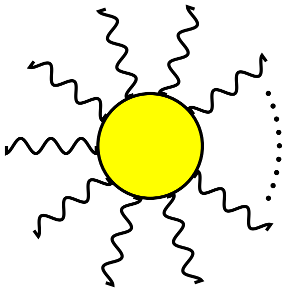
Ward–Takahashi Identities: Diagrammatic Proof

In the first set of notes on Ward–Takahashi identities I have stated the general WT identities and showed how they follow from the electric current conservation. In the present notes, I prove the WT identities in the language of Feynman diagrams. For simplicity, I limit this proof to the basic QED, comprised of EM and electron fields, and nothing else.

There are Ward–Takahashi identities for off-shell amplitudes involving any numbers of photonic or electronic external legs, but the identities for amplitudes with 0 or 2 electronic legs are particularly important, so let me restate them here:

- No electrons, N photons amplitudes

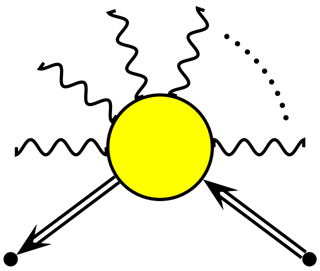


$$= iV_N^{\mu_1\ldots\mu_N}(k_1,\ldots,k_N) \xrightarrow[\text{shorthand}]{} iV_N^{1,\ldots,N}.$$

The V_N are amputated amplitudes, meaning no external leg bubbles in the diagrams, and the external legs themselves are not included in the amplitudes. Ward–Takahashi identities for the V_N are simply

$$\forall i, \quad (k_i)_{\mu_i} \times V_N^{\mu_1\ldots\mu_N}(k_1,\ldots,k_N) \;=\; 0. \tag{1}$$

- 2 electrons, N photons amplitudes, which include S_N include the *dressed* propagators for the 2 electron's external lines, but the N photon's external lines are amputated:



$$= S_N^{\mu_1\ldots\mu_N}(p',p;k_1,\ldots k_N) \xrightarrow[\text{shorthand}]{} S_N^{1,\ldots,N}(p',p).$$

By convention, the photon momenta k_i are treated as incoming, while the electron momenta follow the charge arrows: p is incoming while p' is outgoing, hence $p' - p = k_1 + \cdots k_N$. Ward–Takahashi identities for the S_N amplitudes are recursive relations relating S_N to S_{N-1}, namely

$$\forall i, \quad (k_i)_{\mu_i} \times S_N^{1,\ldots,N}(p',p) \;=\; eS_{N-1}^{1,\ldots,\cancel{i},\ldots,N}(p',p+k_i) \;-\; eS_{N-1}^{1,\ldots,\cancel{i},\ldots,N}(p'-k_i,p). \tag{2}$$

Outline:

1. Proof of (2) at the tree level.
2. Proof of (1) at the one-loop level.
3. Multi-loop amplitudes (in the bare perturbation theory).
4. Taking care of the counterterms (an outline rather than a complete proof)

(1) **Lemma 1:** *the identity* (2) *holds at the tree level.*

Proof by induction in N: first prove (2) for $N = 1$ and $N = 2$, then show that if the identity holds for some N, it also holds for $N + 1$.

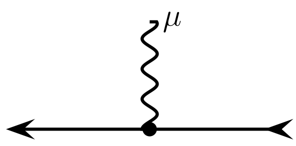
Let's start with $N = 1$. At the tree level

$$S_0(p' = p) \;=$$

$$= \frac{i}{\not p - m} \tag{3}$$

while

$$S_1^\mu(p', p; k) \;=$$



$$= \frac{i}{\not p' - m}\left(ie\gamma^\mu\right)\frac{i}{\not p - m}\,. \tag{4}$$

Multiplying this expression by the k_μ produces

$$k_\mu \times S_1^\mu \;=\; -ie\,\frac{1}{\not p' - m}\left(k_\mu \times \gamma^\mu = \not k\right)\frac{1}{\not p - m}\,, \tag{5}$$

but thanks to momentum conservation

$$k^\mu \;=\; p'^\mu \;-\; p^\mu \quad\Longrightarrow\quad \not{k} \;=\not{p}' \;-\; \not{p} \;=\; (\not{p}' \;-m) \;-\; (\not{p}-m). \tag{6}$$

Consequently

$$\frac{1}{\not{p}'-m}\times\not{k}\times\frac{1}{\not{p}-m} \;=\; \frac{1}{\not{p}-m} \;-\; \frac{1}{\not{p}'-m} \tag{7}$$

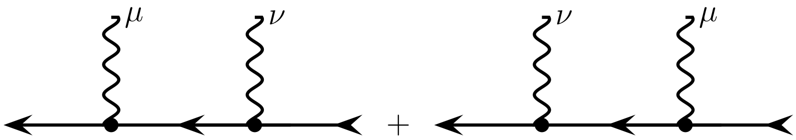
and therefore

$$\begin{aligned} k_\mu\times S_1^\mu(p',p;k) \;&=\; \frac{-ie}{\not{p}-m} \;+\; \frac{ie}{\not{p}'-m} \\ &=\; -eS_0(p,p) \;+\; eS_0(p',p') \\ &=\; -eS_0(p'-k,p) \;+\; eS_0(p',p+k) \quad \text{since } p'-p=k. \end{aligned} \tag{8}$$

This proves the tree-level WT identity (2) for $N=1$.

