Renormalization Group Techniques

Introduction

Consider the physical coupling in the $\lambda\phi^4$ theory. Up to now, we have defined $\lambda_{\rm phys}$ — which we shall henceforth call λ_0 — in terms of a low-energy scattering process, for example elastic scattering at threshold,

$$\lambda_0 \;=\; -\mathcal{M}^{\rm elastic}(s=4M^2,t=0). \tag{1}$$

However, when we organize the perturbation theory as a power series in such a low-energy coupling, the amplitudes of the high-energy processes run into the large-logarithm problem: they become power series in

$$\frac{\lambda_0}{16\pi^2}\times\log\frac{E^2}{M^2}\quad\text{rather than just}\quad\frac{\lambda_0}{16\pi^2}\,, \tag{2}$$

and when the energy is high enough so that $\log(E^2/M^2)\gg 1$ while $(\lambda_0/16\pi^2)$ is not too small, the expansion parameter (2) becomes $O(1)$.

To see how this works, consider the one-loop elastic amplitude

$$\mathcal{M}^{\rm elastic}(s,t,u) \;=\; -\lambda_0\;-\;\frac{\lambda_0^2}{32\pi^2}\Big(J(t/m^2)+J(u/m^2)+J(s/m^2)+2\Big)\;+\;O(\lambda^3) \tag{3}$$

where

$$J(t/m^2) \;=\; \int\limits_0^1 dx\,\log\frac{m^2-tx(1-x)}{m^2}\,. \tag{4}$$

For $E\gg m$ and $\theta\not\approx 0,\pi$ (in the center-of-mass frame),

$$\text{all of}\left\{\begin{aligned} -t\;&\approx\;2E^2(1-\cos\theta)\\ -u\;&\approx\;2E^2(1+\cos\theta)\\ s\;&=\;4E^2\end{aligned}\right\}\;\gg\;m^2, \tag{5}$$

hence

$$J(t/m^2)\;\approx\;\int\limits_0^1 dx\left(\log\frac{-t}{m^2}\;+\;\log x(1-x)\right)\;=\;\log\frac{-t}{m^2}\;-\;2 \tag{6}$$

and likewise for the $J(u/m^2)$ and $J(s/m^2)$. Therefore,

$$\begin{aligned} J(t/m^2) + J(u/m^2) + J(s/m^2) + 2 &= \log\frac{-t}{m^2} + \log\frac{-u}{m^2} + \log\frac{-s-i\epsilon}{m^2} - 4 \\ &= 3\log\frac{E^2}{m^2} + f_1(\theta) \end{aligned} \tag{7}$$

where

$$f(\theta) = \log\sin^2\theta + \log(16) - i\pi - 4. \tag{8}$$

Altogether, at the one-loop level

$$\mathcal{M}^{\text{elastic}}(E,\theta) = -\lambda_0 - \frac{\lambda_0^2}{32\pi^2}\left(3\log\frac{E^2}{m^2} + f_1(\theta)\right) + O(\lambda^3). \tag{9}$$

The two-loop-level calculation is more complicated — and I am not going to do it in this class — but the net result has form

$$\begin{aligned} \mathcal{M}^{\text{elastic}}(E,\theta) &= -\lambda_0 - \frac{\lambda_0^2}{32\pi^2}\left(\log\frac{E^2}{M^2} + f_1(\theta)\right) \\ &\quad - \frac{\lambda_0^3}{(32\pi^2)^2}\left(9\log^2\frac{E^2}{M^2} + \left(6f_1(\theta) - \frac{34}{3}\right)\times\log\frac{E^2}{M^2} + f_2(\theta)\right) \\ &\quad - O\left(\frac{\lambda_0^4}{(32\pi^2)^3}\right) \end{aligned} \tag{10}$$

where the $f_2(\theta)$ — just like the $f_1(\theta)$ — is some kind of $O(1)$ function of the scattering angle. Similar formulae obtain at higher loop orders as well, and at each loop order the *leading log* term — *i.e.*, the term with the highest power of the $\log(E^2/m^2)$ — is

$$-\lambda_0 \times \left(\frac{3\lambda_0}{32\pi^2}\times\log\frac{E^2}{M^2}\right)^{\#\text{loops}}. \tag{11}$$

The renormalization group techniques avoid the large-logarithm problem at high energies by reorganizing the perturbation theory to expand in powers of the energy-dependent effective coupling $\lambda_{\text{eff}}(E)$ — also called the *running coupling* $\lambda(E)$. Then, the amplitude for any process at a high energy scale E becomes a power series in $\lambda(E)/16\pi^2$ with $O(1)$ coefficients,

$$\mathcal{M}^{n\,\text{particle}}(\text{momenta}) \;=\; \big(\lambda(E)\big)^{(n-2)/2} \times \sum_{L=0}^{\infty} \left(\frac{\lambda(E)}{16\pi^2}\right)^L \times F_L(\text{momenta}/E) \tag{12}$$

where the L-loop functions $F_L(\text{momenta}/L)$ are $O(1)$ ad do not grow with $\log E$; instead, the large logarithms are hidden in the formula for the $\lambda(E)$ in terms of the low-energy coupling λ_0,

$$\lambda(E) \;=\; \lambda_0 \;+\; \frac{\lambda_0^2}{16\pi^2} \times \frac{3}{2}\log\frac{E^2}{M^2} \;+\; \frac{\lambda_0^3}{(16\pi^2)^2}\left(\frac{9}{4}\log^2\frac{E^2}{M^2} \;-\; \frac{17}{6}\log\frac{E^2}{M^2}\right) \;+\; \cdots. \tag{13}$$

For example, the elastic amplitude (10) becomes

$$\mathcal{M}^{\text{elastic}}(E,\theta) \;=\; -\lambda(E) \;-\; \frac{\lambda^2(E)}{32\pi^2} \times f_1(\theta) \;-\; \frac{\lambda^3(E)}{(32\pi^2)^2} \times f_2(\theta) \;-\; \cdots. \tag{14}$$

Moreover, the expansion (13) obtains by solving a simple differential equation — called the *renormalization group equation* —

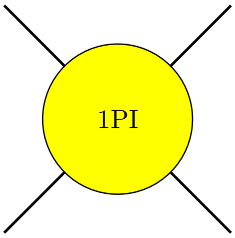
$$\frac{d\lambda(E)}{d(\log E)} \;=\; \beta(\lambda(E)) = \sum_{n=1}^{\infty} b_n \times \frac{\lambda^{n+1}(E)}{(16\pi^2)^n} \tag{15}$$

for some $O(1)$ numbers b_n. For the theory at hand, $b_1 = +3$, $b_2 = -\frac{17}{3}$, *etc.*, each b_n obtaining from an n-loop calculation.

Running Coupling and Running Counterterms

The precise definition of the running coupling $\lambda(E)$ is usually done in terms of some particular amplitude at energy scale E, for example the elastic scattering amplitude at s, $-t$, and $-u$ being particular multiples of $E^2 \gg M^2$. However, in the counterterm perturbation theory we often calculate subgraphs — to be eventually plugged into bigger graphs — and these subgraphs have off-shell external legs. Consequently, it is better to define the running coupling in terms of some completely off-shell amplitude, for example the 1PI 4-scalar amplitude

$$V(p_1,p_2,p_3,p_4) \;=$$



$$\to\; -\lambda(E) \tag{16}$$

when

$$p_1^2 \;=\; p_2^2 \;=\; p_3^2 \;=\; p_4^2 \;=\; -E^2 \quad \text{and} \quad s \;=\; t \;=\; u \;=\; -\tfrac{4}{3}E^2. \tag{17}$$

In the counterterm perturbation theory based on such a running coupling,

$$V^{\rm net}(p_1,\ldots,p_4) \;=\; \bigl(V^{\rm tree} = -\lambda(E)\bigr) \;+\; V^{\rm loops}(p_1,\ldots,p_4) \;-\; \delta^\lambda(E) \tag{18}$$

where the counterterm $\delta^\lambda(E)$ is also energy-scale dependent. Or rather, its finite part is energy-scale dependent to compensate for the energy-scale dependence of the $\lambda(E)$ itself. Indeed, in terms of $\delta^\lambda(E)$ eq. (16) amounts to

$$\delta^\lambda(E) \;=\; V^{\rm loops}(p_1,\ldots,p_4) \quad \text{for momenta as in eq. (17).} \tag{19}$$

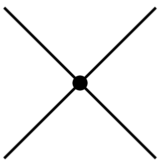
The running physical coupling $\lambda(E)$ and the running counterterm $\delta^\lambda(E)$ are parts of the reorganized perturbation theory where $\lambda(E)$, $M^2(E)$, $\delta^\lambda(E)$, $\delta^Z(E)$, and $\delta^M(E)$ all depend on the energy scale E. In terms of the Feynman rules, we now have:

- Propagator

  $$\rule{5cm}{0.5pt} \;=\; \frac{i}{p^2 - M^2(E) + i0} \tag{20}$$

  for a running mass $M(E)$.

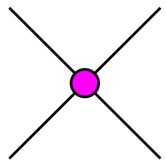
- Physical vertex

  $$=\; -i\lambda(E) \tag{21}$$

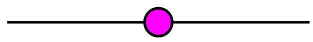
  for a running coupling $\lambda(E)$.

