ldots,j_N}$ is the N-dimensional Levi–Civita tensor, and likewise

$$\bar\theta_{i_1}\cdots\bar\theta_{i_N} = \epsilon_{i_1,\ldots,i_N}\times\bar\theta_1\cdots\bar\theta_N = \epsilon_{i_1,\ldots,i_N}\times(-1)^{N(N-1)/2}\,\bar\theta_N\cdots\bar\theta_1\,. \tag{33}$$

Consequently,

$$\Bigl[\exp\bigl(-\Theta^\dagger A\Theta\bigr)\Bigr]^{\text{highest}}_{\text{component}} = \frac{+1}{N!}\,A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\times\epsilon_{j_1,j_2,\ldots,j_N}\epsilon_{i_1,i_2,\ldots,i_N}\times\theta_1\cdots\theta_N\bar\theta_N\cdots\bar\theta_1 \tag{34}$$

and therefore

$$\int\!d^N\bar\theta\int\!d^N\theta\,\exp\Bigl(-\Theta^\dagger A\Theta\Bigr) = \frac{+1}{N!}\,A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\times\epsilon_{j_1,j_2,\ldots,j_N}\epsilon_{i_1,i_2,\ldots,i_N} = \det(A), \tag{35}$$

which proves the theorem (22).

Note: unlike the bosonic Gaussian integral

$$\int\!d^Nz^*\int\!d^Nz\,\exp\Bigl(-z_i^*A_{ij}z_j\Bigr) = \frac{(2\pi)^N}{\det(A)}\,, \tag{36}$$

the fermionic Gaussian integral (22) is directly rather than inversely proportional to the determinant $\det(A)$. However, both types of Gaussian integrals can be easily generalized to Gaussian+ integrals such as

$$\int\!d^Nz^*\int\!d^Nz\,\exp\Bigl(-z_i^*A_{ij}z_j\Bigr)\times z_kz_\ell^* = \frac{(2\pi)^N}{\det(A)}\times\bigl(A^{-1}\bigr)_{k\ell}\,, \tag{37}$$

$$\int d^N z^* \int d^N z \, \exp\Bigl(-z_i^* A_{ij} z_j\Bigr) \times z_k z_\ell z_m^* z_n^* \;=\; \frac{(2\pi)^N}{\det(A)} \times \tag{38}$$
$$\times \Bigl((A^{-1})_{km} (A^{-1})_{\ell n} \,+\, (A^{-1})_{kn} (A^{-1})_{\ell m} \Bigr),$$

etc., for the bosonic variables — as we saw last lecture, — and similarly

$$\int d^N \bar{\theta} \int d^N \theta \, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \times \theta_k \bar{\theta}_\ell \;=\; \det(A) \times \bigl(A^{-1}\bigr)_{k\ell}\,, \tag{39}$$

$$\int d^N \bar{\theta} \int d^N \theta \, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \times \theta_k \theta_\ell \bar{\theta}_m \bar{\theta}_n \;=\; \det(A) \times \tag{40}$$
$$\times \Bigl(-(A^{-1})_{km} (A^{-1})_{\ell n} + (A^{-1})_{kn} (A^{-1})_{\ell m} \Bigr),$$

etc., for the fermionic variables. Indeed,

$$\begin{aligned}
&\int d^N \bar{\theta} \int d^N \theta \, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \times \theta_k \bar{\theta}_\ell \\
&\qquad = \frac{\partial}{\partial A_{\ell k}} \int d^N \bar{\theta} \int d^N \theta \, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \\
&\qquad = \frac{\partial}{\partial A_{\ell k}} \det(A) \;=\; \det(A) \times \bigl(A^{-1}\bigr)_{k\ell}\,,
\end{aligned} \tag{41}$$

likewise

$$\begin{aligned}
&\int d^N \bar{\theta} \int d^N \theta \, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \times \theta_k \theta_\ell \bar{\theta}_m \bar{\theta}_n \\
&\qquad = -\frac{\partial}{\partial A_{n\ell}} \frac{\partial}{\partial A_{mk}} \int d^N \bar{\theta} \int d^N \theta \, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \\
&\qquad = -\frac{\partial}{\partial A_{n\ell}} \frac{\partial}{\partial A_{mk}} \det(A) \;=\; -\frac{\partial}{\partial A_{n\ell}} \Bigl(\det(A) \times \bigl(A^{-1}\bigr)_{km} \Bigr) \\
&\qquad = \det(A) \times \Bigl(-(A^{-1})_{km} (A^{-1})_{\ell n} + (A^{-1})_{kn} (A^{-1})_{\ell m} \Bigr),
\end{aligned} \tag{42}$$

and similarly for more $(\theta, \bar{\theta})$ pairs outside the exponential.