For $N=2$, there are two tree diagrams for the S_2 amplitude, and we must add them up to make the WT identity work — each diagram by itself does not satisfy any useful WT-like identities. Indeed, at the tree level

$$S_2^{\mu\nu}(p',p;\,k_1,k_2) \;=$$



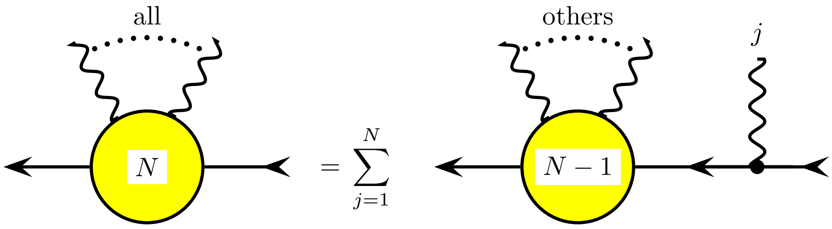
$$\begin{aligned} &=\; \frac{i}{\not{p}'-m}\,\big(ie\gamma^\mu\big)\,\frac{i}{\not{p}'-\not{k}_1-m}\,\big(ie\gamma^\nu\big)\,\frac{i}{\not{p}-m} \\ &\quad+\; \frac{i}{\not{p}'-m}\,\big(ie\gamma^\nu\big)\,\frac{i}{\not{p}+\not{k}_1-m}\,\big(ie\gamma^\mu\big)\,\frac{i}{\not{p}-m}\,. \end{aligned} \tag{9}$$

Multiplying this expression by the $(k_1)_\mu$ and using eqs. (7), we obtain

$$
\begin{aligned}
(k_1)_\mu \times S_2^{\mu\nu}(p', p;\, k_1, k_2) \;&=\; \frac{i}{\not{p}' - m}\left(ie\,\not{k}_1\right)\frac{i}{\not{p}' - \not{k}_1 - m}\left(ie\gamma^\nu\right)\frac{i}{\not{p} - m} \\
&\quad + \frac{i}{\not{p}' - m}\left(ie\gamma^\nu\right)\frac{i}{\not{p} + \not{k}_1 - m}\left(ie\,\not{k}_1\right)\frac{i}{\not{p} - m} \\
&= \left(\cancel{\frac{ie}{\not{p}' - m}} - \frac{ie}{\not{p}' - \not{k}_1 - m}\right) \times \left(ie\gamma^\nu\right)\frac{i}{\not{p} - m} \\
&\quad + \frac{i}{\not{p}' - m}\left(ie\gamma^\nu\right) \times \left(\frac{ie}{\not{p} + \not{k}_1 - m} - \cancel{\frac{ie}{\not{p} - m}}\right) \\
&= e\,\frac{i}{\not{p}' - m}\left(ie\gamma^\nu\right)\frac{i}{\not{p} + \not{k}_1 - m} \;-\; e\,\frac{i}{\not{p}' - \not{k}_1 - m}\left(ie\gamma^\nu\right)\frac{i}{\not{p} - m} \\
&= e \times S_1^\nu(p', p + k_1; k_2) \;-\; e \times S_1^\nu(p' - k_1, p; k_2),
\end{aligned}
\tag{10}
$$

which proves the Lemma for $N = 2$.

For $N > 2$ there are $N!$ tree diagrams according to $N!$ orderings of the N photons' vertices along the electron line. To make the WT identities work for all N photons we must sum all the $N!$ diagrams, although fewer diagrams will make the identity work for any one particular photon.[⋆] But instead of writing down all the $N!$ diagrams, let me simply organize them into N blocks of $(N-1)!$ diagrams according to which photon's vertex is closest to the incoming end of the electron line. Diagrammatically,



(11)

⋆ Specifically, pick any one ordering of the $N-1$ photons for the S_{N-1} amplitudes on the RHS of the identity (2), say $1, 2, \ldots, (N-1)$. Then to make the identity work, the S_N on the LHS of the identity should sum over N orderings — for all possible insertions of the extra photon (whose k_μ multiplies the S_N) into the fixed order of the other photons, namely $(N, 1, 2, \ldots (N-1))$, $(1, N, 2, 3, \ldots, (N-1))$, all the way to $(1, 2, \ldots, (N-2), N, (N-1))$, and finally $(1, 2, \ldots, (N-1), N)$.

which gives us a recursive formula for the tree-level S_N amplitudes,

$$S_N^{1,\ldots,N}(p', p;\,) \;=\; \sum_{j=1}^{N} S_{N-1}^{\ldots\not{j}\ldots}(p', p+k_j) \times \left(ie\gamma^{\mu_j}\right) \frac{i}{\not{p}-m}\,. \tag{12}$$

