Extra Dimensions of Theoretical Physics

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- Everyday life happens in 4 dimensions: 3 of space and 1 of time.
- All experimentally verified physics — from the cosmic scale 10^{10} ly $\sim 10^{26}$ m down to the smallest distances 10^{-19} m resolved by the present day accelerators — also happens in the same 4 dimensions.
- But many theories of high-energy physics beyond the Standard Model — *i.e.* of distances shorter that 10^{-19} m — involve extra dimensions besides the usual $3+1$. This lecture is about these extra dimensions.

Main types of Extra Dimensions:

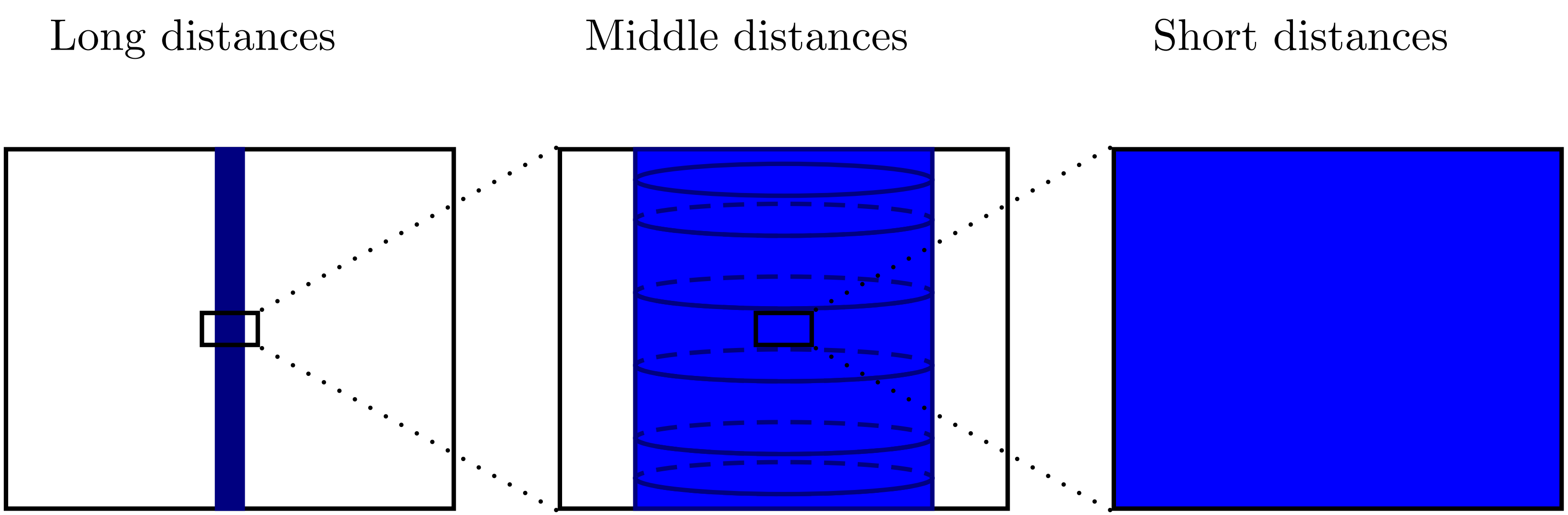
1. Kaluza–Klein: We live in $d > 4$ dimensions, but the extra $d - 4$ dimensions are hidden because they are too small to resolve.

2. Brane World: All known types of particles — quarks, leptons, photons, gluons, *etc.* — are confined to a $3 + 1$ dimensional *brane* embedded into a higher-dimensional spacetime. And maybe some unknown particle types live on *other branes.* But gravity and perhaps some other fields live in the whole spacetime, including the extra dimensions.

3. Holography: Some quantum field theories in the ordinary $3 + 1$ dimensions but in the strongly coupled regime are dual to supergravity theories in $4 + 1$ or even higher dimensions.

Invisible Dimensions à la Kaluza–Klein

Three views of a garden hose.



Only one apparent dimension.

Two dimensions, one infinite, one finite.

Two dimensions, both appear infinite.