Functional Integrals for Free Fermionic Fields

A 'classical' Dirac field $\Psi_\alpha(x)$ is odd–Grassmann–number valued. That is, for each spacetime point x and each Dirac component α there is an independent complex Grassmann variable $\Psi_\alpha(x)$ and its conjugate $\Psi^\dagger_\alpha(x)$, and all such variables anticommute with each other. The Dirac action

$$S[\overline{\Psi}(x),\Psi(x)] \;=\; \int d^4x\,\overline{\Psi}(i\,\partial\!\!\!/ - m)\Psi \tag{43}$$

is bi-linear in Ψ and $\overline{\Psi}$, so the Functional integral over these fields

$$\iiint\mathcal{D}[\overline{\Psi}(x)]\iiint\mathcal{D}[\Psi(x)]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr) \;=\; \operatorname{Det}(\partial\!\!\!/ + im) \tag{44}$$

simply generalizes the Gaussian fermionic integrals from the previous section to the infinite-dimensional family of independent fermionic variables. Likewise, the correlation functions of the fermionic fields obtain from the generalization of the Gaussian+ integrals (39), (40), *etc.*:

$$\begin{aligned}
\langle\Omega|\,\mathbf{T}\Psi(y)\overline{\Psi}(x)\,|\Omega\rangle \;&=\; \frac{\iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr)\times\Psi(y)\overline{\Psi}(x)}{\iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr)}\\
&=\; \langle x|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|y\rangle \;=\; \int\frac{d^4p}{(2\pi)^4}\,e^{ip(x-y)}\times\frac{i}{p\!\!\!/ - m}
\end{aligned} \tag{45}$$

(Dirac indices suppressed), likewise

$$\begin{aligned}
\langle\Omega|\,\mathbf{T}\Psi(w)\Psi(z)\overline{\Psi}(y)\overline{\Psi}(x)\,|\Omega\rangle \;&=\; \frac{\iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr)\times\Psi(w)\Psi(z)\overline{\Psi}(y)\overline{\Psi}(x)}{\iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr)}\\
&=\; -\,\langle x|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|z\rangle\times\langle y|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|w\rangle\\
&\quad +\;\langle x|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|w\rangle\times\langle y|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|z\rangle\,.
\end{aligned} \tag{46}$$

etc., *etc.*

For closer similarity with functional integrals over the bosonic fields, let's analytically continue the Dirac fields to the Euclidean spacetime and introduce the sources. In Euclidean spacetime, all 4 Dirac matrices γ_E^μ are Hermitian, specifically

$$\gamma_E^4 \;=\; \gamma_M^0\,, \quad \vec{\gamma}_E \;=\; -i\vec{\gamma}_M\,, \quad \Longrightarrow \quad \{\gamma_E^\mu,\gamma_E^\nu\} \;=\; 2\delta^{\mu\nu} \tag{47}$$

and also

$$\partial\!\!\!/_E \;=\; \gamma_M^0(-i\partial_0)_M \;+\; (-i\vec{\gamma}_M)\cdot\nabla_M \;=\; -i\,\partial\!\!\!/_M\,. \tag{48}$$