- Counterterm vertices



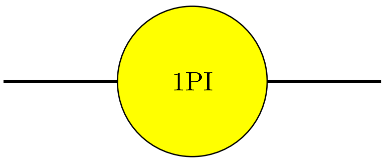
$$= -i\delta^{\lambda}(E) \tag{22}$$

and



$$= -i\delta^{M}(E) \;+\; i\delta^{Z}(E)\times p^2. \tag{23}$$

To define the running mass^2 and the running counterterms $\delta^Z(E)$ and $\delta^M(E)$, we use the off-shell 1PI 2-scalar amplitude



$$= \Sigma^{\rm net}(p^2;E) \;=\; \Sigma^{\rm loops}(p^2;E) \;+\; \delta^M(E) \;-\; \delta^Z(E)\times p^2 \tag{24}$$

and hence the dressed propagator

$$= \frac{i}{p^2 - M^2(E) - \Sigma^{\rm net}(p^2;E) + i0}\,. \tag{25}$$

Instead of focusing on the behavior of this dressed propagator near its pole at the particle's mass^2, we focus on it behavior at high spacelike momenta $p^2 = -E^2$ and demand

$$\Sigma^{\rm net} \;=\; 0 \quad\text{and}\quad \frac{\partial\Sigma^{\rm net}}{\partial p^2} \;=\; 0 \quad\text{for}\quad p^2 \;=\; -E^2, \tag{26}$$

which determines the finite parts of the running $\delta^Z(E)$ and $\delta^M(E)$ counterterms as

$$\begin{aligned} \delta^Z(E) \;&=\; \frac{\partial\Sigma^{\rm loops}}{\partial p^2}\bigg|_{p^2=-E^2}\,, \\ \delta^M(E) \;+\; E^2\times\delta^Z(E) \;&=\; \Sigma^{\rm loops}(p^2=-E^2). \end{aligned} \tag{27}$$

Finally, energy-scale dependence of the $\delta^M(E)$ makes the propagator mass^2 $M^2(E)$ also depend on the energy scale so as to keep the pole mass fixed at the particle mass.

Note: the specific conditions (19) and (27) we have used above to fix the finite parts of the running counterterms — and hence to precisely define the running coupling $\lambda(E)$ and the propagator mass $M(E)$ — are just an example of a *renormalization scheme*. There are many other renormalization schemes used for precise definitions of the running couplings, masses, and counterterms. In general, the running couplings defined according to different renormalization schemes are related to each other as

$$\lambda_1(E) \;-\; \lambda_2(E) \;=\; O(\lambda^2(E)), \tag{28}$$

and this difference becomes important at the two-loop and higher orders of the perturbation theory. We shall return to this issue later in class; for the impatient here are my notes on the subject.

Anomalous Dimensions of Fields

Classically, the scalar field $\Phi(x)$ scales with energy as E^{+1}, but in the quantum theory the bare field $\hat{\Phi}_{\text{bare}}(x) = \sqrt{Z}\hat{\Phi}(x)$ has a slightly different scaling dimension,

$$\hat{\Phi}_{\text{bare}}(x) \;\sim\; E^{\Delta}, \quad \Delta \;=\; 1 \;+\; O(\lambda^2). \tag{29}$$

The classical $\Delta_{\text{cl}} = 1$ is called the *canonical dimension* of the scalar field while the quantum correction $\Delta - \Delta_{\text{cl}}$ is called the *anomalous dimension*.

To see where the anomalous dimension comes from, consider the energy dependence of the $\delta^Z(E)$ counterterm and hence of $Z(E) = 1 + \delta^Z(E)$. Let's define

$$\gamma(E) \;\stackrel{\text{def}}{=}\; \frac{1}{2}\,\frac{d\,\log Z(E)}{d\,\log E}\,, \tag{30}$$

which usually is a slowly varying function of energy. For simplicity, let's approximate $\gamma(E) =$ const, hence

$$\log Z(E) \;=\; \text{const} \;+\; 2\gamma \times \log E \quad\Longrightarrow\quad Z(E) \;=\; \text{const} \times E^{2\gamma}. \tag{31}$$

Now consider the two-point correlation functions for the bare fields and for the renormalized fields,

$$\mathcal{F}_2^{\text{bare}}(x-y) \;=\; \langle\Omega|\,\mathbf{T}\hat{\Phi}_{\text{bare}}(x)\hat{\Phi}_{\text{bare}}(y)\,|\Omega\rangle \quad \text{and} \quad \mathcal{F}_2(x-y) \;=\; \langle\Omega|\,\mathbf{T}\hat{\Phi}(x)\hat{\Phi}(y)\,|\Omega\rangle\,. \tag{32}$$

Since $\hat{\Phi}_{\text{bare}}(x) = \sqrt{Z(E)} \times \Phi(x)$, the bare-field correlation function (32) differs from the renormalized-field correlation function by a factor of $Z(E)$,

$$\mathcal{F}_2^{\text{bare}}(x-y) \;=\; Z(E) \times \mathcal{F}_2(x-y). \tag{33}$$

Likewise, the Fourier transforms of the two correlation functions to the momentum space differ by a factor of $Z(E)$,

$$\mathcal{F}_2^{\text{bare}}(p) \;=\; Z(E) \times \mathcal{F}_2(p). \tag{34}$$

But the $\mathcal{F}_2(p)$ is the dressed propagator of the renormalized scalar field,

$$\mathcal{F}_2(p) \;=\; \overline{\overline{\qquad\qquad\qquad}} \;=\; \frac{i}{p^2 - M^2(E) - \Sigma_{\text{tot}}(p^2;E)}\,, \tag{35}$$

hence for the bare field

$$\mathcal{F}_2^{\text{bare}}(p) \;=\; \frac{iZ(E)}{p^2 - M^2(E) - \Sigma_{\text{tot}}(p^2;E)}\,. \tag{36}$$

Note that the bare-field correlation function on the LHS of this formula does not know or care about the energy scale E at which we renormalize fields, so the E dependence on the RHS must somehow cancel out. Consequently, eq. (36) should be valid for any E and any p unrelated to each other. Nevertheless, it becomes particularly useful for $E^2 = -p^2$ because at this renormalization point the counterterms $\delta^Z(E)$ and $\delta^M(E)$ are set so that the net $\Sigma_{\text{tot}}(p^2;E) = 0$, hence eq. (36) becomes

$$\mathcal{F}_2^{\text{bare}}(p) \;=\; \frac{iZ(E^2=-p^2)}{p^2 - M^2(E)}\,. \tag{37}$$

Moreover, for $p^2 \gg M^2$ we may neglect the mass term in the denominator and approximate

$$\mathcal{F}_2^{\text{bare}}(p) \;\approx\; \frac{iZ(E^2=-p^2)}{p^2} \;=\; -i(\text{const}) \times \frac{(-p^2)^\gamma}{(-p^2)} \tag{38}$$

where the second equality follows from eq. (31) for the $Z(E)$.

To interpret eq. (38) in terms of the bare field's anomalous dimension, consider correlation functions of local operators of known scaling dimensions. Take any local operator $\hat{\mathcal{O}}(x)$ which scales with energy as E^{Δ}; then the correlation function of this operator with itself (or rather with its hermitian conjugate $\hat{\mathcal{O}}^{\dagger}(y)$) scales with the distance $x-y$ as

$$\langle\Omega|\,\mathbf{T}\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^{\dagger}(y)\,|\Omega\rangle \;\propto\; |x-y|^{-2\Delta} \quad \text{for } x-y\to 0. \tag{39}$$

Fourier transforming this formula into momentum space in $D=4$ dimensions, we get

$$\int d^4(x-y)\,e^{-p(x-y)}\times\langle\Omega|\,\mathbf{T}\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^{\dagger}(y)\,|\Omega\rangle \;\propto\; |p|^{2\Delta-4} \;\propto\; (-p^2)^{\Delta-2} \quad \text{for } p\to\infty. \tag{40}$$

Thus, comparing this formula to eq. (38), we immediately see that for the bare field $\hat{\Phi}_{\text{bare}}(x)$

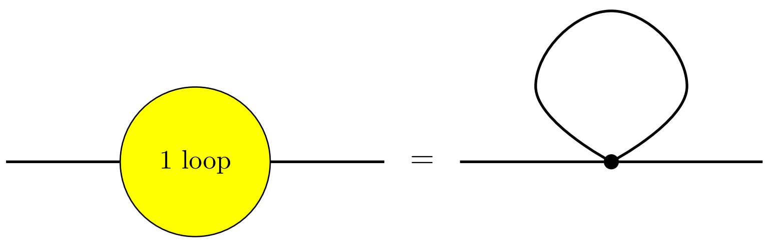
$$\Delta\,-\,2\;=\;\gamma\,-\,1 \quad\Longrightarrow\quad \Delta\;=\;1\,+\,\gamma. \tag{41}$$

In other words,

$$\gamma\;=\;\frac{1}{2}\,\frac{d\,\log Z(E)}{d\,\log E} \tag{30}$$

is the anomalous dimension of the scalar field.

Now let's calculate the anomalous dimension γ to the leading order in perturbation theory. In most quantum field theories, the contribution to γ comes at the one-loop order, but in the $\lambda\phi^4$ theory the one-loop contribution vanishes and the leading contribution comes at the two-loop order, hence $\gamma=O(\lambda^2)$ instead of $O(\lambda)$. Indeed, at the one-loop order



is p independent (42)

hence $\delta^{Z}_{1\,\text{loop}}=0$, $Z_{1\,\text{loop}}=1$, and therefore $\gamma_{1\,\text{loop}}=0$.