5D Kaluza–Klein Theory

$$\boxed{\text{5D spacetime}} \;=\; \boxed{\text{4D Minkowski}} \otimes \boxed{\text{A circle}}$$

The momentum P_4 in the compact direction is quantized, $P_4 = \dfrac{n}{R}$ (in the $\hbar = c = 1$ units), hence a massless 5D particle species becomes a discrete family of 4D particles. Indeed, in 5D

$$E^2 = \vec{P}^2 + P_4^2,$$

so from the 4D point of view $|P_4|$ acts as a mass,

$$\boxed{\text{4D mass}} \;=\; |P_4| \;=\; \frac{|n|}{R}.$$

At low energies ($E \ll R^{-1}$) only the massless zero mode gets excited $\Longrightarrow$ only one 4D particle (species). But at higher energies ($E \gtrsim R^{-1}$) higher P_4 modes become excited too $\Longrightarrow$ a whole family of 4D particles.

The original purpose of Kaluza & Klein was to unify electromagnetism and gravity into a 5D gravity theory:

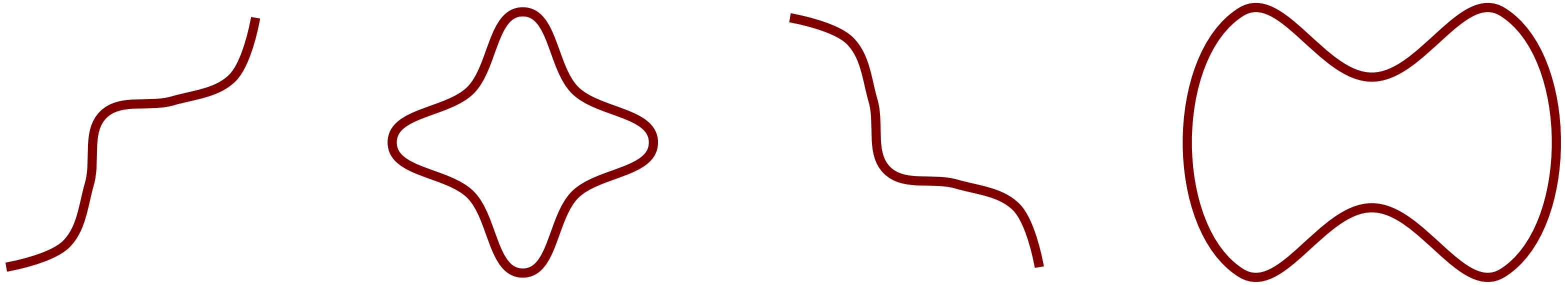
$$\boxed{\text{5D metric } G_{MN}} = \boxed{\text{4D } g_{\mu\nu}} \oplus \boxed{\text{4D vector } g_{4\mu}} \oplus \boxed{\text{4D scalar } g_{44}}$$

- Good news: For small enough circle radius R, the fifth dimension becomes hidden, and all we see are the massless zero modes, namely the 4D gravitons, photons, and scalars.

- Bad news: The electric charge is related to the same P_4 momentum in the circular direction as the 4D mass, $Q = ne = Re \times P_4$, so all the charged particles are superheavy. Energies high enough to make them would probe the hidden dimension.

String Theory

All particles are made from tiny bits or loops of string:



- Each vibrational/rotational quantum state of the string give rise to a particle species, thus infinite but discrete particle spectrum.
- A finite number of massless or almost-massless particle species is governed by an effective quantum field theory.
- The rest of particles are superheavy,

$$M \gtrsim \sqrt{\text{string tension}} \sim 10^{18}\ \text{GeV}.$$

PROBLEM: Quantum string theories need spacetimes of too many dimensions (usually $9+1$), or extremely high curvatures ($R_{\text{curvature}} \sim 10^{-34}$ m), or both.

SIMPLEST SOLUTION: Use $9+1$ dimensions altogether, but only $3+1$ of them are large; the remaining 6 dimensions form a compact manifold of very small size, $L \sim 10^{-32}$ m to 10^{-34} m. The 6 small dimensions are hidden from the present-day experiments à la Kaluza–Klein.

- Massless modes of the string in 10 dimensions include the gravitons and a bunch of other particles of different spins. The wave equations of all these particles in 6 compact dimensions have a bunch of zero-energy eigenstates. These zero-energy eigenstates give rise to massless particles in the $3+1$ large dimensions.

By ‘massless’ particles in 4D I mean all the particles whose mass is much smaller than the Kaluza–Klein scale of 10^{16} to 10^{18} GeV. Some of these particles remain exactly massless like the photon or the graviton, while other particles get masses from the HIggs mechanism or from the quark & gluon confinement in QCD.