This recursive formula will help prove the induction step: suppose all the S_{N-1} amplitudes on the RHS of eq. (12) obey the WT identity (2), then the S_N amplitude on the LHS also obey the WT identity. Indeed, multiplying both sides of eq. (12) by the $(k_i)_{\mu_i}$ we obtain

$$\begin{aligned}(k_i)_{\mu_i} \times S_N^{1,\ldots,N}(p', p) \;=\; & \sum_{j\neq i} (k_i)_{\mu_i} \times S_{N-1}^{\ldots\not{j}\ldots}(p', p+k_j) \times \left(ie\gamma^{\mu_j}\right) \frac{i}{\not{p}-m} \\ & + S_{N-1}^{\ldots\not{i}\ldots}(p', p+k_i) \times \left(ie\,\not{k}_i\right) \frac{i}{\not{p}-m}\end{aligned} \tag{13}$$

where on the RHS I have separated the $j=i$ term in the $\sum_j$ from the other terms. For each $j\neq i$ term we may use the induction hypotheses for the S_{N-1} amplitudes, thus

$$(k_i)_{\mu_i} \times S_{N-1}^{\ldots\not{j}\ldots}(p', p+k_j) \;=\; eS_{N-2}^{\ldots\not{i}\ldots\not{j}\ldots}(p', p+k_j+k_i) \;-\; eS_{N-2}^{\ldots\not{i}\ldots\not{j}\ldots}(p'-k_i, p+k_j). \tag{14}$$

Now let's use the recursive formula (12) in reverse, to go from the $\sum_{j\neq i} S_{N-2}$ to the S_{N-1}. Specifically,

$$\sum_{j\neq i} eS_{N-2}^{\ldots\not{i}\ldots\not{j}\ldots}(p'-k_i, p+k_j) \times \left(ie\gamma^{\mu_j}\right) \frac{i}{\not{p}-m} \;=\; eS_{N-1}^{\ldots\not{i}\ldots}(p'-k_i, p) \tag{15}$$

and likewise

$$\sum_{j\neq i} eS_{N-2}^{\ldots\not{i}\ldots\not{j}\ldots}(p', p+k_i+k_j) \times \left(ie\gamma^{\mu_j}\right) \frac{i}{\not{p}+\not{k}_i-m} \;=\; eS_{N-1}^{\ldots\not{i}\ldots}(p', p+k_i). \tag{16}$$

Note that in the last formula the incoming electron propagator has a different momentum from what we had in eq. (13) — $p+k_i$ instead of p, — but since this propagator is the same for all j, we can correct for it using an overall factor:

$$\frac{i}{\not{p} - m} \;=\; \frac{1}{\not{p} + \not{k}_i - m} \times \left(1 + \not{k}_i \frac{1}{\not{p} - m}\right) \tag{17}$$

and hence

$$\sum_{j \neq i} e S_{N-2}^{\ldots\cancel{i}\ldots\cancel{j}\ldots}(p', p + k_i + k_j) \times \left(ie\gamma^{\mu_j}\right) \frac{i}{\not{p} - m} \;=\; e S_{N-1}^{\ldots\cancel{i}\ldots}(p', p_i) \times \left(1 + \not{k}_i \frac{1}{\not{p} - m}\right). \tag{18}$$

Altogether, eqs. (14), (15), and (18) tell us that the sum on the first line of eq. (13) amounts to

$$\begin{aligned} \text{first line} \;&=\; \sum_{i \neq j} (k_i)_{\mu_i} \times S_{N-1}^{\ldots\cancel{j}\ldots}(p', p + k_j) \times \left(ie\gamma^{\mu_j}\right) \frac{i}{\not{p} - m} \\ &=\; e S_{N-1}^{\ldots\cancel{i}\ldots}(p', p + k_i) \times \left(1 + \not{k}_i \frac{1}{\not{p} - m}\right) \;-\; e S_{N-1}^{\ldots\cancel{i}\ldots}(p' - k_i, p). \end{aligned} \tag{19}$$

As to the $j = i$ term on the second line of eq. (13), it does not need the induction hypotheses, we may simply add it as it is to eq. (19):

$$\begin{aligned} (k_i)_{\mu_i} \times S_N^{1,\ldots,N}(p', p) \;&=\; e S_{N-1}^{\ldots\cancel{i}\ldots}(p', p + k_i) \times \left(1 + \cancel{\not{k}_i \frac{1}{\not{p} - m}}\right) \;-\; e S_{N-1}^{\ldots\cancel{i}\ldots}(p' - k_i, p) \\ &\quad + S_{N-1}^{\ldots\cancel{i}\ldots}(p', p + k_i) \times \cancel{\left(ie\,\not{k}_i\right) \frac{i}{\not{p} - m}} \\ &=\; e S_{N-1}^{\ldots\cancel{i}\ldots}(p', p + k_i) \;-\; e S_{N-1}^{\ldots\cancel{i}\ldots}(p' - k_i, p), \end{aligned} \tag{20}$$

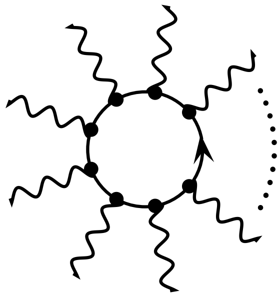
which proves the induction step and hence the whole Lemma 1.

(2) **Lemma 2:** *Ward–Takahashi identity* (1) *holds at the one-loop level.*

Now let's put the 2-electron S_N amplitudes aside for a moment and focus on the no-external-electrons amplitudes V_N. Since there are no tree diagrams for any of the V_N, our starting point is the one-loop level, hence the present Lemma.