The two-loop calculation of the $\Sigma(p^2)$ and $d/\Sigma/dp^2$ was done back in homework set#14 where you (should have) obtained

$$\frac{d\Sigma(p^2)}{dp^2} = -\frac{\lambda^2}{12(4\pi)^4} \iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \\ \times \left(\frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} + \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2}\right). \tag{HW14.13}$$

For $-p^2 \gg m^2$, the expression on the second line here becomes

$$\begin{aligned} (\cdots) &= \frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} - 2\log\frac{(-p^2)}{m^2} + \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta\zeta)^2} + O(m^2/p^2) \\ &= \frac{1}{\epsilon} + 2\log\frac{\mu^2}{(-p^2)} + f(\xi,\eta,\zeta) + O(m^2/p^2) \qquad (43) \\ \text{for} f &= -2\gamma_E + 2\log(4\pi) + \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta\zeta)^2} = O(1), \qquad (44) \end{aligned}$$

or in terms of some energy scale E^2 in the ballpark of $-p^2$,

$$(\cdots) = \frac{1}{\epsilon} + 2\log\frac{\mu^2}{E^2} - 2\log\frac{(-p^2)}{E^2} + f(\xi,\eta,\zeta) + O(m^2/E^2),$$

hence using

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} = \frac{1}{2} \tag{HW14.18.a}$$

we obtain

$$\frac{d\Sigma^{2\,\text{loops}}(p^2)}{dp^2} = -\frac{\lambda^2}{24(4\pi)^4}\left(\frac{1}{\epsilon} + 2\log\frac{\mu^2}{E^2} + 2\log\frac{E^2}{-p^2} + \text{const} + O(m^2/E^2)\right). \tag{45}$$

Therefore, in the off-shell renormalization scheme

$$\delta^Z_{2\,\text{loops}}(E) = \left.\frac{d\Sigma^{2\,\text{loops}}(p^2)}{dp^2}\right|_{p^2=-E^2} = -\frac{\lambda^2}{24(4\pi)^4}\left(\frac{1}{\epsilon} + 2\log\frac{\mu^2}{E^2} + \text{const}\right). \tag{46}$$

Now let's switch from the counterterm calculation to the energy dependence of the $Z(E) = 1 + \delta^Z(E)$. Expanding in powers of λ — and hence of the $\delta^Z = O(\lambda)^2$ — before we take the $\epsilon \to 0$ limit, we have

$$\log\left(Z = 1 + \delta^Z\right) \;=\; \delta^Z \;-\; \tfrac{1}{2}\left(\delta^Z\right)^2 \;+\; \cdots, \tag{47}$$

so to the leading 2-loop order

$$\log Z(E) \;=\; -\frac{\lambda^2}{24(4\pi)^4}\left(\frac{1}{\epsilon} \;+\; 2\log\frac{\mu^2}{E^2} \;+\; \text{const}\right) +\; O(\lambda^3). \tag{48}$$

Note that the divergent part of the RHS here is E-independent, so its derivative WRT $\log E$ is finite,

$$\frac{d\log Z^{2\,\text{loops}}}{d\log E} \;=\; -\frac{\lambda^2}{24(4\pi)^4}\times(-4) \;=\; +\frac{\lambda^2}{6(4\pi)^4}\,, \tag{49}$$

which gives us a finite anomalous dimension

$$\gamma(E) \;=\; +\frac{\lambda^2(E)}{12(4\pi)^4} \;+\; O(\lambda^3). \tag{50}$$

of the scalar field.

Or at least, $\gamma(E)$ is finite to the leading order of the perturbation theory, but what about the higher orders? Loop expansion of the counterterm $\delta^Z(E)$ has general form

$$\delta^Z(E) \;=\; \lambda^2(E)\times A_2(\epsilon,E) \;+\; \lambda^3(E)\times A^3(\epsilon,E) +\; \lambda^4(E)\times A_4(\epsilon,E) \;+\; \cdots, \tag{51}$$

hence

$$\begin{aligned}\log(Z(E) = 1 + \delta^Z) \;&=\; \delta^Z(E) \;-\; \tfrac{1}{2}\left(\delta^Z(E)\right)^2 \;+\; \cdots\\ &=\; \lambda^2\times A_2 \;+\; \lambda^3\times A_3 \;+\; \lambda^4\times\left(A_4 - \tfrac{1}{2}A_2^2\right) \;+\; \cdots,\end{aligned} \tag{52}$$

and therefore

$$\begin{aligned}2\gamma \;=\; \frac{\partial\log Z}{\partial\log E} \;=\; &\lambda^2\times\frac{\partial A_2}{\partial\log E} \;+\; 2\lambda\frac{d\lambda}{d\log E}\times A_2 \;+\; \lambda^3\times\frac{\partial A_3}{\partial\log E} \;+\; 3\lambda^2\frac{d\lambda}{d\log E}\times A_3\\ &+\; \lambda^4\times\frac{\partial}{\partial\log E}\left(A_4 - \tfrac{1}{2}A_2^2\right) \;+\; 4\lambda^3\frac{d\lambda}{d\log E}\times\left(A_4 - \tfrac{1}{2}A_2^2\right) \;+\; \cdots.\end{aligned} \tag{53}$$

In the next section we shall learn that

$$\frac{d\lambda(E)}{d\log E} \;=\; \beta(\lambda(E)) \;=\; b_1\times\lambda^2(E) \;+\; b_2\times\lambda^3(E) \;+\; b_3\times\lambda^4(E) \;+\; \cdots \tag{54}$$

for some finite numeric constants $b_1, b_2, b_3, \ldots$. Plugging this formula into eq. (53) and collecting similar powers of $\lambda(E)$, we obtain

$$\begin{aligned} 2\gamma(E) \;&=\; \lambda^2(E)\times\frac{\partial A_2}{\partial\log E}\;+\;\lambda^3(E)\times\left(\frac{\partial A_3}{\partial\log E}\,+\,2b_1\times A_2\right)\\ &+\;\lambda^4(E)\times\left(\frac{\partial A_4}{\partial\log E}\,-\,A_2\times\frac{\partial A_2}{\partial\log E}\,+\,3b_1\times A_3\,+\,2b_2\times A_2\right)\;+\;\cdots. \end{aligned} \tag{55}$$

Earlier in this section we saw that while the A_2 coefficient is UV divergent, its derivative WRT $\log E$ is finite, so the leading 2-loop term in $\gamma(E)$ is finite. At the next $O(\lambda^3)$ level, the situation is more complicated: the 3–loop coefficient A_3 is UV divergent and its derivative $\partial A_3/\partial\log E$ is also UV divergent, but the divergence cancels out from the combination

$$\frac{\partial A_3}{\partial\log E}\,+\,2b_1\times A_2 \tag{56}$$

which multiplies λ^3 in the expansion (55). Similar situation obtains at the 4-loop order: while each term in the coefficient

$$\frac{\partial A_4}{\partial\log E}\,-\,A_2\times\frac{\partial A_2}{\partial\log E}\,+\,2b_1\times A_3\,+\,3b_2\times A_2 \tag{57}$$

is UV divergent, the divergence cancels out from their sum so the net coefficient is finite.

I am not going to explicitly demonstrate this calculation in class since explicit 3-loop and 4-loop calculations are way too hard and time-consuming. Instead, let me simply state a **Theorem:** *the anomalous dimension* $\gamma(E)$ *obtains as a power series in the running coupling* $\lambda(E)$ *with finite constant coefficients,*

$$\gamma(E)\;=\;\sum_{n_2}^{\infty}C_n\times\lambda^n(E)\quad\text{for some finite constants } C_n\,. \tag{58}$$

Let me conclude this section with a few general remarks. First, due to energy dependence of the running coupling $\lambda(E)$, the anomalous dimension (58) slowly changes with $\log E$. Consequently, the scaling of the bare quantum field with energy is not exactly power-like but has a more general form

$$\hat\Phi_{\rm bare}(x)\;\propto\;E^{1+\gamma(E)} \tag{59}$$

However, $\gamma(E)$ changes rather slowly even on the logarithmic scale of energy, so whenever E changes by not too many orders of magnitude we may approximate the anomalous dimension by a constant.

Second, in theories involving multiple fields, each field has its own anomalous dimension

$$\gamma_i(E) \;=\; \frac{1}{2}\frac{\partial\log(1+\delta_i^Z(E))}{\partial\log E} \tag{60}$$

hence

$$\text{each } \hat{\Phi}_i^{\rm bare}(x) \;\propto\; E^{1+\gamma_i}, \tag{61}$$

$$\text{each } \hat{\Psi}_i^{\rm bare}(x) \;\propto\; E^{\frac{3}{2}+\gamma_i}, \tag{62}$$

$$\text{each } \hat{A}_{\mu,a}^{\rm bare}(x) \;\propto\; E^{1+\gamma_a}. \tag{63}$$

General Theorem: *each of these anomalous dimensions obtains as a power series in the running couplings of the theory with finite coefficients,*

$$\gamma_i(E) \;=\; \gamma_i(\lambda(E), g(E), \ldots). \tag{64}$$

For example, in the Yukawa theory

$$\begin{aligned}
\gamma^\psi(E) \;&=\; \sum_{n,m} C^\psi_{n,m}\times\lambda^n(E)g^{2m}(E)\\
&=\; \Bigl(C^\psi_{1,0}\lambda + C^\psi_{0,1}g^2\Bigr)^{\rm 1\,loop} \;+\; \Bigl(C^\psi_{2,0}\lambda^2 + C^\psi_{1,1}\lambda g^2 \;+\; C^\psi_{0,2}g^4\Bigr)^{\rm 2\,loops} \;+\;\cdots,\\
\gamma^\phi(E) \;&=\; \sum_{n,m} C^\phi_{n,m}\times\lambda^n(E)g^{2m}(E),\\
&=\; \Bigl(C^\phi_{1,0}\lambda + C^\phi_{1,0}g^2\Bigr)^{\rm 1\,loop} \;+\; \Bigl(C^\phi_{2,0}\lambda^2 + C^\phi_{1,1}\lambda g^2 \;+\; C^\phi_{0,2}g^4\Bigr)^{\rm 2\,loops} \;+\;\cdots,
\end{aligned} \tag{65}$$

for some finite coefficients $C^\psi_{n,m}$ and $C^\phi_{n,m}$ obtaining at the $(n+m)$ loop orders.

Renormalization Group Equations and Beta–Functions

The renormalization group equations (RGEs) are differential equations for the energy dependence of the running couplings. For the $\lambda\phi^4$ theory with a single coupling $\lambda(E)$ there is one RGE of the form

$$\frac{d\lambda(E)}{d\log E} \;=\; \beta(\lambda(E)) \;=\; b_1\times\lambda^2(E) \;+\; b_2\times\lambda^3(E) \;+\;\cdots. \tag{54}$$

In this section we shall derive this equation and learn how to calculate the β-function and its expansion in powers of λ.