Biggest problem of the String Theory: The six compact dimensions may have may different geometries, and each geometry yields a different spectrum of ‘massless’ particles in 4D and different interactions between them. Finding the 6D geometry which yields the 4D Physics which agrees with the real world is like finding a needle in a Texas-sized field of haystacks!

Branes and Brane-worlds

A *brane* is an extended object which supports some kinds of waves which propagate only along that brane but don't spread out into the surrounding space. For example, a 2D membrane in 3D space can vibrate, and the vibration waves propagate only in the 2 dimensions of the membrane and do not spread out in the third dimension of the embedding space.

Terminology: a p-brane has p space dimensions plus 1 time.
— A 0–brane is a particle.
— A 1–brane is a string.
— A 2–brane is a membrane. (Hence the name 'brane'.)

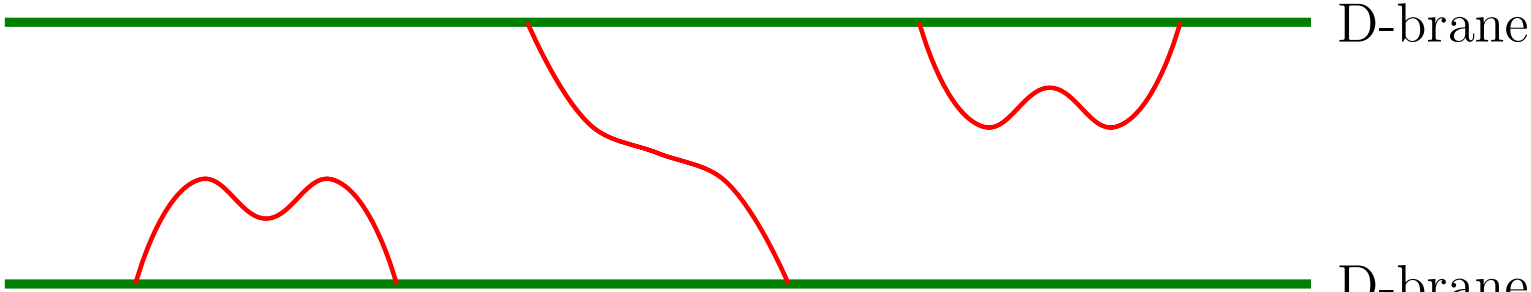
- In spaces with $(d-1) > 3$ dimensions we may also have 3–branes, 4–branes, *etc.*, which do not have special names.

- In quantum field theory, branes obtain from topologically nontrivial configurations of scalar, vector, or gravitational fields, which trap zero modes of some other fields. These zero modes give rise to the waves which propagate only along the brane.

- Examples: domain walls, vortices, magnetic monopoles, Yang–Mills instantons, gravitational instantons, *etc.*

- String theory has several kinds of branes. The NS-branes stem from the closed string sector; similar to the QFT branes, they are singular configurations of the background gravitational, vector, and scalar fields which trap zero modes of closed strings.

- The D-branes stem from the open-string sector — they are places where the open strings are allowed to begin or end:

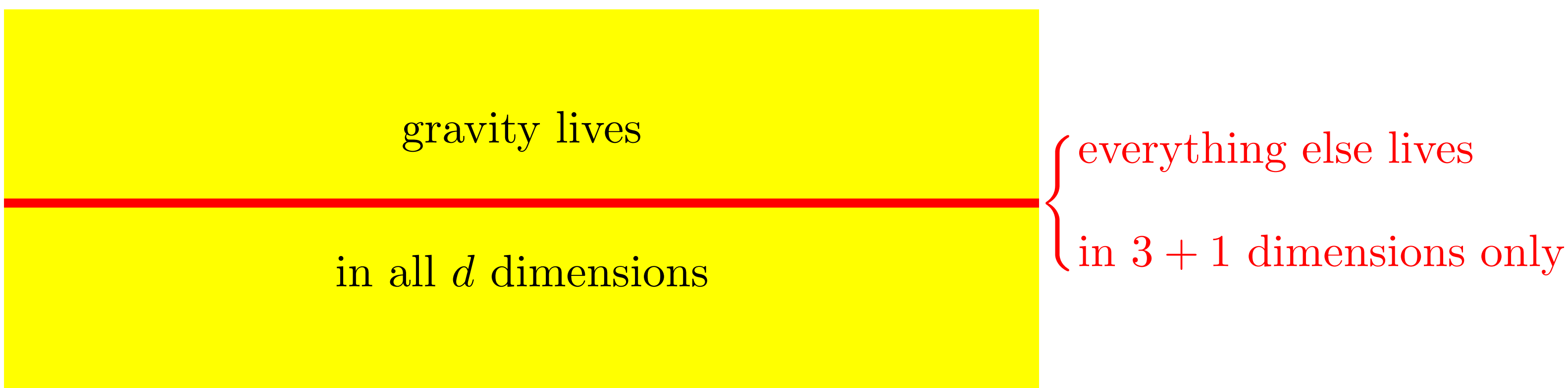


A string beginning and ending on the same D-brane may have zero classical length, so its quantum states include zero-energy modes, hence low-frequency waves living on the D-brane.

- Other types of branes include stacks of N coincident D-branes, bound states of NS-branes and D-branes, black branes, *etc.*, *etc.*

- Each brane type carries different kind of waves and hence different types of quanta living on the brane.

BRANE WORLD is a setup where we live on a 3–brane in some higher-dimensional space. That is, all the particles we are made from, all the particles we can make at the accelerators, the photons and the cosmic rays coming to us from across the Universe — everything but gravity lives in the 3+1 dimensions of the brane and does not feel the extra dimensions. Only the gravitational fields extend through the entire $d > 4$ dimensional spacetime.



PROBLEM: If the gravity lives in almost-flat $d > 4$ dimensions, Newton's Law of Gravity becomes

$$F = \frac{GM_1M_2}{r^2} \quad\Longrightarrow\quad F = \frac{G'M_1M_2}{r^{d-2}}$$

SOLUTION: Compactify the extra dimensions à la Kaluza–Klein, or warp them. Then Newton's Law becomes

$$F = \frac{GM_1M_2}{r^2} \quad \text{for } r \gg L \quad \text{but} \quad F = \frac{G'M_1M_2}{r^{d-2}} \quad \text{for } r \ll L$$

where L is the size (or curvature radius) of the extra dimensions.

EXPERIMENTAL LIMIT: $L < 30\ \mu\text{m}$, obtains from Cavendish-like experiments using micro-cantilever or torsion pendulum probes.

(A) Strong gravity at the TeV scale

- In 4 spacetime dimensions, gravity between elementary particles becomes strong when particle energies reach the Planck scale $M_p \sim G^{-1/2} \sim 10^{19}$ GeV (in $\hbar = c = 1$ units). In $d > 4$ dimenensions, gravity becomes strong at a lower energy scale

$$M_p' \sim (G')^{\frac{-1}{d-2}} \sim \left(G \times L^{d-4}\right)^{\frac{-1}{d-2}}.$$

- In a brane world with 6 nucleus-sized extra dimensions, this strong-gravity scale could be as low as a few TeV, within reach of the LHC. Thus, proton-proton collisions at the LHS would be able to create microscopic black holes, which would then decay into unusually large numbers of elementary particles.

- Alas, the LHC did not observe any black holes (or rather their decay products) nor any other strong-gravity effects, which puts a tighter limit on the extra dimensions' sizes.
- However, brane worlds with smaller extra dimensions are OK. Also, we may have one or two large extra dimensions (up to a few microns) while the rest of the extra dimensions are extremely small, which would push the strong-gravity scale beyond a few TeV.
- But perhaps a future accelerator will reach the strong-gravity regime and make microscopic black holes. Or something even weirder.