Consequently,

$$iS_M \;=\; i\!\int\! d^4x_M\,\overline{\Psi}(i\,\partial\!\!\!/_M - m)\Psi \;=\; \int\! d^4x_E\,\overline{\Psi}(-\,\partial\!\!\!/_E - m)\Psi \;=\; -\int\! d^4x_E\,\mathcal{L}_E \tag{49}$$

for

$$\mathcal{L}_E \;=\; \overline{\Psi}(\partial\!\!\!/_E + m)\Psi. \tag{50}$$

As to the sources, since $\Psi_\alpha(x)$ and $\overline{\Psi}_\alpha(x)$ are independent fermionic fields, we have independent sources for both of them, $\eta_\alpha(x)$ and $\overline{\eta}_\alpha(x)$. Altogether, the Euclidean action including the source terms is

$$S_E[\Psi,\overline{\Psi};\eta,\overline{\eta}] \;=\; \int\! d^4x_E\Big(\mathcal{L}_E \;-\; \overline{\eta}\Psi \;-\; \overline{\Psi}\eta\Big), \tag{51}$$

the partition function is

$$Z[\eta,\overline{\eta}] \;=\; \iiint\!\mathcal{D}[\overline{\Psi}]\iiint\!\mathcal{D}[\Psi]\;\exp\bigl(-S_E[\Psi,\overline{\Psi};\eta,\overline{\eta}]\bigr), \tag{52}$$

and its logarithm (or rather $-\log(Z)$) is the generating functional of the connected correlation functions,

$$G_2^{\rm conn}(x;y) \;=\; -\frac{\delta^2\log Z[\eta,\bar{\eta}]}{\delta\bar{\eta}(x)\,\delta\eta(y)}\,, \quad etc. \tag{53}$$

For the free fermions, this generation functional — or rather its dependence on η and $\bar{\eta}$ sources — can be completed exactly by completing the action (51) to a full square: For any given $\eta(x)$ and $\bar{\eta}(x)$, let

$$\Psi'(x) \;=\; \Psi(x) - (\not{\partial} + m)^{-1}\eta(x), \quad \overline{\Psi}'(x) \;=\; \overline{\Psi}(x) - \overline{\eta}(x)(\overleftarrow{\not{\partial}} + m)^{-1}, \tag{54}$$

then

$$S_E \;=\; \int d^4x_E \Big(\overline{\Psi}(\not{\partial} + m)\Psi \,-\, \overline{\eta}\Psi \,-\, \overline{\Psi}\eta\Big) \;=\; \int d^4x_E \Big(\overline{\Psi}'(\not{\partial} + m)\Psi' \,-\, \overline{\eta}(\not{\partial} + m)^{-1}\eta\Big) \tag{55}$$

and consequently

$$\begin{aligned} Z[\eta, \overline{\eta}] \;&=\; \iiint \mathcal{D}[\overline{\Psi}] \iiint \mathcal{D}[\Psi]\, \exp\big(-S_E[\Psi, \overline{\Psi}; \eta, \overline{\eta}]\big) \\ &= \iiint \mathcal{D}[\overline{\Psi}'] \iiint \mathcal{D}[\Psi']\, \exp\left(-\int d^4x_E\, \overline{\Psi}'(\not{\partial}_m)\Psi'\right) \times \exp\left(+\int d^4x_e\, \overline{\eta}(\not{\partial} + m)^{-1}\eta\right) \\ &= \exp\left(+\int d^4x_e\, \overline{\eta}(\not{\partial} + m)^{-1}\eta\right) \times Z[0, 0]. \end{aligned} \tag{56}$$

Or in terms of the generating functional of the connected correlators,

$$-\log Z[\eta, \bar{\eta}] \;=\; -\log Z_0 \,-\, \int d^4x_e\, \overline{\eta}(\not{\partial} + m)^{-1}\eta \quad \text{(exactly)}. \tag{57}$$

Thus, the free Dirac fields have only one connected correlation function, namely the free propagator

$$G_2^{\text{conn}}(x, y) \;=\; \frac{\delta}{\delta\overline{\eta}(y)}\frac{\delta}{\delta\eta(x)}\big(-\log Z\big) \;=\; +\langle y|\, (\not{\partial} + m)^{-1}\, |x\rangle \;=\; \int \frac{d^4p_E}{(2\pi)^4}\, e^{ip(x-y)} \times \frac{1}{i\,\not{p}_E + m}\,. \tag{58}$$

