RELATIVISTIC ELECTROMAGNETIC FIELDS

In 3D terms, the electric and the magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are both vector fields, so altogether there are 6 field components. Relativistically, these 6 components form an anti-symmetric tensor field $F^{\mu\nu}(x) = -F^{\nu\mu}(x)$ according to

$$F^{00} = 0, \quad F^{i0} = +E^i, \quad F^{0j} = -E^j, \quad F^{ij} = -\epsilon^{ijk}B^k, \quad \text{for } i,j = 1,2,3. \quad (1)$$

In terms of the $F^{\mu\nu}$ tensor, the homogeneous pair of Maxwell equations

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad (2)$$

(in rationalized Gauss units) can be written as

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0, \quad (3)$$

or equivalently as

$$\partial^{[\lambda} F^{\mu\nu]} = 0 \quad (4)$$

where $[\lambda\mu\nu]$ denote total antisymmetrization of the 3 Lorentz indices, or in terms of the 4D Levi–Civita tensor as

$$\epsilon_{\alpha\lambda\mu\nu}\partial^\lambda F^{\mu\nu} = 0. \quad (5)$$

Indeed, using the convention $\epsilon_{0123} = +1$ (but $\epsilon^{0123} = -1$), we have for $\alpha = 0$

$$\tfrac{1}{2}\epsilon_{0\lambda\mu\nu}\partial^\lambda F^{\mu\nu} = \tfrac{1}{2}\epsilon^{ijk}\partial^i F^{jk} = (-\nabla^i)(-B^i) = \nabla \cdot \mathbf{B}, \quad (6)$$

while for $\alpha = i = 1,2,3$

$$
\begin{aligned}
\tfrac{1}{2}\epsilon_{i\lambda\mu\nu}\partial^\lambda F^{\mu\nu} &= \tfrac{1}{2}\epsilon_{i0jk}\partial^0 F^{jk} + \tfrac{1}{2}\epsilon_{ij0k}\partial^j F^{0k} + \tfrac{1}{2}\epsilon_{ijk0}\partial^j F^{k0} \\
&= -\tfrac{1}{2}\epsilon^{ijk}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)\left(-\epsilon^{jk\ell}B^\ell\right) + \tfrac{1}{2}\epsilon^{ijk}\left(-\nabla^j\right)\left(-E^k\right) - \tfrac{1}{2}\epsilon^{ijk}\left(-\nabla^j\right)\left(+E^k\right) \\
&= +\frac{1}{c}\frac{\partial}{\partial t}B^i + \epsilon^{ijk}\nabla^j E^k.
\end{aligned}
$$
$$(7)$$

Thus, for $\alpha = 0$ eq. (5) amounts to $\nabla \cdot \mathbf{B} = 0$ while for $\alpha = 1,2,3$ it amounts to the Induction

Law,

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \;+\; \nabla \times \mathbf{E} \;=\; 0. \tag{8}$$

As to the inhomogeneous pair of Maxwell equations

$$\nabla \cdot \mathbf{E} \;=\; \rho \quad\text{and}\quad \nabla \times \mathbf{B} \;-\; \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \;=\; \frac{1}{c}\mathbf{J} \tag{9}$$

(in rationalized Gauss units*), in relativistic notations they become

$$\partial_\mu F^{\mu\nu} \;=\; \frac{1}{c}J^\nu \tag{10}$$

where $J^\mu = (c\rho, \mathbf{J})$ is the electric current 4–vector. Indeed, for $\nu = 0$ we have

$$\partial_\mu F^{\mu 0} \;=\; \partial_i F^{i0} \;=\; \nabla^i E^i \;=\; \nabla \cdot \mathbf{E} \tag{11}$$

so eq. (10) becomes the Gauss Law $\nabla \cdot \mathbf{E} = \frac{1}{c}J^0 = \rho$, while for $\mu = j = 1,2,3$ we have

$$
\begin{aligned}
\partial_\mu F^{\mu j} \;&=\; \partial_0 F^{0j} \;+\; \partial_i F^{ij}\\
&=\; \frac{1}{c}\frac{\partial}{\partial t}\bigl(-E^j\bigr) \;+\; \nabla^i\bigl(-\epsilon^{ijk}B^k \;=\; +\epsilon^{jik}B^k\bigr)\\
&=\; \left(\nabla \times \mathbf{B} \;-\; \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right)^j
\end{aligned}
\tag{12}
$$

so that eq. (10) becomes the Maxwell–Ampere Law (9).