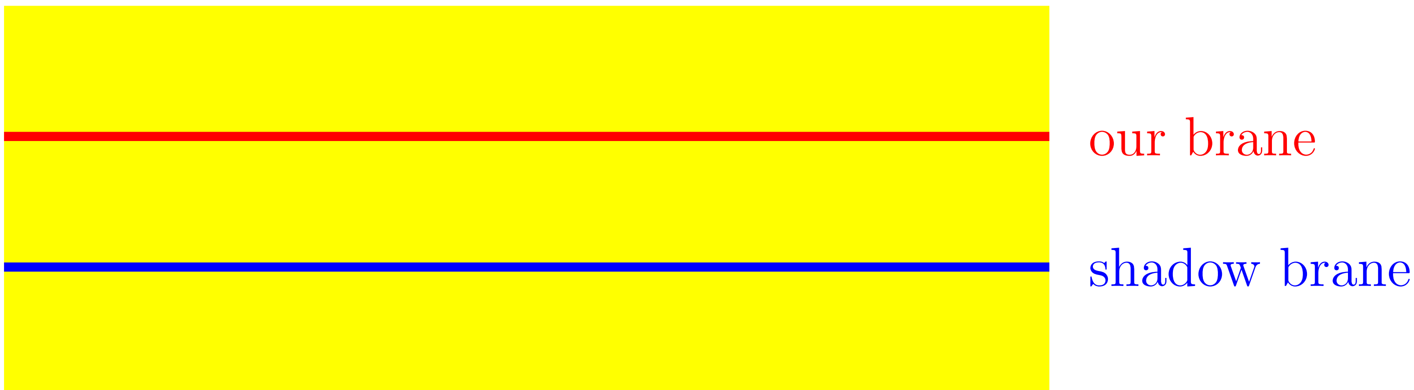
(B) Superluminal gravity waves

If the world 3–brane folds upon itself in the surrounding space, then gravity waves would be able to travel across the bulk space in less time than the light waves would travel along the 3–brane:

- Ruled out by GW170817: gravity waves (LIGO and Virgo) and a gamma-ray burst (Fermi and Integral) from a neutron start merger 130 millions light years away were detected within 2 seconds from each other!

(C) Shadow Matter

What if there are several 4D brane worlds within the same higher-dimensional spacetime? That is, there are several 3–branes parallel to each other in the surrounding space:



All the familiar particles of the Standard Model live on the same world-brane; let's call it *our brane*. The other branes (called *shadow branes*) carry *shadow matter* made from particle species we have not yet discovered.

- The only force between out kind of matter and the shadow matter is gravity. Likewise, the only force between particles living on different shadow branes is gravity. The familiar EM, strong, and weak forces are limited to particles living on our brane.
- However, if a shadow brane is exactly like our brane, then it would carry quark-like and lepton-like particles with strong-like, EM-like, and weak-like forces between them — but only between the particles living on the same brane.

• Shadow matter just like the ordinary matter and with similar energy density would make the Universe expand too fast. This is ruled out by the precise measurements of the cosmic microwave background.

• But shadow matter with a lower energy density would be OK.

- Instead of mirroring our kind of matter, the shadow matter on another type of a world-brane could be quite different. And in some models, such shadow matter may play important roles.
- For example, in a supersymmetric model shadow matter may be responsible for the spontaneous supersymmetry breakdown.
- Or perhaps the dark matter of astronomy and cosmology is a kind of shadow matter.
- The possibilities are endless...

Combining Kaluza–Klein and Branes

- Many modern string models combine Kaluza–Klein-like hidden dimensions with branes. There are $9+1$ dimensions altogether, but only $3+1$ dimensions are large while the other six form a compact manifold $\mathcal{M}$ of extremely small size $L \sim 10^{-34}$ m. In this 10D spacetime there are several 5–branes, 6–branes, and/or 7–branes; for each such p–brane, 3 space dimensions (and one time) are large, while the remaining $p-3$ space dimensions span a topologically stable submanifold of $\mathcal{M}$. Also, many branes intersect each other over subspaces of smaller dimensions.

- In this setup, the gravitons live in the $9+1$ dimensional bulk, the vector particles like photons or gluons live in $p+1$ dimensions of various branes, while the fermions like quarks and leptons live on the *brane intersections*.

Holography

- In optics, *holography* is the way to encode all visible information of a 3D object in a 2D picture. In high-energy theoretical physics, *holography* encodes all gravitational properties — including the quantum information — of some curved spacetime in a quantum field theory living in a lower dimension.
- For example, holography maps gravity in a 5D Anti-de-Sitter space to a 4D conformally invariant quantum field theory, hence holography is often called the *AdS/CFT correspondence*.
- Other examples involve gauge theories without conformal symmetries, hence yet another name for the holography — *gauge-gravity duality*.

- Holography maps strongly coupled gauge theories — which are poorly understood since the perturbation theory breaks down in the strong coupling regime — onto semi-classical gravity theories in spaces of low curvature (in Planck units) which we know how to work with. In this way, holography provides us with a non-perturbative tool set for handling the strongly coupled gauge theories like QCD.