The key to the renormalization group equation (54) is the relation between the running coupling $\lambda(E)$ and the bare coupling λ_{bare},

$$\lambda(E) + \delta^\lambda(E,\text{cutoff}) \;=\; Z^2(E,\text{cutoff})\times\lambda_{\text{bare}}(\text{cutoff}) \;=\; \bigl(1+\delta^Z(E,\text{cutoff})\bigr)^2\times\lambda_{\text{bare}}(\text{cutoff}). \tag{66}$$

Note that the bare coupling λ_{bare} depends on the UV cutoff but not on the renormalization energy E, so when we take a derivative of both sides of eq. (66) WRT to $\log E$, we get

$$\begin{aligned}
\frac{d\lambda}{d\log E} + \frac{\partial\delta^\lambda}{\partial\log E} &= \frac{\partial Z^2\lambda_{\text{bare}}}{\partial\log E} = \frac{\partial Z^2}{\partial\log E}\times\lambda_{\text{bare}} \\
&= \left(2\frac{\partial\log Z}{\partial\log E}\times Z^2\right)\times\lambda_{\text{bare}} \\
&= \left(2\frac{\partial\log Z}{\partial\log E} = 4\gamma\right)\times\left(Z^2\times\lambda_{\text{bare}} = \lambda+\delta^\lambda\right) \\
&= 4\gamma\times(\lambda+\delta^\lambda),
\end{aligned} \tag{67}$$

hence

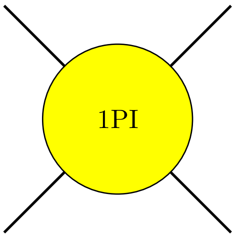
$$\frac{d\lambda}{d\log E} = 4\gamma\times(\lambda+\delta^\lambda) - \frac{\partial\delta^\lambda}{\partial\log E}. \tag{68}$$

At the leading order of perturbation theory $\gamma = O(\lambda^2)$ and $\delta^\lambda = O(\lambda^2) \ll \lambda$, hence the first term in eq. (68) is $O(\lambda^3)$ while the second term is $O(\lambda^2)$. Hence, to the leading order

$$\frac{d\lambda(E)}{d\log E} = -\frac{\partial\delta^\lambda}{\partial\log E} + O(\lambda^3). \tag{69}$$

Now let's calculate the δ^λ counterterm to the leading one-loop order. The running $\lambda(E)$ coupling is defined by

$$V(p_1,p_2,p_3,p_4) =$$



$$\to -\lambda(E) \tag{16}$$

when

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = -E^2 \quad \text{and} \quad s = t = u = -\tfrac{4}{3}E^2, \tag{17}$$

hence

$$\delta^\lambda(E) = V^{\text{loops}}(p_1, \ldots, p_4) \quad \text{for momenta as in eq. (17).} \tag{19}$$

At the one-loop level

$$\begin{aligned} V^{1\,\text{loop}} &= \frac{\lambda^2}{32\pi^2}\left(3\left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m^2}\right) - J(s/m^2) - J(u/m^2) - J(t/m^2)\right) \\ &\to \frac{\lambda^2}{32\pi^2}\left(3\left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m^2}\right) - 3\left(J(-\tfrac{4}{3}E^2/m^2) \approx \log\frac{4E^2}{3m^2} - 2\right)\right) \\ &= \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2} + \text{const}\right), \end{aligned} \tag{70}$$

hence

$$\delta^\lambda_{1\,\text{loop}} = \frac{3\lambda^2}{32\pi^2} \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2} + \text{const}\right). \tag{71}$$

Taking the derivative of this counterterm WRT $\log E$ we get

$$\frac{\partial\delta^\lambda_{1\,\text{loop}}}{\partial\log E} = \frac{3\lambda^2}{32\pi^2} \times (-2) = -\frac{3\lambda^2}{16\pi^2}, \tag{72}$$

so according to eq. (69)

$$\frac{d\lambda(E)}{d\log E} = +\frac{3\lambda^2(E)}{16\pi^2} + O(\lambda^3). \tag{73}$$

Now consider the sub-leading contributions to the

$$\beta(\lambda(E)) \stackrel{\text{def}}{=} \frac{d\lambda(E)}{d\log E} = 4\gamma \times \left(\lambda + \delta^\lambda\right) - \frac{\partial\delta^\lambda}{\partial\log E}. \tag{68}$$

At the $O(\lambda^3)$ order, we have

$$\beta^{\text{order}\,\lambda^3} = 4\gamma_{2\,\text{loops}} \times \lambda - \frac{\partial}{\partial\log E}\delta^\lambda_{2\,\text{loops}} \tag{74}$$

where $\delta^\lambda_{2-\text{loops}}$ has a finite derivative WRT $\log E$, hence finite $O(\lambda^3)$ term in $\beta(\lambda)$. But at the next $O(\lambda^4)$ order we get

$$\beta^{\text{order }\lambda^4} = 4\gamma_{3\,\text{loops}} \times \lambda + 4\gamma_{2\,\text{loops}} \times \delta^\lambda_{1\,\text{loop}} - \frac{\partial}{\partial \log E}\delta^\lambda_{3\,\text{loops}} \tag{75}$$

where the second term $4\gamma \times \delta^\lambda$ is UV-divergent. However, the three-loop counterterm $\delta^\lambda_{3\,\text{loops}}$ is not only UV divergent itself but its derivative WRT $\log E$ is also UV divergent, so the UV divergences of the second and the third terms in eq. (75) cancel each other.

For obvious reasons, I am not going to calculate the 3-loop-order counterterm in order to explicitly verify the cancellation of infinities in eq. (75). Instead, let me simply state the **Theorem:** *the beta-function is a power series in the running coupling $\lambda(E)$ with finite coefficients,*

$$\beta(\lambda(E)) \stackrel{\text{def}}{=} \frac{d\lambda(E)}{d\log E} = \sum_{n=1}^{\infty} b_n \times \lambda^{n+1}(E) \quad \text{for finite } b_n\,, \tag{76}$$

each b_n obtaining at the n–loop order of the perturbation theory.

Solving the Renormalization Group Equation

Given the beta-function $\beta(\lambda)$, the renormalization group equation

$$\frac{d\lambda(E)}{d\log E} = \text{given } \beta(\lambda) \tag{77}$$

is fairly easy to solve:

$$\frac{d\lambda}{\beta(\lambda)} = d\log E, \tag{78}$$

hence

$$\int\limits_{\lambda(E_1)}^{\lambda(E_2)} \frac{d\lambda}{\beta(\lambda)} = \log\frac{E_2}{E_1}\,. \tag{79}$$

In particular, in the one-loop approximation $\beta(\lambda) \approx (3/16\pi^2)\lambda^2$ we have

$$\frac{d\lambda}{\beta(\lambda)} = \frac{16\pi^2}{3}\frac{d\lambda}{\lambda^2} = d\left(-\frac{16\pi^2}{3\lambda}\right), \tag{80}$$

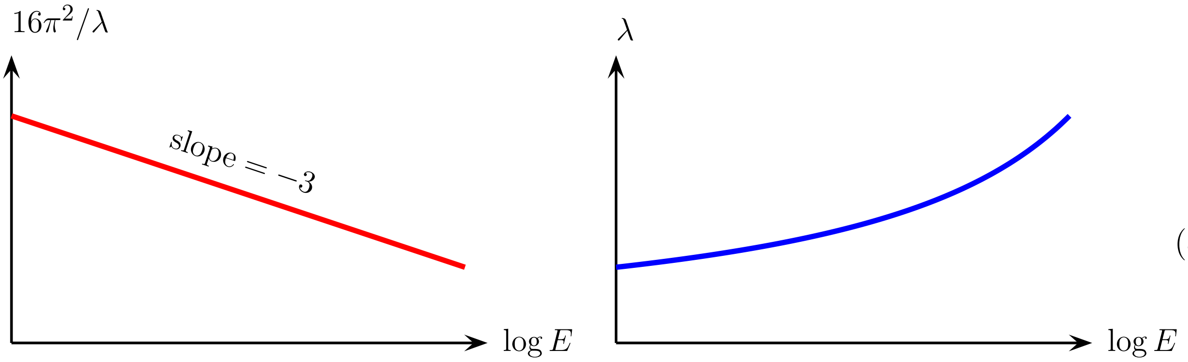
hence

$$\frac{16\pi^2}{3\lambda(E_1)} \; - \; \frac{16\pi^2}{3\lambda(E_2)} \; = \; \log\frac{E_2}{E_1} \tag{81}$$

or

$$\frac{16\pi^2}{\lambda(E)} \; = \; \text{const} \; - \; 3\log E. \tag{82}$$

Graphically,



(83)

Note however that $\beta^{1\,\text{loop}} = (3/16\pi^2)\lambda^2$ only for $E \gg M$. Below the threshold, especially for $E \ll M$,

$$\begin{aligned} \delta^\lambda_{1\,\text{loop}} \;&=\; V_{1\,\text{loop}}(s = t = u = -\tfrac{4}{3}E^2) \\ &=\; \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon} \,-\, \gamma_E \,+\, \log\frac{4\pi\mu^2}{M^2} \,-\, \left(J\left(-\frac{4E^2}{3M^2}\right) \,\approx\, \frac{2E^2}{9M^2}\right)\right), \end{aligned} \tag{84}$$

hence

$$\beta_{1\,\text{loop}} \;=\; -\frac{d\delta^\lambda_{1\,\text{loop}}}{d\log E} \;=\; +\frac{\lambda^2}{24\pi^2}\times\frac{E^2}{M^2} \;\ll\; \frac{3\lambda^2}{16\pi^2}\,. \tag{85}$$

Consequently, well below the threshold

$$E\,\frac{d\lambda}{dE} \;=\; \beta(\lambda) \;=\; \frac{\lambda^2}{24\pi^2}\times\frac{E^2}{M^2}\,, \tag{86}$$

hence

$$\frac{d\lambda}{\lambda^2} \;=\; \frac{1}{24\pi^2}\,\frac{E\,dE}{M^2}\,, \tag{87}$$

$$d\left(-\frac{1}{\lambda}\right) \;=\; d\left(\frac{1}{48\pi^2}\,\frac{E^2}{M^2}\right), \tag{88}$$

and therefore

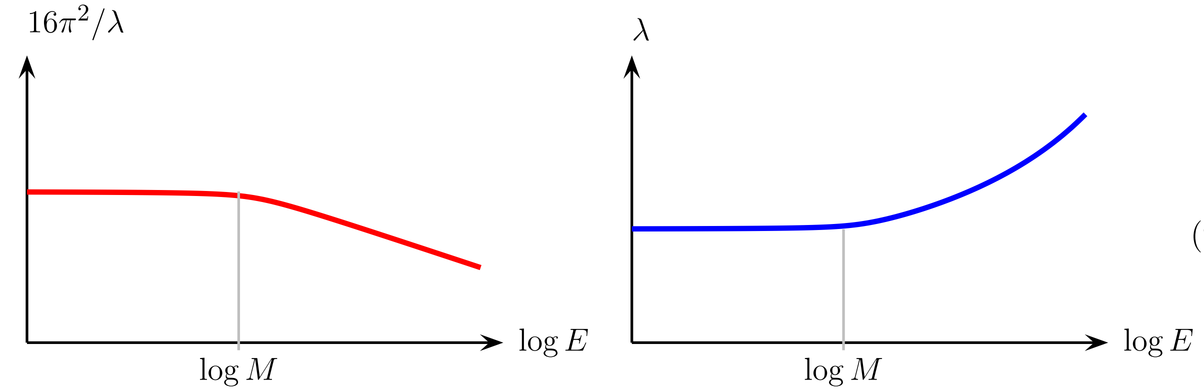
$$\frac{16\pi^2}{\lambda(E)} \;=\; \frac{16\pi^2}{\lambda(0)} \;-\; \frac{E^2}{3M^2} \;\approx\; \frac{16\pi^2}{\lambda(0)} \tag{89}$$

In other words, below the threshold we may neglect the running of the coupling and treat it as a constant low-energy coupling $\lambda_0 = \lambda(E=0)$.