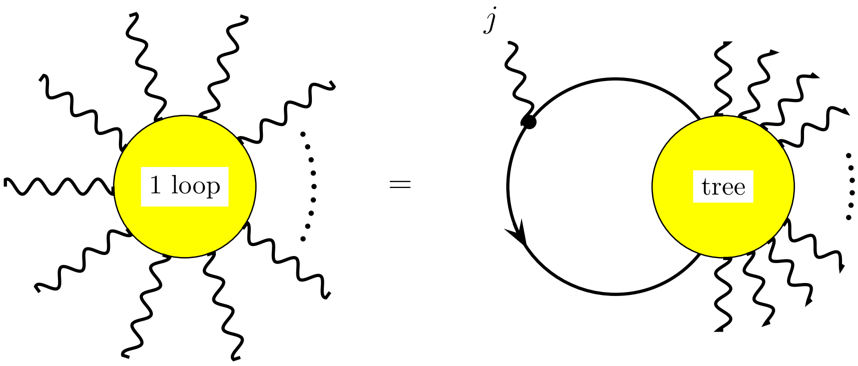
At the one-loop level, the V_N come from electron loops going through N photonic vertices,

$$iV_N^{1\,\text{loop}} \;=$$



$$+ \text{ photon permutations.} \tag{21}$$

Note that only the *cyclic order* of the photon vertices is relevant to the electron loop, so we may always keep one particular photon — say photon $\#j$ — at the beginning of the loop, and then we should sum over $(N-1)!$ permutations of the other $N-1$ photons. Schematically,



(22)

which translates to

$$i\;{}^{1\,\text{loop}}V_N^{1,\ldots,N} \;=\; -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big[\big(ie\gamma^{\mu_j}\big)\times{}^{\text{tree}}S_{N-1}^{\cdots\not{j}\cdots}(p,p+k_j)\Big], \qquad \text{same } \forall j. \tag{23}$$

Thanks to this relation, we may use Lemma 1 to prove the present Lemma 2. Indeed,

$$(k_i)_{\mu_i}\times iV_N^{1,\ldots,N} \;=\; -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big[\big(ie\gamma^{\mu_j}\big)\times(k_i)_{\mu_i}\times S_{N-1}^{\cdots\not{j}\cdots}(p,p+k_j)\Big] \tag{24}$$

$$\langle\!\langle \text{ for some } j\neq i \,\rangle\!\rangle$$

$$= \; -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\big(ie\gamma^{\mu_j}\big)\times\begin{pmatrix} eS_{N-2}^{\cdots\not{i}\cdots\not{j}\cdots}(p+k_i,p+k_j) \\ -\; eS_{N-2}^{\cdots\not{i}\cdots\not{j}\cdots}(p,p+k_j-k_i)\end{pmatrix}\right] \tag{25}$$

$$\text{⚠} \quad = \quad -e\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big[\big(ie\gamma^{\mu_j}\big)\times eS_{N-2}^{\cdots\not{i}\cdots\not{j}\cdots}(p+k_i,p+k_j)\Big]$$

$$+\,e\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big[\big(ie\gamma^{\mu_j}\big)\times eS_{N-2}^{\cdots\not{i}\cdots\not{j}\cdots}(p,p+k_j-k_i)\Big] \tag{26}$$

$$\text{⚠} \quad = \quad 0 \tag{27}$$

because the two integrals (26) are related by a constant shift of the integration variable, $p \to p - k_i$.

This argument appears to prove Lemma 2, but the caution signs in eqs. (26) and (27) warn of a loophole in the last two steps in our argument. Specifically, we have turned an integral of a difference into a difference of two integrals, and then we have shifted the integration variable in just one of these integrals. When all the integrals converge, such manipulations work fine, *but using them for divergent integrals is dangerous and may easily produce wrong results.*

In Quantum Field Theory, a divergent momentum integral is a short-hand notation for a long procedure: first, we impose a UV cutoff, then we re-calculate the integrand using the Feynman rules of the cut-off theory, then we take the integral, and finally we go back to the original theory by taking the $\Lambda \to \infty$ or the $D \to 4$ limit. For the problem at hand, we need a UV regulator that

- Renders all the integrals (26) finite (for a large but finite Λ, or for $D < 4$);
- Allows shifting of the momentum integration variables;
- Does not change the QED Feynman rules in a way that screws up the tree-level Ward–Takahashi identities (2).

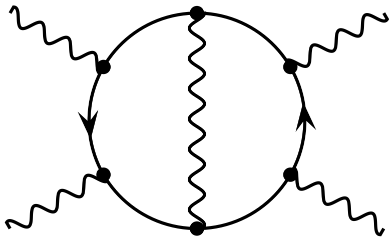
Fortunately, QED does have UV regulators that satisfy all these criteria — for example, the dimensional regularization — so eqs. (26) and (27) work as written and the Ward–Takahashi identities (1) hold true.

Likewise, other gauge theories with true-vector currents $\overline{\Psi}\gamma^\mu\Psi$ obey Ward–Takahashi identities similar to the (1). However, the chiral gauge theories — in which the left-handed and the right-handed Weyl fermions may have different charges or belong to different multiplets — do not allow dimensional regularization or any other UV regulators that would make eqs. (26)

and (27) work for $N = 3$ (or $N = 4$ for some non-abelian theories). Consequently, some of the WT identities suffer from the *anomalies* — I shall explain them later in class, probably in April — and if those anomalies do not cancel, the gauge theory fails as a quantum theory.