- For example, holography allows us to calculate the hydrodynamic properties of the quark gluon plasma. In particular, the ratio of the plasma's shear viscosity η to its entropy density s comes out to be

$$\frac{\eta}{s} \approx \frac{\hbar}{4\pi k_B} \times \text{ a calculable } O(1) \text{ number},$$

much lower than η/s of everyday fluids like water.

- On the other hand, holography maps gravity in highly curved spaces — for which quantum gravity effects become important — onto weakly coupled quantum field theories for which we have well-understood perturbation theory. In this way, holographic duals of perturbative quantum field theories act as non-perturbative definitions of the quantum gravity theory.

- In particular, holography helps us understand the quantum information issues involving black holes, such as entanglement between particles falling into the black home and the outside world, or what happens when the black hole evaporates.

Holography and String Theory

- Holographic duality between a specific QFT and between gravity in a specific geometry is usually established via string theory. One starts with a string model (typically, a bunch of branes in some embedding space) and then finds that its low-energy degrees of freedom can be described by some QFT or by gravity in some geometry, depending on some adjustable parameter.

- For an example, consider the original AdS/CFT duality (Maldacena, 1997). In string theory, it starts with a stack of $N \gg 1$ coincident 3–branes in approximately flat 10D spacetime.

- If the product $N \times g_{\text{string}}$ happens to be small, then the space surrounding the branes remains approximately flat, and the low-energy degrees of freedom are the zero modes of the strings connecting the 3–branes. Specifically, there N^2 gluon-like vector particles, $2N^2$ quark-like fermions, and $6N^2$ spinless particles.

- The interactions between all these particles are governed by the super–Yang–Mills theory with a $U(N)$ gauge group, a non-abelian gauge theory like QCD but different in many details.

- Unlike QCD, the super–Yang–Mills theory (SYM) is scale-invariant: it has the same coupling strength α at all distance scales, from very short to very long. Moreover, SYM is invariant under conformal symmetries of spacetime which change the distance scale by different factors in different places.

- In the opposite regime of $N \times g_{\text{string}} \gg 1$, the stack of N branes collapses under its own gravity into a *black brane:* In the 6 dimensions $\perp$ to the brane it looks like a black hole, but the $3+1$ dimensions of the original 3–brane stack remain infinite.

- The Einstein metric of the 10D spacetime surrounding the black brane is warped:

$$ds^2 \;=\; \frac{d\mathbf{x}_{3d}^2 \;-\; dt^2}{f(r)} \;+\; f(r) \times \left(dr^2 \;+\; r^2 d\Omega_5^2\right)$$

where r and Ω_5 are the radial and the angular coordinates of the 6 dimensions $\perp$ to the black brane,