OOH, well above the threshold we have

$$\frac{16\pi^2}{\lambda(E)} \;=\; \text{const} \;-\; 3\log E, \tag{82}$$

while close to the threshold — *i.e.*, at $E = O(M)$ — the running coupling $\lambda(E)$ interpolates between the low-energy constant (89) and the high-energy behavior (82); graphically



(90)

From the high-energy point of view, we may treat the low-energy coupling (89) as a boundary condition for the high-energy renormalization group equation, thus

$$\frac{d\lambda(E)}{d\log E} \;=\; \frac{3\lambda^2}{16\pi^2} \quad\text{and}\quad \lambda(E_0) \;=\; \lambda_0 \tag{91}$$

for $E_0 = M \times O(1)$ constant, hence

$$\text{for } E \gg M,\quad \frac{16\pi^2}{\lambda(E)} \;=\; \frac{16\pi^2}{\lambda_0} \;-\; 3\log\frac{E}{E_0} \;=\; \frac{16\pi^2}{\lambda_0} \;-\; 3\log\frac{E}{M} \;+\; O(1)\text{ constant.} \tag{92}$$

The $O(1)$ constant term here — called the *threshold correction* — follows from the careful calculation of the $\delta^\lambda(E)$ in the threshold region $E = O(M)$; for the problem at hand it's equivalent to setting $E_0 \approx 2.35M$ instead of $E_0 = M$.

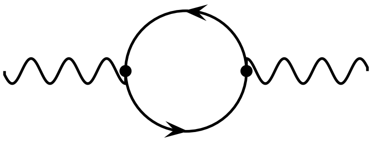
Renormalization of QED

Consider the basic QED comprised of the EM field $A^\mu(x)$ coupled to the electron field $\Psi(x)$ and nothing else. Each field has its own anomalous dimensions, thus

$$\begin{aligned} \gamma_\gamma &= \frac{1}{2}\frac{d\log Z_3}{d\log E}\,, \qquad & \hat{A}^\mu_{\rm bare}(x) &\propto E^{1+\gamma_\gamma}, \\ \gamma_e &= \frac{1}{2}\frac{d\log Z_2}{d\log E}\,, \qquad & \hat{\Psi}_{\rm bare}(x) &\propto E^{\frac{3}{2}+\gamma_e}. \end{aligned} \tag{93}$$

Let's calculate these anomalous dimensions.

Few weeks ago — see my notes on electric charge renormalization — we saw that the one-loop two-photon amplitude is

$$\Sigma^{\mu\nu}_{1\,\rm loop}(k) \;=\;$$  $$=\; \left(g^{\mu\nu}k^2 - k^\mu k^\nu\right) \times \Pi_{1\,\rm loop}(k^2) \tag{94}$$

$$\text{for}\quad \Pi_{1\,\rm loop}(k^2) \;=\; -\frac{\alpha}{3\pi}\left(\frac{1}{\epsilon} \,-\, \gamma_E \,+\, \log\frac{4\pi\mu^2}{m_e^2} \,+\, I(k^2/m_e^2)\right) \tag{95}$$

$$\begin{aligned} \text{where}\quad I(k^2/m_e^2) &= -6\int_0^1 dx\, x(1-x)\times\log\frac{m_e^2 - x(1-x)k^2}{m_e^2} \\ &\to \frac{5}{3} \,-\, \log\frac{-k^2}{m_e^2} \quad \text{for } -k^2 \gg m_e^2. \end{aligned} \tag{96}$$

$$\begin{aligned} \text{hence}\quad \Pi_{1\,\rm loop}(k^2) &\to -\frac{\alpha}{3\pi}\left(\frac{1}{\epsilon} \,-\, \gamma_E \,+\, \log\frac{4\pi\mu^2}{m_e^2} \,+\, \frac{5}{3} \,-\, \log\frac{(-k^2)}{m_e^2}\right) \\ &= -\frac{\alpha}{3\pi}\left(\frac{1}{\epsilon} \,+\, \log\frac{\mu^2}{(-k^2)} \,+\, \text{const}\right). \end{aligned} \tag{97}$$

In the off-shell renormalization scheme, we set

$$\Pi_{\rm net}(k^2) \;=\; \Pi_{\rm loops}(k^2) \;-\; \delta_3(E) \;=\; 0 \quad \text{for } k^2 = -E^2, \tag{98}$$

hence at the one-loop level and for $E \gg m_e$,

$$\delta_3^{\text{order}\,\alpha}(E) \;=\; \Pi^{1\,\rm loop}(k^2 = -E^2) \;=\; -\frac{\alpha}{3\pi}\left(\frac{1}{\epsilon} \,+\, \log\frac{\mu^2}{E^2} \,+\, O(1)\ \text{constant}\right). \tag{99}$$

Taking a derivative WRT $\log E$ we get

$$\frac{\partial\delta_3^{\text{order }\alpha}}{\partial\log E} = -\frac{\alpha}{3\pi}\times(-2) = +\frac{2\alpha}{3\pi} \tag{100}$$

and hence the anomalous dimension of the bare EM field is

$$\gamma_\gamma = \frac{1}{2}\frac{d\log Z_3}{d\log E} = \frac{1}{2}\frac{d\delta_3}{d\log E} + O(\alpha^2) = +\frac{\alpha}{3\pi} + O(\alpha^2). \tag{101}$$

For future reference, let me also give you the two-loop result,

$$\gamma_\gamma = +\frac{\alpha}{3\pi} + \frac{\alpha^2}{4\pi^2} + O(\alpha^3). \tag{102}$$

Now consider the electron field. In your homework set#18, you should calculate

$$\delta_2^{\text{order }\alpha} = -\frac{\xi\alpha}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2} + O(1)\text{ constant}\right) \tag{103}$$

where ξ is the gauge-fixing parameter. Consequently, the anomalous dimensions of the bare electron field is gauge-dependent

$$\gamma_e = +\frac{\xi\alpha}{4\pi} + O(\alpha^2). \tag{104}$$

This is an example of a **General Rule:**

- ⋆ Neutral gauge-invariant fields and operators like $\hat{F}^{\mu\nu}(x)$ or $\hat{\overline{\Psi}}(x)\hat{\Psi}(x)$ have gauge-invariant anomalous dimensions.⋆
- ⋆ But the charged fields and other gauge-dependent operators have gauge-dependent anomalous dimensions!

⋆ For example, $\hat{F}^{\mu\nu}(x)$ has gauge-invariant anomalous dimension (102). Since the canonical dimension of the $\hat{F}^{\mu\nu}(x)$ operator is $\Delta_{\text{can}} = 2$, its net scaling dimension is $\Delta = 2 + \gamma_\gamma$.

Next, the beta-function for the QED running coupling $e(E)$. Since QED vertex involves one EM field and two electron fields, the bare and the renormalized QED couplings are related to each other as

$$Z_2(E)\sqrt{Z_3(E)} \times e_{\text{bare}} \;=\; e(E) \;+\; e(E)\delta_1(E) \;=\; e(E) \times Z_1(E). \tag{105}$$

Both $Z_2(E)$ and $Z_1(E)$ factors here are gauge-dependent, but fortunately the Ward identity

$$Z_1(E) \;=\; Z_2(E) \tag{106}$$

holds in any gauge. Therefore, we may remove these equal factors from the two sides of eq. (105) and get

$$\sqrt{Z_3(E)} \times e_{\text{bare}} \;=\; e(E). \tag{107}$$

On the LHS here, the bare electric charge depends on the UV cutoff but not on the running energy scale E, hence

$$\frac{de(E)}{d\log E} \;=\; e_{\text{bare}} \times \frac{d\sqrt{Z_3}}{d\log E} \;=\; e_{\text{bare}} \times \sqrt{Z_3} \times \frac{1}{2}\,\frac{d\log Z_3}{d\log E} \;=\; e(E) \times \gamma_e(E). \tag{108}$$

In other words, QED beta function is

$$\beta_e(e) \;=\; e \times \gamma_\gamma(e), \quad \text{exactly.} \tag{109}$$

In light of eq. (102) for the EM fields anomalous dimension, this formula yields

$$\beta_e \;\stackrel{\text{def}}{=}\; \frac{de}{d\log E} \;=\; \left(\frac{e^3}{12\pi^2}\right)_{1\,\text{loop}} \;+\; \left(\frac{e^5}{64\pi^4}\right)_{2\,\text{loops}} \;+\; O(e^7). \tag{110}$$