$\star$ The rationalized Gauss units — also called the Lorentz–Heaviside units — use factors of 4π similar to the MKSA units instead of the ordinary Gauss units. On the other hand, the rationalized Gauss units have factors of $1/c$ similar to the ordinary Gauss units and no factors of ϵ_0 or mu_0. For example, the Coulomb force in the rationalized Gauss units is $F_{12} = q_1 q_2/(4\pi r^2)$.

Consistency of the inhomogeneous Maxwell equations (10) requires the electric current J^ν to obey the continuity equation

$$\partial_\nu J^\nu \;=\; \frac{\partial \rho}{\partial t} \;+\; \nabla \cdot \mathbf{J} \;=\; 0. \tag{13}$$

Indeed,

$$
\begin{aligned}
\frac{1}{c}\,\partial_\nu J^\nu \;&=\; \partial_\nu \partial_\mu F^{\mu\nu} \quad \langle\!\langle\, \text{by eq. (10)}\,\rangle\!\rangle \\
&=\; \tfrac{1}{2}\partial_\mu \partial_\nu F^{\mu\nu} \;-\; \tfrac{1}{2}\partial_\mu \partial_\nu F^{\nu\mu} \quad \langle\!\langle\, \text{by antisymmetry } F^{\mu\nu} = -F^{\nu\mu}\,\rangle\!\rangle \\
&=\; \tfrac{1}{2}\partial_\mu \partial_\nu F^{\mu\nu} \;-\; \tfrac{1}{2}\partial_\nu \partial_\mu F^{\mu\nu} \quad \langle\!\langle\, \text{relabeling } \mu \leftrightarrow \nu \text{ in the second term}\,\rangle\!\rangle \\
&=\; \tfrac{1}{2}[\partial_\nu, \partial_\mu]F^{\mu\nu} \;=\; 0. \quad \langle\!\langle\, \text{since } \partial_\nu \text{ and } \partial_\mu \text{ commute}\,\rangle\!\rangle.
\end{aligned}
\tag{14}
$$

Physically, the continuity equation (13) means *local conservation of the electric charge.* That is, not only the net charge of the whole Universe is conserved, but the charge inside any closed volume changes *only* due to the net current through that volume's surface. In other words, the electric charges cannot suddenly jump from one space point to another but must flow with the current.

In high-energy-physics terminology, the continuity equation (13) for the current and the local conservation of the charge are usually conflated, and we call the electric current J^ν itself a *conserved current.* The same terminology applies to the other kinds of locally conserved charges — like the baryon number or the lepton number, — we refer to the corresponding currents which obey continuity equations $\partial_\nu J^\nu = 0$ as *conserved currents.* And the continuity equation itself is called the current conservation equation.

Now consider the scalar potential $\Phi(\mathbf{x}, t)$ and the vector potential $\mathbf{A}(\mathbf{x}, t)$ for the EM fields. In terms of these potentials, the tension fields are

$$\mathbf{B} \;=\; \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} \;=\; -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \;-\; \nabla\Phi. \tag{15}$$