Note: in the Euclidean spacetime, the Dirac propagator is

$$\longleftarrow\!\!\!-\!\!\!-\!\!\!- \;=\; \frac{1}{i\,\not{p}_E + m} \;=\; i \times \frac{i}{\not{p}_M - m} \tag{59}$$

where the overall factor of i between Euclidean and Minkowski propagators is common to all field types, scalars, vectors, spinors, *etc.*, *etc.* As to the denominator here, due to different

$\gamma\mu$ matrices in Euclidean and Minkowski spaces, we have

$$\not{p}_E \;=\; \gamma_E^4 p_E^4 \;+\; \vec{\gamma}_E \cdot \vec{p} \;=\; \gamma^0(ip^0) \;+\; (-i\vec{\gamma}) \cdot \vec{p} \;=\; i\gamma^\mu p_\mu \;=\; i\not{p}_M \tag{60}$$

and therefore

$$\left(\frac{1}{i\not{p}+m}\right)_E \;=\; \left(\frac{1}{-\not{p}+m}\right)_M \;=\; i \times \left(\frac{i}{\not{p}-m}\right)_M, \tag{61}$$

or equivalently

$$\left(\frac{-i\not{p}+m}{p^2+m^2}\right)_E \;=\; i \times \left(\frac{\not{p}+m}{p^2-m^2}\right)_M. \tag{62}$$

Similar to the scalar propagator, the pole of the Minklowski-space propagator here is regulated by $p_E^2 = -(p_M^2 + i\epsilon)$ rathaer than simply $p_E^2 = -p_M^2$, thus

$$\left(\frac{-i\not{p}+m}{p^2+m^2}\right)_E \;=\; i \times \left(\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\right)_M : \tag{63}$$

Fermionic Functional Integrals in QED

In the simplest version of Quantum ElectroDynamics — EM and electron fields, and nothing else — the Euclidean Lagrangian is

$$\mathcal{L}_E \;=\; +\tfrac{1}{4}\big(F^{\mu\nu}\big)_e^2 \;+\; \overline{\Psi}(\not{D}_e+m)\Psi \tag{64}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, the Euclidean action including the source terms is

$$S_E \;=\; \int d^4x_E\Big(\mathcal{L}_E \;-\; J_\mu A_\mu \;-\; \overline{\Psi}\eta \;-\; \overline{\eta}\Psi\Big), \tag{65}$$

and the partition action is

$$\begin{aligned} Z[J_\mu,\eta,\overline{\eta}] \;=& \iiint \mathcal{D}[A_\mu]\,\exp\left(-\int d^4x_E\Big(\tfrac{1}{4}F_{\mu\nu}^2 \;-\; J_\mu A_\mu\Big)_e\right) \times \\ & \times \iiint \mathcal{D}[\overline{\Psi}] \iiint \mathcal{D}[\Psi]\,\exp\left(-\int d^4x_E\Big(\overline{\Psi}(\not{D}_e+m)\Psi \;-\; \overline{\eta}\Psi \;-\; \overline{\Psi}\eta\Big)\right). \end{aligned} \tag{66}$$

The functional integral over the EM fields $A_\mu(x)$ has its own issues, and I address it in a separate set of notes. For the moment, let's focus on the fermionic functional integral in a background of given EM fields $A^\mu(x)$. Thus, we identify the integral on the second line of eq. (66) as a fermionic partition function

$$\widehat{Z}[A_\mu,\eta,\bar{\eta}] \;=\; \iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\left(-\int\! d^4x_E\Big(\overline{\Psi}(\not{D}_e+m)\Psi\;-\;\overline{\eta}\Psi\;-\;\overline{\Psi}\eta\Big)\right). \tag{67}$$