The energy dependence of QED coupling is often expressed in terms of the running

$$\alpha(E) \;\stackrel{\text{def}}{=}\; \frac{e^2(E)}{4\pi}\,. \tag{111}$$

In terms of this $\alpha(E)$, the beta-function becomes

$$\beta_\alpha \stackrel{\text{def}}{=} \frac{d\alpha(E)}{d\log E} = \frac{2e}{4\pi} \times \frac{de}{d\log E} = \frac{2e}{4\pi} \times \beta_e\,, \tag{112}$$

or in terms of the EM field's anomalous dimension

$$\begin{aligned}\beta_\alpha &= \frac{2e}{4\pi} \times e\gamma_\gamma = 2\alpha \times \gamma_\gamma \\ &= \frac{2\alpha^2}{3\pi} + \frac{\alpha^3}{2\pi^2} + O(\alpha^4).\end{aligned} \tag{113}$$

Now let's solve the renormalization group equation for the running QED coupling. Similar to what we had for the 4-scalar coupling $\lambda(E)$, the general solution of the

$$\frac{d\alpha(E)}{d\log E} = \beta_\alpha(\alpha(E)) \tag{114}$$

equation is

$$\int\limits_{\alpha(E_1)}^{\alpha(E_2)} \frac{d\alpha}{\beta_\alpha(\alpha)} = \log\frac{E_2}{E_1}\,. \tag{115}$$

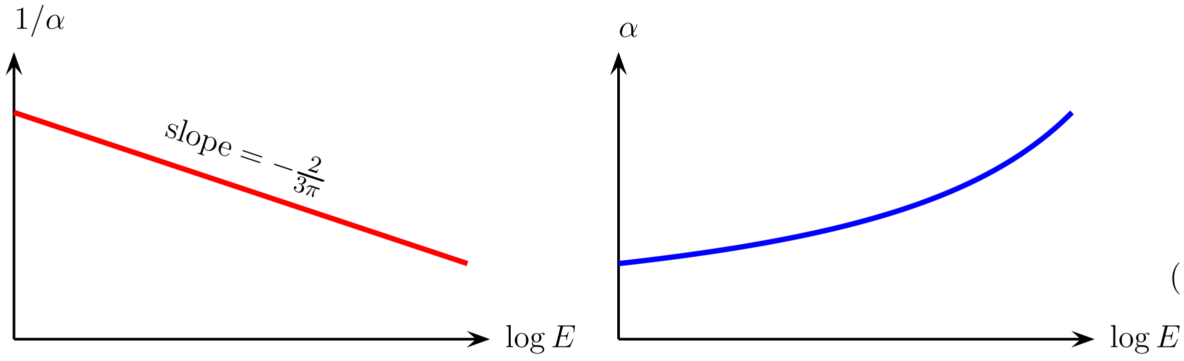
At the one-loop approximation to the β_α we have

$$\frac{d\alpha}{\beta_\alpha} = \frac{3\pi}{2}\,\frac{d\alpha}{\alpha^2} = d\left(-\frac{3\pi}{2\alpha}\right), \tag{116}$$

hence

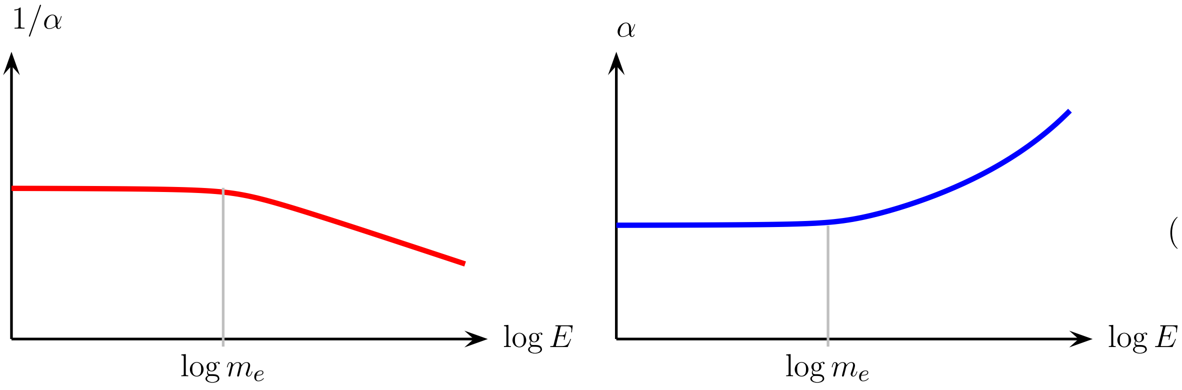
$$\frac{1}{\alpha(E)} = \frac{1}{\alpha(E_{\text{ref}})} - \frac{2}{3\pi}\log\frac{E}{E_{\text{ref}}} \tag{117}$$

for some reference energy $E_{\text{ref}} \gg m_e$. Graphically,



(118)

Similar to the scalar case, the beta-function (113) and hence eq. (117) apply only to energies much higher than the electron's mass. OOH, at low energies $E \ll m_e$ the beta function shrinks as $O(E^2/m_e^2)$ so $\alpha(E)$ is approximately constant, the same as the zero-energy value $\alpha_0 \approx 1/137$. Finally, at the intermediate energies $E \sim m_e$ the running $\alpha(E)$ interpolates between the low-energy constant $\alpha(E) = \alpha_0$ and the high-energy formula (117), so altogether we have



(119)

From the high-energy point of view, the interpolation at $E \sim m_e$ provides the boundary condition for the high-energy RGE (114), namely $\alpha(E_0) = E_0$ at $E_0 = m_e \times$ an $O(1)$ number. Thus, the solution (117) becomes

$$\text{for } E \gg m_e: \quad \frac{1}{\alpha(E)} = \frac{1}{\alpha_0} - \frac{2}{3\pi}\left(\log\frac{E}{m_e} + O(1)\text{ constant}\right) \tag{120}$$

where the red constant term is the so-called *threshold correction*. Calculating this threshold correction involves careful analysis of the $\Pi^{\text{loops}}(k^2)$ for $k^2 \sim m_e^2$ to extract the sub-leading terms besides $\log(-k^2)$ when $-k^2 \to \infty$. Fortunately, we have already done this calculation for the one-loop QED a while ago — see my notes on the electric charge renormalization, eq. (61) on page 12, — thus

$$\frac{1}{\alpha(E)} = \frac{1}{\alpha_0} - \frac{2}{3\pi}\left(\log\frac{E}{m_e} - \frac{5}{6}\right). \tag{121}$$

For large $\log(E/m_e)$ but small α, the threshold correction has a similar magnitude to the effect of the two-loop correction to the beta function, so let's calculate the latter. Using

$$\beta_\alpha \;=\; \frac{2\alpha^2}{3\pi}\left(1\,+\,\frac{3\alpha}{4\pi}\,+\,O(\alpha^2)\right), \tag{122}$$

we have

$$\frac{1}{\beta_\alpha} \;=\; \frac{3\pi}{2\alpha^2}\left(1\,+\,\frac{3\alpha}{4\pi}\,+\,O(\alpha^2)\right)^{-1} \;=\; \frac{3\pi}{2\alpha^2}\left(1\,-\,\frac{3\alpha}{4\pi}\,+\,O(\alpha^2)\right) \;=\; \frac{3\pi}{2\alpha^2}\,-\,\frac{9}{8\alpha}\,+\,O(1) \tag{123}$$

and therefore

$$\int\frac{d\alpha}{\beta_\alpha(\alpha)} \;=\; -\frac{3\pi}{2\alpha}\,-\,\frac{9}{8}\log\alpha\,+\,O(\alpha)\;+\;\text{const.} \tag{124}$$

In light of eq. (115), this means

$$-\frac{3\pi}{2}\left(\frac{1}{\alpha(E)}\,-\,\frac{1}{\alpha_0}\right)\,-\,\frac{9}{8}\log\frac{\alpha(E)}{\alpha_0}\;+\;O(\alpha(E)-\alpha_0)\;=\;\log\frac{E}{m_e}\;+\;\text{const,} \tag{125}$$

where the constant on the RHS may be approximated by its one-loop-level value of $-\frac{5}{6}$. Furthermore, inside the $\log(\alpha(E)/\alpha_0)$ in the sub-leading second term on the LHS we may use the one-loop approximation to the α/α_0 ratio

$$\frac{\alpha(E)}{\alpha_0} \;\approx\; \left(1\,-\,\frac{2\alpha_0}{3\pi}\left(\log\frac{E}{m_e}-\frac{5}{6}\right)\right)^{-1}, \tag{126}$$

hence

$$\frac{1}{\alpha(E)} \;\approx\; \frac{1}{\alpha_0}\,-\,\frac{2}{3\pi}\left(\log\frac{E}{m_e}-\frac{5}{6}\right)+\,\frac{3}{4\pi}\log\left(1\,-\,\frac{2\alpha_0}{3\pi}\left(\log\frac{E}{m_e}-\frac{5}{6}\right)\right). \tag{127}$$

Types of RG Flows

Missing section, to be written later. Rough outline:

1. RG flow for $\beta > 0$: Landau pole and UV incompleteness.
2. RG flow for $\beta < 0$: Asymptotic freedom in UV v. strong IR coupling; QCD example; discuss Λ_{QCD} and confinement.
3. Fixed points for $\beta(g^*) = 0$: scale invariance and conformal symmetry; UV-stable and IR-stable fixed points; $\lambda\phi^4$ in $D < 4$ dimensions; conformal window of QCD (Banks–Zaks).

Multiple Couplings

Some quantum field theories — like $\lambda\phi^4$, QED, or QCD — have just one independent coupling $\lambda(E)$ or $\alpha(E)$. But many theories have multiple couplings; for example, the Yukawa theory from homework set#15 and set#16 has two independent couplings $g(E)$ and $\lambda(E)$, while the Standard Model has three gauge couplings $\alpha_1(E)$, $\alpha_2(E)$, $\alpha_3(E)$, the Higgs self-coupling $\lambda(E)$, and a bucketful of Yukawa couplings of the Higgs to the quarks and the leptons. In theories like that, each running coupling $g_a(E)$ has its own beta-function β_a which depends not only on the $g_a(E)$ but also on all the other couplings of the theory,

$$\frac{dg_a(E)}{d\log E} \;=\; \beta_a(\text{all of the } g_1, g_2, \ldots, g_N). \tag{128}$$

Consequently, instead of a simple RGE being a simple first-order differential equation, we get a system of several *coupled* differential equations. Which obviously makes them much harder to solve. I shall give an example of two coupled RGEs for the Yukawa theory later in this section.