As long as the $\mathbf{E}$ and $\mathbf{B}$ have this form for any $\mathbf{A}(\mathbf{x}, t)$) and $\Phi(\mathbf{x}, t)$, the homogeneous Maxwell equations (2) are automatically satisfied. Conversely, any $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ field which obey the homogeneous Maxwell equations for all $(\mathbf{x}, \mathbf{t})$ can always be written in the

form (15) for some potentials $\mathbf{A}(\mathbf{x}, t))$ and $\Phi(\mathbf{x}, t)$. However, such potentials are not unique: they may be *gauge transformed* according to

$$\begin{aligned}
\mathbf{A}(\mathbf{x}, t) &\rightarrow \mathbf{A}'(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t) + \nabla\Lambda(\mathbf{x}, t), \\
\Phi(\mathbf{x}, t) &\rightarrow \Phi'(\mathbf{x}, t) = \Phi(\mathbf{x}, t) - \frac{1}{c}\frac{\partial}{\partial t}\Lambda(\mathbf{x}, t),
\end{aligned} \tag{16}$$

for any $\Lambda(\mathbf{x}, t)$ — as long as it's the same $\Lambda(\mathbf{x}, t)$ in both eqs. (16) — without changing the tension fields $\mathbf{E}$ and $\mathbf{B}$,

$$\mathbf{E}'(\mathbf{x}, t) = \mathbf{E}(\mathbf{x}, t) \quad \text{and} \quad \mathbf{B}'(\mathbf{x}, t) = \mathbf{B}(\mathbf{x}, t). \tag{17}$$

In relativistic notations, the 3–scalar potential Φ and the 3–vector potential $\mathbf{A}$ combine into the 4–vector potential $A^\mu = (\Phi, \mathbf{A})$ while eqs. (15) for the tension fields become

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = -F^{\nu\mu}(x). \tag{18}$$

Indeed, for the electric components of the $F^{\mu\nu}$ tensor eq. (18) amounts to

$$E^i = F^{i0} = \partial^i A^0 - \partial^0 A^i = -\nabla^i \Phi - \frac{1}{c}\frac{\partial}{\partial t}A^i \tag{19}$$

while for the magnetic components we have

$$\epsilon^{ijk} B^k = -F^{ij} = -\partial^i A^j + \partial^j A^i = +\nabla^i A^j - \nabla^j A^i = \epsilon^{ijk}\epsilon^{k\ell m}\nabla^\ell A^m = \epsilon^{ijk}(\nabla \times \mathbf{A})^k. \tag{20}$$

In relativistic notations, the tensor field (18) automatically obeys the homogeneous Maxwell equation (4) as

$$\partial^{[\lambda} F^{\mu\nu]} = \partial^{[\lambda}\partial^\mu A^{\nu]} - \partial^{[\lambda}\partial^\nu A^{\mu]} = 2\partial^{[\lambda}\partial^\mu A^{\nu]} = 0 \tag{21}$$

because $\partial^{[\lambda}\partial^{\mu]} = [\partial^\lambda, \partial^\mu] = 0$ — the spacetime derivatives commute with each other, so antisymmetrizing them yields an automatic zero. Conversely, by the Poincare Lemma (which

generalizes Green, Stokes, and Gauss theorems of 3D vector calculus), any antisymmetric tensor field $F^{\mu\nu}(x)$ that obeys the homogeneous Maxwell equation throughout the 4D space-time can be written in the form (18) for some 4–vector potential $A^\mu(x)$. But of course such 4–vector potential is not unique: a *gauge transformed* potential

$$A'^\mu(x) \;=\; A^\mu(x) \;-\; \partial^\mu\Lambda(x) \qquad \langle\!\langle\,\text{note the sign}\,\rangle\!\rangle \tag{22}$$

for any scalar $\Lambda(x)$ yields exactly the same $F^{\mu\nu}$ tensor as the original A^μ. Indeed,

$$F'^{\mu\nu} \;=\; \partial^\mu\big(A^\nu - \partial^\nu\Lambda\big) - \partial^\nu\big(A^\mu - \partial^\mu\Lambda\big) \;=\; \big(\partial^\mu A^\nu - \partial^\nu A^\mu\big) - \big[\partial^\mu, \partial^\nu\big]\Lambda \;=\; F^{\mu\nu} - 0. \tag{23}$$