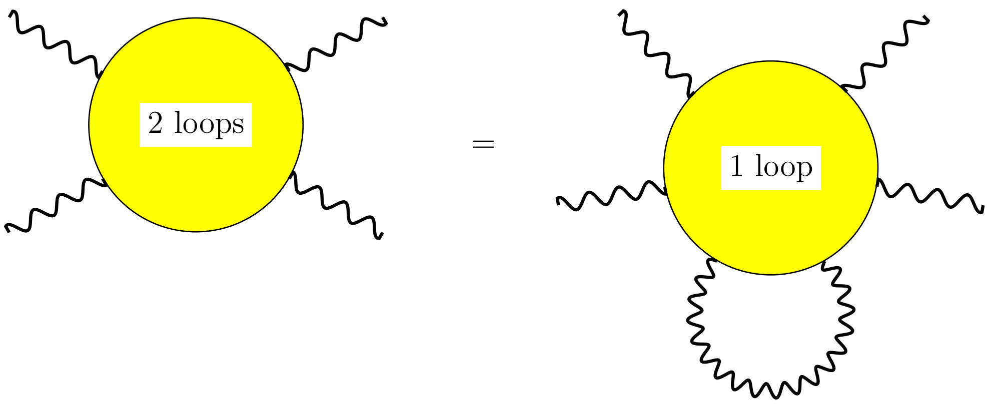
(3) Going Beyond One Loop

In §2 we have proved the Ward–Takahashi identities (1) at the one loop level, now let's extend the proof to the multi-loop diagrams. For starters, consider the two-loop diagrams with one electronic loop and one internal photon propagator (which makes for the second loop), for example



(28)

When evaluating such a diagram, let us integrate over the electron's momentum before we integrate over the momentum of the internal photon. The first stages of this evaluation — the Dirac traceology and integrating over the p_e — are exactly similar to working a one-loop diagram with $N + 2$ external photons instead of N. Also, totaling up similar diagrams with different cyclic orders of the photonic vertices on the electronic line — including the vertices belonging to the internal photon — works exactly similar to the one-loop diagrams. Consequently



(29)

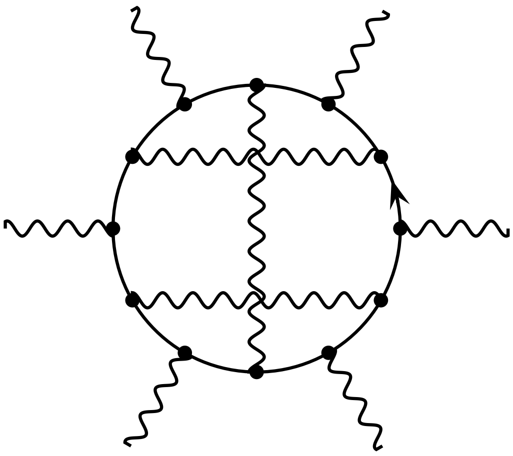
which means

$$
\begin{aligned}
{}^{2\,\text{loops}}V_N^{\mu_1\ldots\mu_N}(k_1,\ldots,k_N) &= \int\frac{d^4\hat{k}}{(2\pi)^4}\,\frac{-i}{\hat{k}^2+i0}\left(g_{\nu\rho}+(\xi-1)\frac{\hat{k}_\nu\hat{k}_\rho}{\hat{k}^2}\right) \\
&\qquad\qquad \times\,{}^{1\,\text{loop}}V_{N+2}^{\mu_1\ldots\mu_N,\nu\rho}(k_1,\ldots,k_N,+\hat{k},-\hat{k}).
\end{aligned}
\tag{30}
$$

Thanks to this relation, the Ward–Takahashi identity (1) for the one-loop V_{N+2} immediately implies a similar identity for the two-loop V_N,

$$
\begin{aligned}
(k_i)_{\mu_i}\times{}^{1\,\text{loop}}V_{N+2}^{\mu_1\ldots\mu_N,\nu\rho}(k_1,\ldots,k_N,+\hat{k},-\hat{k}) &= 0 \\
&\Downarrow \\
(k_i)_{\mu_i}\times{}^{2\,\text{loops}}V_N^{\mu_1\ldots\mu_N}(k_1,\ldots,k_N) &= 0.
\end{aligned}
\tag{31}
$$

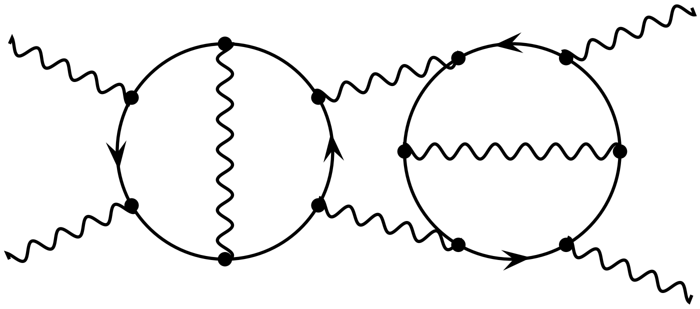
The same argument applies to the multi-loop diagrams that have one electron loop and several internal photon propagators, for example



(32)