$$f(r) \;=\; \sqrt{1+(L/r)^4}$$

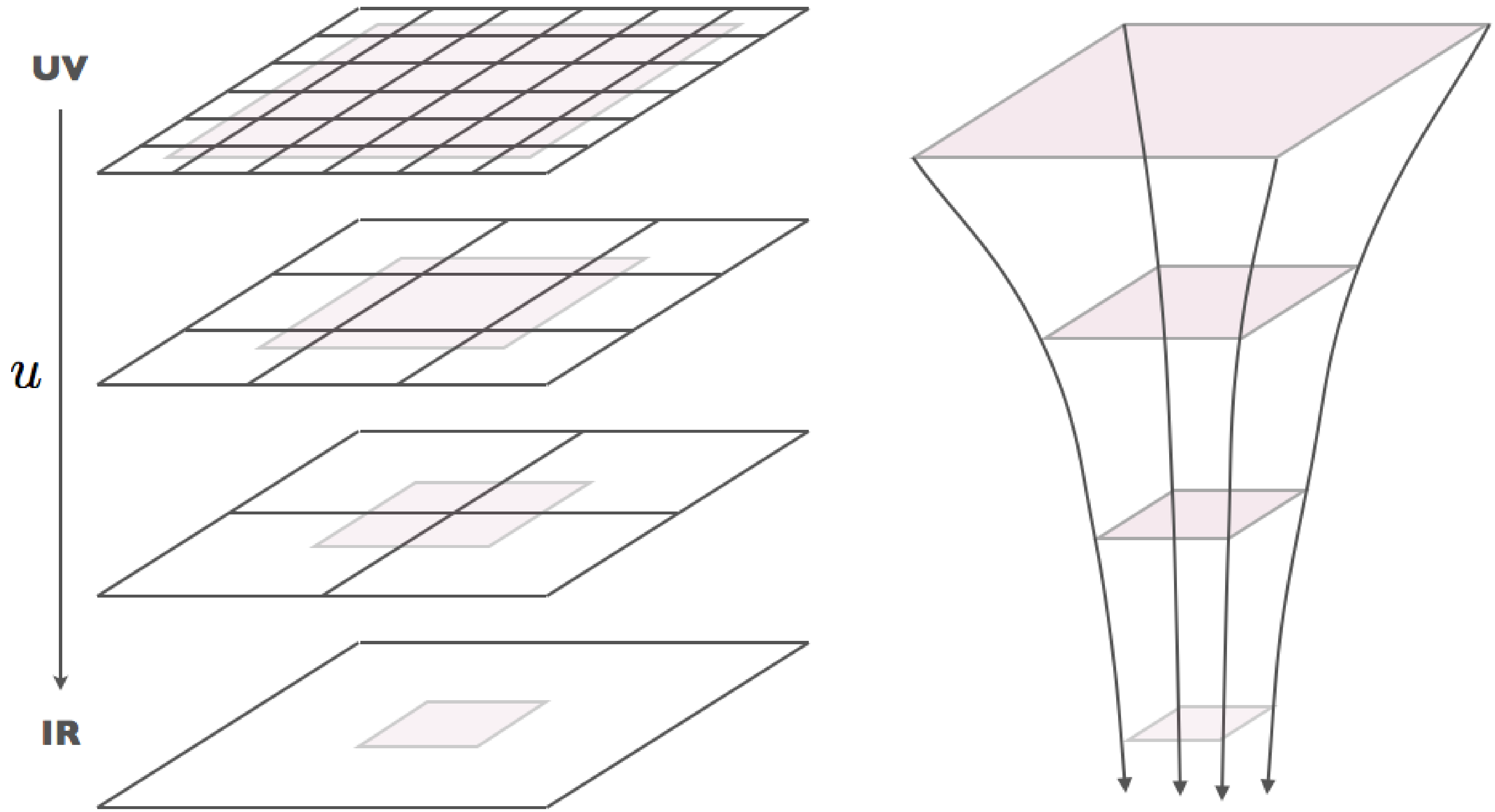
is the warp factor, and $L \propto (N g_{\text{string}})^{1/4}$ is the Schwarzschild radius.

- In the near-horizon limit $r \ll L$, this 10D metric becomes a direct product of the 5D Anti–de–Sitter space (AdS_5) and the 5D sphere S^5:

$$ds^2 \approx \boxed{\frac{L^2}{r^2} \times dr^2 + \frac{r^2}{L^2} \times \left(d\mathbf{x}_{3d}^2 - dt^2\right)} + L^2 \times d\Omega_5^2$$

- Geometric symmetries of the $\mathrm{AdS}_5 \otimes \mathrm{S}^5$ spacetime form the $SO(4,2) \times SO(6)$ group. The SYM field theory has a similar symmetry group, although the $SO(6)$ factor acts non-geometrically; instead, it transforms different species of scalar and fermionic fields into each other. As to the $SO(4,2)$ factor, it's the conformal symmetry group in $3+1$ dimensions, comprising translations, rotations, Lorentz boosts, dilatations, and the special conformal transforms.

- The r coordinate of the Anti–de–Sitter space geometrizes the renormalization group flow from the UV at $r \to \infty$ to the IR at $r \to 0$:



Applications of Holography

Although the original holographic duality was limited to conformally invariant field theories on one side and AdS geometries on the other, it was soon generalized to may other contexts:

- IR-strong gauge theories with many QCD-like features like quark confinement. (Alas, no known holographic dual of the QCD itself.)
- Gauge theories at finite temperature — they are dual to geometries with black holes!
- Hadronic and nuclear physics.
- Condensed matter at or near a quantum critical point.

- Relativistic hydrodynamics.
- Quantum entanglement.
- Complexity theory.

⋆ And great many other issues, too numerous to list here.

In all such holographic dualities, the gravity side has more dimensions than the ‘QFT’ side, although the specific meaning of the extra dimensions may differ for different dualities.

That's all, folks!