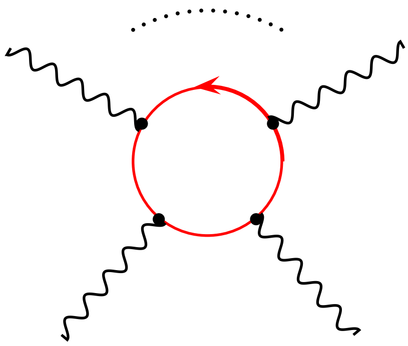
The integral here is Gaussian, so it formally evaluates to

$$\widehat{Z}[A_\mu,\eta,\bar{\eta}] \;=\; \mathrm{Det}(\not{D}_e+m)\times\exp\left(\int\! d^4x_e\,\overline{\eta}\frac{1}{\not{D}_e+m}\eta\right) \tag{68}$$

or in terms of the generating functional $-\log\widehat{Z}$,

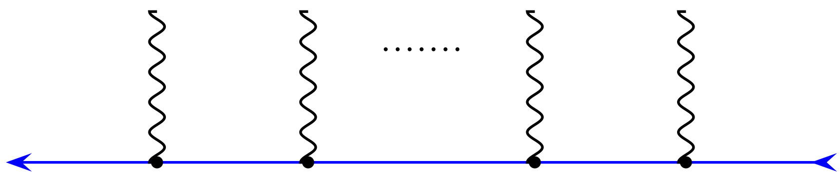
$$-\log\widehat{Z}[A_\mu,\eta,\bar{\eta}] \;=\; -\log\det(\not{D}_e+m)\;-\;\int\! d^4x_e\,\overline{\eta}\frac{1}{\not{D}_e+m}\eta. \tag{69}$$

Physically, the red term generates one loop diagrams where a bunch of external photons are connected to an electron loop



(70)

while the blue term generates tree diagrams where photons are connected to an open electron line



(71)

To see how this works, let's start with the functional determinant $\mathrm{Det}(\not{D}_e+m)$. To evaluate this determinant, we note that

$$\not{D}_e + m = \not{\partial}_e - ie\not{A}_e + m = \left[1 - (ie\not{A}_e)\frac{1}{\not{\partial}_e + m}\right] \times (\not{\partial}_e + m) \tag{72}$$

and hence

$$\mathrm{Det}(\not{D}_e + m) = \mathrm{Det}\left[1 - (ie\not{A}_e)\frac{1}{\not{\partial}_e + m}\right] \times \mathrm{Det}(\not{\partial}_e + m) \tag{73}$$

where the second factor is badly divergent but does not depend on the background EM field $A_\mu(x)$. Therefore, we may treat that second factor as an overall *constant* factor of the partition function. But the first factor in eq. (73) does depend on the EM background, so when we eventually integrate over EM fields $A_\mu(x)$, this factor will appear in the context of

$$\iiint \mathcal{D}[A_\mu]\, \exp\bigl(-S_E[A_\mu]\bigr) \times \mathrm{Det}\left[1 - (ie\not{A}_e)\frac{1}{\not{\partial}_e + m}\right]. \tag{74}$$

which we may interpret as

$$\iiint \mathcal{D}[A_\mu]\, \exp\bigl(-S_E^{\mathrm{eff}}[A_\mu] = -S_E^{\mathrm{tree}}[A_\mu] - \Delta S_E[A_\mu]\bigr) \tag{75}$$

where

$$\Delta S_E[A_\mu] = -\log\mathrm{Det}\left[1 - (ie\not{A}_e)\frac{1}{\not{\partial}_e + m}\right] \tag{76}$$

acts as an extra bit of effective action for the EM field due to electrons living in the EM background.

In the effective theory for the EM fields from which the electrons have been integrated out, ΔS acts as a perturbation giving rise to n-photon vertices for $n = 2, 4, 6, \ldots$. But in terms of the original QED, these effective n-photon vertices stem from the one-loop electron subgraphs. To see how this works, note that for any operator $\hat{\mathcal{O}}$ acting on functions of x_e