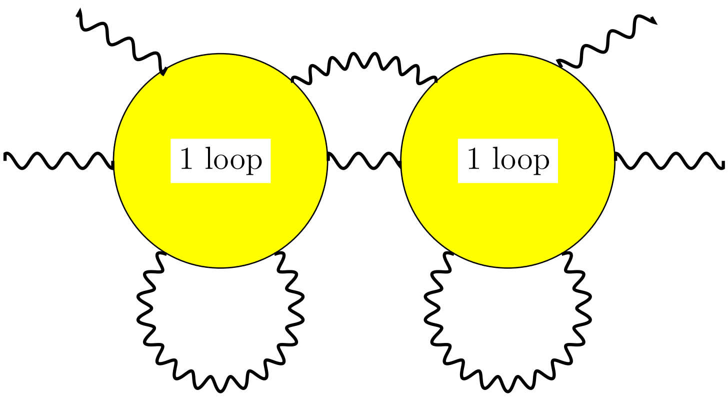
Again, once we total up the diagrams in which the photons — external or internal — attach to the electron line in all the cyclic orders, the net amplitude V_N becomes the integral of the one-loop amplitude V_{N+2m} times the internal photon propagators. Once multiplied by the k_μ of any external photon, the $k_\mu\times V_{N+2m}$ inside the integral vanishes by Lemma 2, which makes the whole integral vanish and hence $k_\mu\times V_N^{\text{multi loop}}=0$.

Now consider diagrams with multiple electronic loops such as



(33)

Let's group such diagrams according to how many internal photons connect each pair of electronic loops (or one electronic loop to itself) or which loop is connected to which external photon; the diagrams in which the same photons are attached to the same electron lines — albeit in a different order — belong to the same group. For example, the diagram (33) belongs to a group of 1800 diagrams that can be summarized as



(34)

For each diagram, we do the Dirac traceology and integrals over the electron momenta before integrating over the photon momenta. We also total up all the diagram in the group before integrating over the photon momenta, which gives

$$V_N[\text{group}] \;=\; \int\limits_{\substack{\text{photon}\\ \text{momenta}}} d^{4n}\hat{k} \prod \binom{\text{photon}}{\text{propagators}} \times \prod_{\substack{\text{electron}\\ \text{loops}}} V_M^{1\,\text{loop}}\,. \tag{35}$$

Each external photon is attached to one of the electronic loops, and the corresponding $V_M^{1\,\text{loop}}$

factor carries that photon's index μ. Consequently,

$$k_\mu \times \left(V_M^{\text{that loop}}\right)^\mu \;=\; 0, \tag{36}$$

which makes the whole integral (35) vanish,

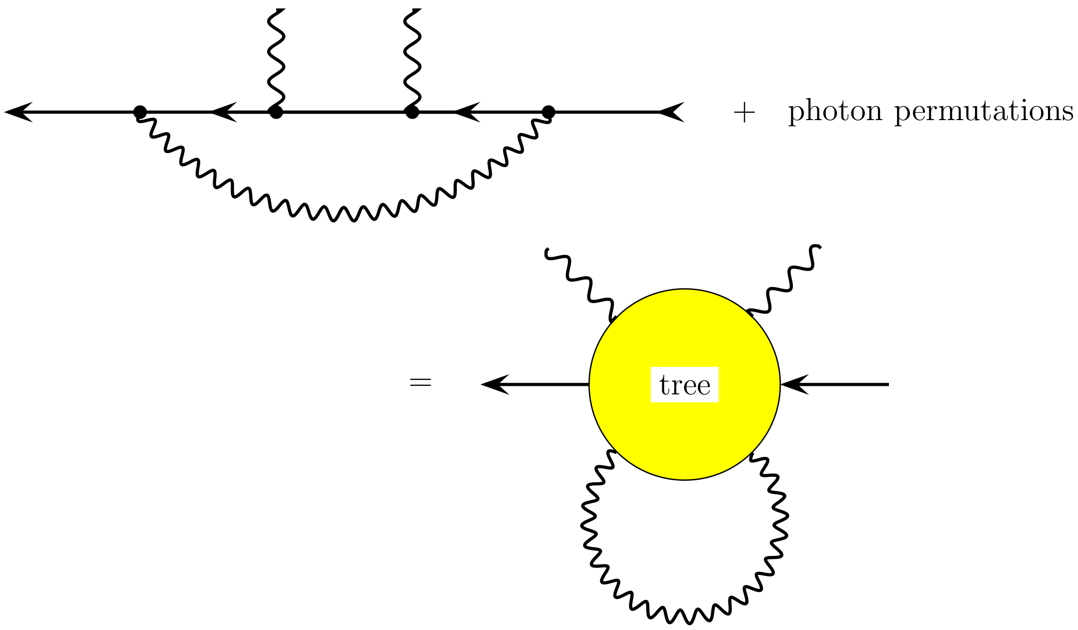
$$k_\mu \times \left(V_N^{\text{whole group}}\right)^\mu \;=\; 0. \tag{37}$$

Finally, combining all the diagram groups which contribute to an L-loop, N-photon amplitude, we prove the WT identity

$$k_\mu \times \left(V_N^{\text{net}}\right)^\mu \;=\; 0. \tag{1}$$

⋆ ⋆ ⋆

Next, let's turn our attention back to the WT identities (2) for the two-electron, N-photon amplitudes S_N. Back in §1 we have proved those identities for the tree-level amplitudes, and now we are going to extend the proof to the loop amplitudes. Let's start with the one-loop amplitudes such as



(38)

Adding up all the photon permutations, we obtain

$$ {}^{1\,\text{loop}}S_N^{\mu_1\ldots\mu_N}(p',p;\,k_1,\ldots,k_N) \;=\; \int\frac{d^4\hat{k}}{(2\pi)^4}\,\text{prop}_{\nu\rho}(\hat{k})\times{}^{\text{tree}}S_{N+2}^{\mu_1\ldots\mu_n\nu\rho}(p',p;\,k_1,\ldots,k_N,+\hat{k},-\hat{k}). \tag{39} $$