But first let us learn how to calculate the beta-functions for a general coupling $g(E)$, which we take to be the coefficient of a product of n fields $\hat{\varphi}_1(x), \ldots, \hat{\varphi}_n(x)$ (or perhaps their derivatives). For such an operator, the relation between the bare and the renormalized coupling is

$$g(E) \;+\; \delta^g(E) \;=\; g_{\text{bare}} \times \prod_{i=1}^{n} \sqrt{Z_i(E)}\,. \tag{129}$$

On the RHS here, the bare coupling g_{bare} depends on the UV cutoff but not on the running energy scale E, hence taking the derivative of both sides of eq. (129) WRT $\log E$ gives us

$$\begin{aligned}
\frac{dg}{d\log E} \;+\; \frac{d\delta^g}{d\log E} \;&=\; g_{\text{bare}} \times \frac{d}{d\log E} \prod_{i=1}^{n} \sqrt{Z_i(E)} \\
&=\; g_{\text{bare}} \times \prod_{i=1}^{n} \sqrt{Z_i(E)} \times \sum_{i=1}^{n} \left(\frac{d\log\sqrt{Z_i}}{d\log E} \;=\; \gamma_i \right) \\
&=\; \left(g \;+\; \delta^g\right) \times \sum_{i=1}^{n} \gamma_i\,.
\end{aligned} \tag{130}$$

Consequently,

$$\beta_g \stackrel{\rm def}{=} \frac{dg}{d\log E} = (\gamma_1+\cdots+\gamma_n)\times\left(g\,+\,\delta^g\right)\,-\,\frac{d\delta^g}{d\log E}\,. \tag{131}$$

For example, in the $\lambda\phi^4$ theory, the operator $\hat{\phi}^4$ is a product of 4 fields $\hat{\phi}(x)$ of the same kind, hence $(\gamma_1+\cdots+\gamma_n)\to 4\gamma$, so eq. (131) becomes

$$\beta_\lambda = 4\gamma\times\left(\lambda+\delta^\lambda\right)\,-\,\frac{d\delta^\lambda}{d\log E}\,, \tag{132}$$

exactly as we saw in the section on the $\lambda\phi^4$ theory (eq. (68) on page 13). For a more interesting example, consider the Yukawa coupling $g\times i\Phi\overline{\Psi}\gamma^5\Psi$. The operator here involves 3 fields, Φ, Ψ, and $\overline{\Psi}$, but since Ψ and $\overline{\Psi}$ have exactly the same anomalous dimension, in this case

$$\gamma_1\,+\,\cdots\,+\,\gamma_n\,=\,2\times\gamma_\psi\,+\,\gamma_\phi\,, \tag{133}$$

hence

$$\beta_g = (2\gamma_\psi+\gamma_\phi)\times\left(g+\delta^g\right)\,-\,\frac{d\delta^g}{d\log E}\,. \tag{134}$$

To illustrate these formulae, let's calculate the beta-functions of the Yukawa theory β_g and β_λ to the one-loop order. In the homework set#16 (solutions) you should have calculated (to one-loop order) the infinite parts of all the counterterms, thus

$$\delta^\lambda = \frac{1}{16\pi^2}\left(\tfrac{3}{2}\lambda^2\,-\,24g^4\right)\times\frac{1}{\epsilon}\,+\,\text{finite}, \tag{S16.11}$$

$$\delta^g = \frac{1}{16\pi^2}\left(g^3\right)\times\frac{1}{\epsilon}\,+\,\text{finite}, \tag{S16.17}$$

$$\delta^Z_\psi = \frac{1}{16\pi^2}\left(-\tfrac{1}{2}g^2\right)\times\frac{1}{\epsilon}\,+\,\text{finite}, \tag{S16.30}$$

$$\delta^m_\psi = \frac{1}{16\pi^2}\left(-g^2m_\psi\right)\times\frac{1}{\epsilon}\,+\,\text{finite}, \tag{S16.32}$$

$$\delta^Z_\phi = \frac{1}{16\pi^2}\left(-2g^2\right)\times\frac{1}{\epsilon}\,+\,\text{finite}, \tag{S16.43}$$

$$\delta^M_\phi = \frac{1}{16\pi^2}\left(\tfrac{1}{2}\lambda M_\phi^2\,+\,4g^2m_\psi^2\right)\times\frac{1}{\epsilon}\,+\,\text{finite}. \tag{S16.45}$$

Moreover, at the one-loop level the energy-scale dependence of any counterterm (except maybe δ_ϕ^M) follows from its UV-divergent part. Indeed, at high off-shell momenta $-p^2 \gg M^2$, all the logarithmically divergent 1PI amplitudes have form

$$\text{(amplitude)} \;=\; \text{(const)} \times \log\frac{\Lambda_{\text{UV}}^2}{E^2} \;+\; f(\text{momenta}/E), \tag{135}$$

or in dimensional regularization

$$\text{(amplitude)} \;=\; \text{(const)} \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2}\right) \;+\; f(\text{momenta}/E). \tag{136}$$

Consequently, when we define the counterterms in terms of such amplitudes at momenta at some fixed multiple of the energy scale $E \gg M$, we end up with

$$\delta \;=\; \text{(const)} \times \log\frac{\Lambda_{\text{UV}}^2}{E^2} \;+\; \text{const} \tag{137}$$

or

$$\delta \;=\; \text{(const)} \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2}\right) \;+\; \text{const}. \tag{138}$$

Therefore, for any such counterterm

$$\frac{d\delta}{d\log E} \;=\; (-2) \times \text{coefficient of the } \frac{1}{\epsilon} \text{ pole in } \delta. \tag{139}$$

But please note: this formula works only at the one-loop order; alas, the higher-loop counterterms are more complicated. Also, it does not work for the quadratically divergent scalar-mass^2 counterterms

Applying eq. (139) to the Yukawa theory, we get

$$\begin{aligned} \frac{d\delta_\psi^Z}{d\log E} &= \frac{g^2}{16\pi^2}\,, \\ \frac{d\delta_\phi^Z}{d\log E} &= \frac{4g^2}{16\pi^2}\,, \end{aligned} \tag{140}$$

— and hence anomalous dimensions

$$\gamma_\psi = \frac{g^2}{32\pi^2} \quad \text{and} \quad \gamma_\phi = \frac{g^2}{8\pi^2}\,, \tag{141}$$

— and also

$$\begin{aligned} \frac{d\delta^g}{d\log E} &= -\frac{g^3}{8\pi^3}\,, \\ \frac{d\delta^\lambda}{d\log E} &= \frac{-3\lambda^2 + 48g^4}{16\pi^2}\,. \end{aligned} \tag{142}$$

Applying these formulae into eqs. (132) and (134) for the beta-functions, we find that at the one-loop level

$$\beta_g = (2\gamma_\psi + \gamma_\phi) \times g - \frac{d\delta^g}{d\log E} = \frac{2g^2 + 4g^2}{32\pi^2} \times g + \frac{g^3}{8\pi^2} = \frac{5g^3}{16\pi^2} \tag{143}$$

and

$$\beta_\lambda = 4\gamma_\phi \times \lambda - \frac{d\delta^\lambda}{d\log E} = \frac{4g^2}{8\pi^2} \times \lambda + \frac{3\lambda^2 - 48g^4}{16\pi^2} = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{16\pi^2}\,. \tag{144}$$

Note that the beta-function β_λ for the 4-scalar coupling depends on both couplings λ and g already at the one-loop level. On the other hand, the Yukawa coupling's beta-function β_g seems to depend only on the Yukawa coupling itself. However, that is an artefact of the one-loop approximation, and at the higher-loop orders β_g does depend on both couplings. For example, at the two-loop level

$$\beta_g(g,\lambda) = \frac{5g^3}{16\pi^2} + \frac{c_1 g^5 + c_2 g^3 \lambda + c_3 g \lambda^2}{(16\pi^2)^2} + \cdots \tag{145}$$

for some non-zero coefficients c_1, c_2, and c_3.

RG Flows in Yukawa Theory

Altogether, the renormalization group equations for the Yukawa theory form a pair of coupled differential equations

$$\frac{dg(E)}{d\log E} = \beta_g(g(E),\lambda(E)) \quad \text{and} \quad \frac{d\lambda(E)}{d\log E} = \beta_\lambda(g(E),\lambda(E)), \tag{146}$$

so let's learn how to solve such coupled equations. But for simplicity's sake, I am going to use the one-loop approximations (143) and (144) for the two β-functions.

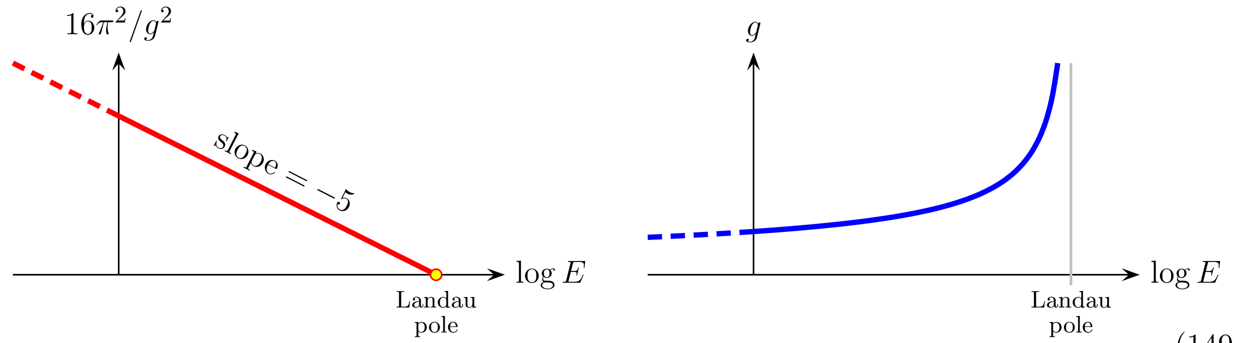
Since $\beta_g^{1\,\text{loop}}(g)$ is independent of λ, we may solve the one-loop RGE for the Yukawa coupling independently of the $\lambda(E)$:

$$\frac{dg}{d\log E} = \frac{5g^3}{16\pi^2} \quad\Longrightarrow\quad d\log E = \frac{dg}{\beta_g} = \frac{16\pi^2\,dg}{5g^3} = d\left(\frac{-8\pi^2}{5g^2}\right), \tag{147}$$

hence

$$\frac{8\pi^2}{g^2(E)} = \frac{8\pi^2}{g^2(E_0)} - 5\log\frac{E}{E_0}\,. \tag{148}$$

Graphically,



(149)

We see that similar to QED or to the $\lambda\phi^4$ theory, the Yukawa coupling keeps increasing with energy, and eventually at some very high energy hits a Landau pole. In the other hand, in the IR direction of the RG flow, the Yukawa coupling gets weaker and weaker until this RG flow is stopped by the scalar's mass or the fermion's mass. But if both masses happen to vanish, then the Yukawa coupling keeps getting weaker, and in the extreme IR limit the fermions become free.