— or rather spinor-valued functions of x_e — we have

$$\begin{aligned}
\log \mathrm{Det}(1-\hat{\mathcal{O}}) = \ \mathrm{Tr}\log(1-\hat{\mathcal{O}}) \ &= \ \mathrm{Tr}\left(-\sum_{n=1}^{\infty}\frac{1}{n}\hat{\mathcal{O}}^n\right) \\
&= \ -\sum_{n=1}^{\infty}\frac{1}{n}\int d^4x^e\ \mathrm{tr}_{\mathrm{Dirac}}\big(\langle x^e|\,\hat{\mathcal{O}}^n\,|x^e\rangle\big) \\
&= \ -\sum_{n=1}^{\infty}\frac{1}{n}\int d^4x_1^e\cdots\int d^4x_n^e\ \mathrm{tr}_{\mathrm{Dirac}}\left(\begin{array}{r}\langle x_n^e|\,\hat{\mathcal{O}}\,|x_{n-1}^e\rangle \times \langle x_{n-1}^e|\,\hat{\mathcal{O}}\,|x_{n-2}^e\rangle \times \\ \cdots \times \langle x_1^e|\,\hat{\mathcal{O}}\,|x_0^e = x_n^e\rangle\end{array}\right).
\end{aligned} \tag{77}$$

Eq. (76) for the $-\Delta S_E[A^\mu]$ which generates n-photon vertices has form of eq. (75) for

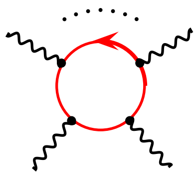
$$\hat{\mathcal{O}} \ = \ (ie\not{A}_e)\frac{1}{\not{\partial}_e+m}. \tag{78}$$

Moreover, $\not{A}_e$ is a function of x_e, hence coordinate-space matrix elements

$$\begin{aligned}
\langle x^e|\,\hat{\mathcal{O}}\,|y^e\rangle \ &= \ \langle x^e|\,(ie\not{A}_e)\frac{1}{\not{\partial}_e+m}\,|y^e\rangle \\
&= \ (ie\not{A}_e(x^e))\times\langle x^e|\,\frac{1}{\not{\partial}_e+m}\,|y^e\rangle \\
&= \ (ie\not{A}_e(y^e))\times G_\psi(x^e-y^e)
\end{aligned} \tag{79}$$

where $G_\psi(x^e-y^e)$ is the free electron's propagator in the Euclidean coordinate space. Plugging this formula into eq. (77) for $-\Delta S_E[A^\mu]$, we arrive at

$$\begin{aligned}
-\Delta S_E[A_\mu] \ &= \ \log\mathrm{Det}\left[1\ -\ (ie\not{A})\frac{1}{\not{\partial}+m}\right] \\
&= \ \sum_{n=1}^{\infty}\frac{-1}{n}\int d^4x_1^e\cdots\int d^4x_n^e\ \mathrm{tr}_{\mathrm{Dirac}}\left(\begin{array}{c}(ie\not{A}(x_n))\times G_\psi(x_n;x_{n-1})\times \\ (ie\not{A}(x_{n-1}))\times G_\psi(x_{n-1};x_{n-2})\times \\ \cdots \\ \times(ie\not{A}(x_2))\times G_\psi(x_2;x_1)\times \\ (ie\not{A}(x_1))\times G_\psi(x_1;x_n)\end{array}\right) \\
&= \ \sum_{n=1}^{\infty} n\text{-photon amputated diagram}
\end{aligned} \tag{80}$$



Indeed, each term on the second line here evaluates (in the Euclidean coordinate space) the n-photon Feynman diagram on the third line. In particular, the $1/n$ factors stems from the cyclic symmetry of each diagram, while the overall minus sign is due to one fermionic loop.

Thus we see that the red term in the fermionic free energy

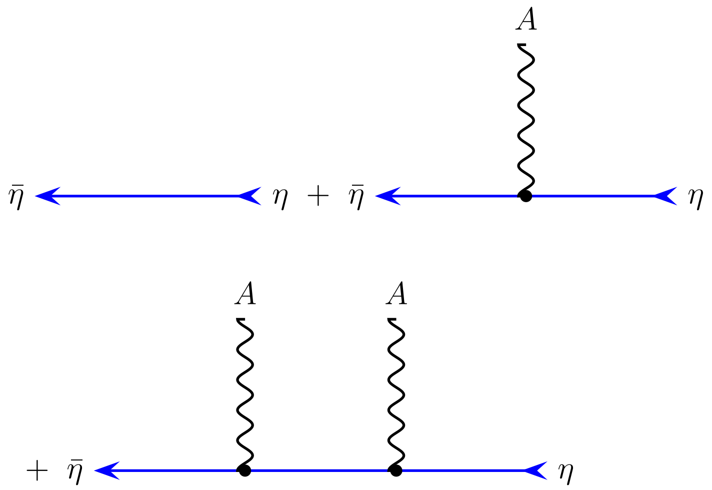
$$-\log \widehat{Z}[A_\mu, \eta, \bar{\eta}] \;=\; -\log\det(\not{D} + m) \;-\; \int d^4x_e\, \overline{\eta} \frac{1}{\not{D}+m} \eta. \tag{69}$$