Consequently, when we multiply this amplitude by the k_μ of an external photon, the WT identity for the tree-level S_{N+2} immediately produces a similar identity for the one-loop-level S_N,

$$ \begin{aligned} k_\mu\times\left(S_{N+2}^{\text{tree}}\right)^{\mu\cdots}(p',p;\cdots) \;&=\; e\left(S_{N+1}^{\text{tree}}\right)^{\cdots}(p',p+k;\cdots) \;-\; e\left(S_{N+1}^{\text{tree}}\right)^{\cdots}(p'-k,p;\cdots)\\ &\Downarrow\\ k_\mu\times\left(S_N^{1\,\text{loop}}\right)^{\mu\cdots}(p',p;\cdots) \;&=\; e\left(S_{N-1}^{1\,\text{loop}}\right)^{\cdots}(p',p+k;\cdots) \;-\; e\left(S_{N-1}^{1\,\text{loop}}\right)^{\cdots}(p'-k,p;\cdots) \end{aligned} \tag{40} $$

Clearly, the same argument applies to the diagrams with more internal photon propagators, so all the multi-photon-loop amplitudes obey similar WT identities (2).

Now let's allow for all kinds of multi-loop diagrams with two external electrons and N external photons. All such diagrams have one open electronic line — which begins at the incoming electron line, goes through a few vertices and propagators, and ends at the outgoing electron line. In addition, there may be any number of closed electronic loops. All these electronic lines — open or closed — are connected to each other by some internal photon propagators; some internal photons may also connect an electron line to itself. Finally, each of the external photons is connected to one of the electron lines, open or closed.

As we did before, we should group such diagrams according to the numbers of electronic lines, the numbers of the internal photons connecting each pair of those lines (or a line to itself), and also according to which external photons attach to which line. Again, all diagrams related by permutations of the photon vertices on the same electron line — open or closed — belong in the same group, and we must add them all up to make the WT identities work. As usual, it's convenient to add them up after evaluating the electron lines and integrating over the electron momenta, but before integrating over the photon's momenta, thus

$$ S_N^{\text{whole group}}(p',p) \;=\; \int\limits_{\substack{\text{photon}\\\text{momenta}}} d^{4L'}\hat{k}\,\prod\binom{\text{photon}}{\text{propagators}}\times\prod_{\substack{\text{electron}\\\text{loops}}}V_M^{1\,\text{loop}}\times S_n^{\text{tree}}(p',p). \tag{41} $$

This formula — plus the Lemmas 1 and 2 — tells us what happens when we multiply such a multi-loop amplitude by a k_μ of an external photon: it depends on whether that photon is connected to an open electron line or to the one of the closed electron loops. For a photon connected to a closed loop we have

$$k_\mu \times \left(V_M^{\text{that loop}}\right)^{\mu\cdots} = 0 \quad\Longrightarrow\quad k_\mu \times \left(S_N^{\text{whole group}}\right)^{\mu\cdots} = 0. \tag{42}$$

On the other hand, for an external photon attached to the open line we have

$$\begin{aligned} k_\mu \times \left(S_n^{\text{tree}}\right)^{\mu\cdots}(p',p;\cdots) &= e\left(S_n^{\text{tree}}\right)^{\cdots}(p',p+k;\cdots) - e\left(S_n^{\text{tree}}\right)^{\cdots}(p'-k,p;\cdots) \\ &\Downarrow \\ k_\mu \times \left(S_N^{\text{whole group}}\right)^{\mu\cdots}(p',p;\cdots) &= e\left(S_{N-1}^{\text{whole group}}\right)^{\cdots}(p',p+k;\cdots) \\ &\quad - e\left(S_{N-1}^{\text{whole group}}\right)^{\cdots}(p'-k,p;\cdots) \end{aligned} \tag{43}$$

because all the other factors in (41) do not depend on *that* external photon or on the external electron momenta p and p'.

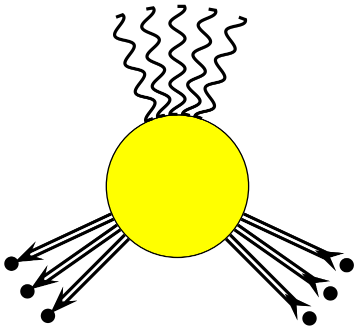
To make the WT identities work for all the external photons, we need to combine the diagrams into bigger groups so that each photon can be attached to any of the electron lines, open or closed. Consequently, for any external photon $\#i$ we have

$$\begin{aligned} (k_i)_{\mu_i} \times S_N^{\text{big group}}(p',p) &= \sum_\ell^{\text{lines}} (k_i)_{\mu_i} \times S_N[i \to \ell](p',p) \\ &\langle\!\langle \text{ in light of eqs. (42) and (43) } \rangle\!\rangle \\ &= (k_i)_{\mu_i} \times S_N[i \to \text{open}](p',p) + 0 \\ &= eS_{N-1}^{\text{big group}}(p',p+k_i) - eS_{N-1}^{\text{big group}}(p'-k_i,p). \end{aligned} \tag{44}$$