The RG flow of the 4-scalar coupling $\lambda(E)$ is more complicated since β_λ depends on both couplings λ and g. To simplify solving the RGE for the $\lambda(E)$, we shall first focus on the *RG flow trajectory through the coupling space* (g^2, λ) — that is, we shall first calculate λ as a function of g^2 — and then plug in the energy-dependence of the $g^2(E)$ according to eq. (148). To get a differential equation for the $\lambda(g^2)$, we start by dividing the RGE for λ by the RGE for g^2, thus

$$\frac{d\lambda(E)}{dg^2(E)} = \frac{d\lambda}{d\log E} \Big/ \frac{2g\,dg}{d\log E} = \frac{\beta_\lambda}{2g\beta_g} = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{10g^4}. \tag{150}$$

Next, we take the coupling ratio λ/g^2 to be a function of g^2, which in turn depends on the energy, thus

$$\frac{\lambda(E)}{g^2(E)} = X(g^2(E)), \tag{151}$$

and rewrite eq. (150) as a first-order differential equation for X: On one hand

$$\frac{d\lambda}{d(g^2)} = \frac{d}{d(g^2)}\big(\lambda = g^2 \times X(g^2)\big) = X + g^2 \times \frac{dX}{d(g^2)} = X + \frac{dX}{d\log g^2}, \tag{152}$$

while on the other hand eq. (150) becomes

$$\frac{d\lambda}{d(g^2)} = \frac{3X^2 + 8X - 48}{10}, \tag{153}$$

hence

$$\begin{aligned} X + \frac{dX}{d\log g^2} &= \frac{3X^2 + 8X - 48}{10}, \\ \frac{dX}{d\log g^2} &= \frac{3X^2 - 2X - 48}{10}, \end{aligned} \tag{154}$$

and therefore

$$\frac{dX}{3X^2 - 2X - 48} = \frac{1}{10} d\log(g^2(E)). \tag{155}$$

Integrating the LHS here we get

$$\int \frac{dX}{3X^2 - 2X - 48} = \frac{1}{2\sqrt{145}} \log \frac{3X - 1 - \sqrt{145}}{3X - 1 + \sqrt{145}} + \text{const}, \tag{156}$$

hence

$$\begin{aligned} \log\frac{3X-1-\sqrt{145}}{3X-1+\sqrt{145}} &= \left(\frac{2\sqrt{145}}{10}=\sqrt{\frac{29}{5}}\right)\times\log(g^2(E))\ +\ \text{const}, \\ \frac{3X-1-\sqrt{145}}{3X-1+\sqrt{145}} &= \text{const}\times\bigl(g^2(E)\bigr)^{\sqrt{29/5}}, \end{aligned} \tag{157}$$

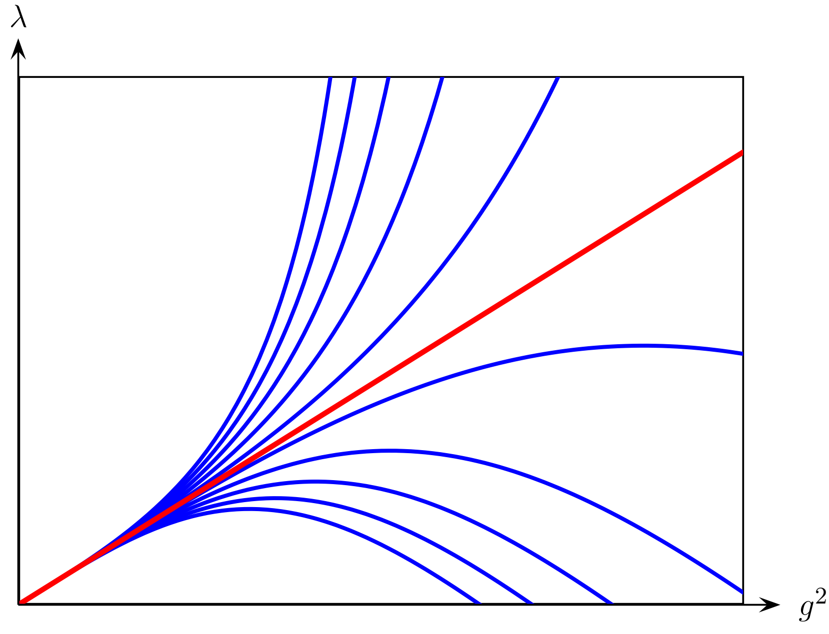
and therefore

$$\frac{3\lambda(E)\ -\ (\sqrt{145}+1)g^2(E)}{3\lambda(E)\ +\ (\sqrt{145}-1)g^2(E)}\ =\ \text{const}\times\bigl(g^2(E)\bigr)^{\sqrt{29/5}}. \tag{158}$$

The overall constant factor on the RHS here follows from the initial conditions to the RGE, namely the coupling values λ_0 and g_0 at some reference energy E_0, thus

$$\frac{3\lambda(E)\ -\ (\sqrt{145}+1)g^2(E)}{3\lambda(E)\ +\ (\sqrt{145}-1)g^2(E)}\ =\ \frac{3\lambda_0\ -\ (\sqrt{145}+1)g_0^2}{3\lambda_0\ +\ (\sqrt{145}-1)g_0^2}\times\left(\frac{g^2(E)}{g_0^2}\right)^{\sqrt{29/5}}. \tag{159}$$

Physically, eq. (159) describes a *trajectory of the renormalization group flow through the (g^2,λ) coupling space*. Here is a plot of several such trajectories for different initial values of (g_0^2,λ_0):



A few noteworthy features of these trajectories:

- Since $\beta_g > 0$ at all g, the UV-bound direction of these trajectories is to the right, while the IR-bound direction is to the left.

- In the IR direction, all trajectories converge to the attractor line (shown in red)

$$\lambda(E) \;=\; \frac{\sqrt{145}+1}{3}\times g^2(E). \tag{160}$$

Note: this attractor line is not a line of fixed points: Once the couplings reach this line, they do not stop evolving with $\log(E)$ but continue to diminish with decreasing energy scale; they simply evolve in lock-step along the attractor line.

- In the UV direction, the trajectories spread out away from the red line.

  ∗ If we start at some point above this line, then in the UV direction the λ/g^2 ratio keeps increasing until eventually $\lambda(g^2(E))$ hits a Landau pole. That is, it hits a Landau pole before the Landau pole of the $g^2(E)$ itself, assuming these Landau poles actually exist beyond the one-loop approximations to the β-functions.

  ∗ On the other hand, if we start at some point below the red line, then the λ/g^2 ratio decreases in the UV direction until $\lambda(E)$ itself starts decreasing and eventually hits zero value. Beyond that point, a negative λ destabilizes the vacuum state of the theory due to unlimited-from-below scalar potential.

  ∗ Similar to a Landau pole, a point where the scalar potential becomes unbounded from below acts as an upper limit on UV energy scales to which the original low-energy may be extrapolated. Beyond this limit we would need a different high-energy theory with more degrees of freedom. In other words, the original low-energy theory is *not* UV-complete.

⋆ ⋆ ⋆

Beyond the Yukawa theory, other QFT's with $n>1$ independent coupling parameters generally have n coupled renormalization group equations:

$$\text{for each } i=1,\ldots,n: \quad \frac{dg_i}{d\log E} \;=\; \beta_i(g_1,\ldots,g_n). \tag{161}$$

A common tool for solving such equations is to eliminate the $\log E$ variable — just like we did above for the Yukawa theory — and reduce the problem to a system of $n-1$ differential equations for the RG flow trajectory through the n-dimensional coupling space. Plotting the flow lines then reveals all kinds of interesting features, such as attractor lines, surfaces, *etc.*, bifurcation points (or lines, *etc.*), or even phase boundaries where the trajectories starting on two sides of a boundary end up in radically different places.

A particularly interesting features are the fixed points attracting all the trajectories — or at least all the trajectories starting within a particular *basin of attraction* — in either UV or IR direction. A theory that has such a fixed point becomes scale-invariant — and usually conformally invariant — in either extreme UV limit or extreme IR limit, depending on the type of a fixed point.

From the β-function point of view, a fixed point is a common zero of all n β-functions,

$$\text{all}\quad \beta_i(g_1^*,\ldots,g_n^*)\ =\ 0. \tag{162}$$

Also, the derivative matrix

$$\mathcal{B}_{ij}\ =\ \left(\frac{\partial\beta_i}{\partial g_j}\right)_{(g_1^*,\ldots,g_n^*)} \tag{163}$$

determines the fixed point's type:

- If the matrix (163) is positive-definite (all eigenvalues are positive), then the fixed point is IR-stable. That is, the RG flow in the IR direction moves the couplings closer and closer to the fixed point.
- OOH, if the matrix (163) is negative-definite (all eigenvalues are negative), then the fixed point is UV stable. That is, the RG flow in the UV direction moves the couplings closer and closer to the fixed point.
- Finally, if the matrix (163) has both positive and negative eigenvalues, then the fixed point is unstable in both UV and IR directions: Either way, the RG flow moves some couplings (or coupling combinations) closer to the fixed point while other couplings or combinations move further away from it.