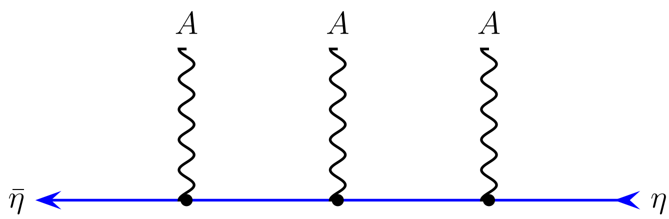
indeed generates the electron loops acting as effective vertices for the photon lines attached to them. As to the blue term involving the fermionic sources η and $\bar{\eta}$, it generates tree diagrams where a bunch of photonic lines are connected to a single open electron line. To see that, we expand

$$\begin{aligned}
\frac{1}{\not{D}+m} &= \frac{1}{(\not{\partial}+m) - (ie\not{A})} \\
&= \frac{1}{\not{\partial}+m} \;+\; \frac{1}{\not{\partial}+m}(ie\not{A})\frac{1}{\not{\partial}+m} \\
&\quad + \; \frac{1}{\not{\partial}+m}(ie\not{A})\frac{1}{\not{\partial}+m}(ie\not{A})\frac{1}{\not{\partial}+m} \\
&\quad + \; \frac{1}{\not{\partial}+m}(ie\not{A})\frac{1}{\not{\partial}+m}(ie\not{A})\frac{1}{\not{\partial}+m}(ie\not{A})\frac{1}{\not{\partial}+m} \\
&\quad + \; \cdots,
\end{aligned} \tag{81}$$

hence

$$\int d^4x_e\, \overline{\eta}\frac{1}{\not{D}+m}\eta \;=\; \tag{82}$$



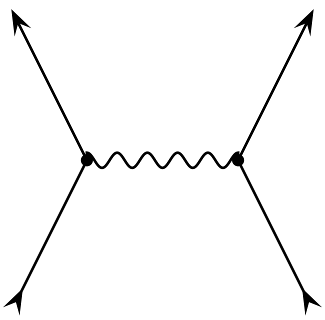
$+$ 

$$+ \cdots \tag{82}$$

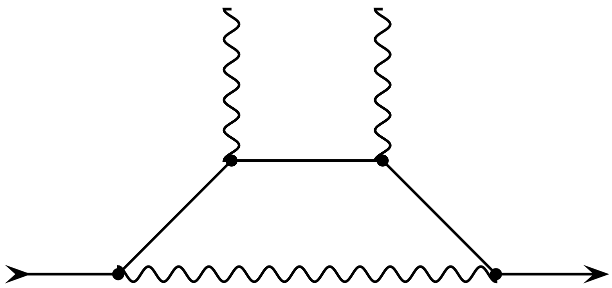
Altogether, the fermionic functional integral

$$\widehat{Z}[A_\mu,\eta,\bar{\eta}] \;=\; \iiiint \mathcal{D}[\overline{\Psi}] \iiiint \mathcal{D}[\Psi] \,\exp\left(-\int d^4x_E \Big(\overline{\Psi}(\not{D}+m)\Psi \,-\, \overline{\eta}\Psi \,-\, \overline{\Psi}\eta\Big)\right). \tag{83}$$

takes care of all the electron lines — open or closed — in QED Feynman rules. However, at this point, all photonic lines are treated as external. To get the photon propagators — and hence diagrams like



or



we need to integrate over the $A_\mu(x)$ fields as well as the fermions. Such functional integrals over the gauge fields pose their own problems due to gauge symmetry and its fixing. These issues are discussed in detail in the next set of my notes.