Lagrangian Formalism

In the Lagrangian formulation of the electromagnetism, the 4 components of the $A^\mu(x)$ potential are treated as independent fields, while the $F^{\mu\nu}(x)$ tensor field is *defined* as

$$F^{\mu\nu}(x) \;\overset{\text{def}}{=}\; \partial^\mu A^\nu(x) \;-\; \partial^\nu A^\mu(x). \tag{24}$$

Consequently, regardless of the Lagrangian, the $F^{\mu\nu}$ field obeys the homogeneous Maxwell equation $\partial^{[\lambda}F^{\mu\nu]} = 0$ as a second-derivative identity. On the other hand, the inhomogeneous Maxwell equation $\partial_\mu F^{\mu\nu} = (1/c)J^\nu$ emerges as an Euler–Lagrange equation stemming from the action

$$S\big[A^\mu(x)\big] \;=\; \frac{1}{c}\int d^4x\,\mathcal{L} \tag{25}$$

$$\text{for}\quad \mathcal{L} \;=\; -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;-\; \frac{1}{c}\,J_\nu A^\nu \tag{26}$$

where the first term technically is merely a compact way of writing

$$-\frac{1}{4}\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big)\big(\partial^\mu A^\nu - \partial^\nu A^\mu\big). \tag{27}$$

In 3D notations, the Lagrangian density (26) is

$$\mathcal{L} = \tfrac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) - \rho\Phi + \frac{1}{c}\mathbf{J}\cdot\mathbf{A} \tag{28}$$

where again the first term is merely a compact form of writing

$$\frac{1}{2}\left(-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi\right)^2 - \frac{1}{2}\left(\nabla\times\mathbf{A}\right)^2. \tag{29}$$

Let's derive the Euler–Lagrange equations form the Lagrangian density (26) and see that they are indeed the inhomogeneous Maxwell equations. Since the $F_{\alpha\beta}$ tensor involves the derivatives of the potentials A_μ but not the potential themselves, it follows that

$$\frac{\partial F_{\alpha\beta}}{\partial A_\nu} = 0 \quad \text{while} \quad \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta. \tag{30}$$

Consequently,

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = -\frac{1}{c}J^\nu \tag{31}$$

while

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial\mathcal{L}}{\partial F_{\alpha\beta}} \times \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = -\frac{2}{4}F^{\alpha\beta} \times \left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta\right) = -\tfrac{1}{2}F^{\mu\nu} + \tfrac{1}{2}F^{\nu\mu} = -F^{\mu\nu}. \tag{32}$$

Consequently, the Euler–Lagrange equations for each components $A_\nu(x)$ of the vector potential, namely

$$\forall\nu: \quad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0, \tag{33}$$

become

$$-\partial_\mu F^{\mu\nu} + \frac{1}{c}J^\nu = 0, \tag{34}$$

which are indeed the inhomogeneous Maxwell equations in 4–vector notations.

Now consider the behavior of the Lagrangian density (26) and the action (25) under gauge transforms of the potentials,

$$A'_\mu(x) \;=\; A_\mu(x) \;-\; \partial_\mu \Lambda(x), \quad F'_{\mu\nu}(x) \;=\; F_{\mu\nu}(x). \tag{35}$$

Obviously the $F_{\mu\nu}F^{\mu\nu}$ term of the Lagrangian (26) is *gauge invariant*, but the source term $J_\nu A^\nu$ is not, thus

$$\mathcal{L}' \;=\; \mathcal{L} \;+\; \frac{1}{c} J^\nu \partial_\nu \Lambda \tag{36}$$

and hence

$$\begin{aligned}
S' \;&=\; S \;+\; \frac{1}{c^2} \int d^4x \, J^\nu(x) \partial_\nu \Lambda(x) \\
\;&=\; S \;-\; \frac{1}{c^2} \int d^4x \, \big(\partial_\nu J^\nu(x) \big) \times \Lambda(x) \;+\; \text{a boundary term.}
\end{aligned} \tag{37}$$

Moreover, for the current which turns off at spacetime infinity, $J^\nu \to 0$ for $x \to 0$, the boundary term vanishes, thus

$$\Delta S \;=\; -\frac{1}{c^2} \int d^4x \, \big(\partial_\nu J^\nu(x) \big) \times \Lambda(x), \tag{38}$$

which means that *the EM action is gauge invariant if and only if the electric current is conserved, $\partial_\nu J^\nu(x) \equiv 0$.*

We have seen that for a non-conserved current with $\partial_\nu J^\nu \neq 0$, the Maxwell equations would be inconsistent. In terms of the least action principle, for a non-conserved current the EM action (25) does not have any local minima. Instead, for any potential configuration $A^\mu(x)$ the action can always be decreased by an appropriate gauge transform. Physically, $\partial_\nu J^\nu \neq 0$ would source a non-existent longitudinal polarization of the EM waves, and the Maxwell equations would be inconsistent precisely because such polarization does not exist!

Now let's count the dynamical EM degrees of freedom (per space point $\mathbf{x}$). Naively, there are 4 independent potentials $A^\mu(x)$ subject to second-order PDEs, so there should be 4 dynamical degrees of freedom. But two separate effects reduce this number to just 2 dynamical degrees of freedom:

1 : The scalar potential $A^0 = \Phi$ appears in the Lagrangian density $\mathcal{L}$ without a *time derivative* $\partial A^0 / \partial t$ — only the A^0 itself and its space derivatives ∇A^0 enter $\mathcal{L}$. Consequently, from the time-evolution point of view, the A^0 is not a dynamical variable but rather an *auxiliary field* enforcing the Gauss Law $\nabla \cdot \mathbf{E} = \rho$ as a time-independent constraint. This leaves us with only three dynamical EM degrees of freedom, namely A^1, A^2, and A^3.

2 : For a gauge-invariant action $S[A]$, we may treat gauge transforms of the potentials $A^\mu(x)$ parametrized by an arbitrary $\Lambda(x)$ as *redundancies*. In other words, the true degrees of freedom are the EM potentials modulo gauge transforms. Or equivalently, we may use this redundancy to impose a gauge-fixing constraint, for example an axial gauge $A^3 \equiv 0$, or Coulomb gauge $\nabla \cdot \mathbf{A} \equiv 0$, or Landau gauge $\partial_\mu A^\mu \equiv 0$. Any such gauge fixing imposes one constraint at every space point $\mathbf{x}$, which eliminates one dynamical degree of freedom.

⋆ And that's how the EM fields end up with $4 - 1 - 1 = 2$ dynamical degrees of freedom. In terms of plane waves, these 2 degrees of freedom correspond to 2 transverse polarizations — but no longitudinal or temporal polarizations. And in the quantum theory, a photon with a given momentum $\mathbf{p}$ has only 2 independent polarization states.

Note that without gauge invariance of the action, the $A^\mu(x)$ fields would not be redundant, and we would end up with 3 rather then 2 dynamical degrees of freedom. Indeed, consider a massive relativistic vector field $A^\mu(x)$ with Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa^2}{2} A_\mu A^\mu - \frac{1}{c} J_\mu A^\mu. \tag{39}$$

As you shall see in homework set 1 (problem#1), for this theory the vector field obeys

$$\partial_\mu A^\mu = \frac{1}{c\kappa^2} \partial_\mu J^\mu \longrightarrow 0 \quad \text{for a conserved current} \tag{40}$$

and hence

$$\left(\partial^2 + \kappa^2\right) A^\mu = \frac{1}{c} J^\mu. \tag{41}$$

In particular, the plane-wave solutions for the A^μ in the absence of the current look like

$$A^\mu(x) \;=\; \big(\text{constant } a^\mu\big) \times \exp(ik^\nu x_\nu) \tag{42}$$

where the wave vector k^ν has $k^\nu k_\nu = \kappa^2$ — and hence the quanta have mass $m = \kappa\hbar/c$ — while the polarization vector a^μ obeys $a_\mu k^\mu = 0$. Consequently, for each allowed wave vector k, there are 3 independent polarizations a^μ, rather than only 2 polarizations of the EM waves.