In other words, the WT (2) identities works for the bigger groups of diagrams, and once we total up all the diagrams (up to some maximal #loops), the identities work for the complete multi-loop amplitudes. *Quod erat demonstrandum.*

⋆ ⋆ ⋆

Besides the identities (1) and (2) for amplitudes involving zero or two external electron lines, there are similar Ward–Takahashi identities for amplitudes with any number of incoming and outgoing electrons. In general, we may have M incoming electron lines, same number of outgoing electron lines, and N external photons,



$$= S_{MN}^{\mu_1 \ldots \mu_N}(p_1', \ldots, p_M'; \, p_1, \ldots, p_M; \, k_1, \ldots, k_N) \tag{45}$$

(Dirac indices suppressed). Similar to the 2-electron amplitudes S_N, all the external photonic lines here are amputated but all the incoming and outgoing electron lines include the dressed propagators.

For all such amplitudes, the Ward–Takahashi identities relate an amplitude contracted with a k_μ of an external photon to amplitudes without that photon. Specifically,

$$\begin{aligned}
&(k_i)_{\mu_i} \times S_{MN}^{\mu_1 \ldots \mu_N}(p_1', \ldots, p_M'; \, p_1, \ldots p_M; \, k_1, \ldots, k_N) \\
&\qquad = \; e \sum_{j=1}^{M} S_{M,N-1}^{\cdots \not\mu_i \cdots}(p_1', \ldots, p_M'; \, p_1, \ldots, p_j + k_i, \ldots, p_M; \, k_1, \ldots \not k_i \ldots, k_N) \\
&\qquad\quad - \; e \sum_{j=1}^{M} S_{M,N-1}^{\cdots \not\mu_i \cdots}(p_1', \ldots, p_j' - k_i, \ldots, p_M'; \, p_1, \ldots p_M; \, k_1, \ldots \not k_i \ldots, k_N).
\end{aligned} \tag{46}$$

The proof of these identities works similarly to what we have in §3, so let me outline it without working through the details. A generic diagram contributing to the amplitude (45) has M open electronic lines, any number of closed electronic loops, a bunch of internal photons connecting all these lines to each other (or to themselves), and each external photon should be connected to one of the electronic lines, open or closed. Combining such diagrams in groups related by permutations of photons attached to the same electronic line, we relate the S[group] to the product of tree-level 2-electron S_N for the open lines and one-loop-level no-electron V_n for the closed loops. Consequently, contracting the S[group] with k_μ of an external photon gives us zero if that photon is attached to a closed line; if it's attached to an open line, we get two terms that look like S of a similar group but without the external photon in question. Finally, adding up the groups where the photon in question is attached to all possible electronic lines, we obtain the WT identity (46).

(4) Taking Care of the Counterterms.

In §3 we have proved the identities (1) and (2) for groups of tree or loop diagrams, but we have not considered the diagrams containing the counterterms vertices. In effect, we have worked in the bare perturbation theory, so the identities (2) we have proved thus far amount to

$$k_\mu \times S^{\mu}_{1,\text{bare}}(p', p; k) \;=\; e_{\text{bare}} S_{0,\text{bare}}(p') \;-\; e_{\text{bare}} S_{0,\text{bare}}(p), \tag{47}$$

and likewise for $N = 2, 3, 4, \ldots$. Nevertheless, as we saw in §4 of the first set of notes on WT identities, eq. (47) in the bare perturbation theory implies

$$Z_1 \;=\; Z_2 \tag{48}$$

for the renormalized QED, and hence

$$\delta_1 \;=\; \delta_2 \tag{49}$$

for the counterterms. Note: eq. (49) is exact, both infinite and finite terms of the δ_1 and δ_2 counterterms must be equal to each other.

The identity (49) — as well as all the other Ward–Takahashi identities (1) and (2) — can be proved directly from the counterterm perturbation theory, without invoking the bare theory at all, but the proof is a bit more convoluted than what we had thus far in these notes. To save time and aggravation, let me skip the gory details of this proof and give you only the outline.

- ⋆ The proof works order by order in $\alpha = e^2/4\pi$ by induction in power of α. That is, given $\delta_1 = \delta_2$ to order α^L, we prove the identities (2) and (1) and hence $\delta_1 = \delta_2$ to order α^{L+1}.

⋆ The induction base is verifying $\delta_1 = \delta_2$ — including the finite parts of the counterterms — to order α. We shall do this later in class.

⋆ To prove the induction step, we *assume* $\delta_1 = \delta_2$ to order α^L and then consider the diagrams containing the counterterm vertices of all kinds as well as the physical vertices.

- We start with the tree diagrams — but containing the counterterm vertices — which contribute to the S_N amplitudes (2 electrons, N photons). We show that IF $\delta_1 = \delta_2$ THEN the Ward–Takahashi identities (2) work for for such diagrams, or rather groups of diagrams related by vertex permutations. This part of the proof proceeds by induction in N similarly to what we had in part (1) of these notes.

- Given the identities (2) for tree diagrams (but including the counterterm vertices), we proceed similar to part (2) and prove the photonic identities (1) for one-loop diagrams including the counterterms, and then we follow part (3) to extend the WT identities to the multi-loop diagrams.

- Since out induction assumptions is $\delta_1 = \delta_2$ only up to order α^L, the above arguments establish Ward–Takahashi identities (2) and (1) for complete amplitudes (all contributing diagrams) up to order α^L. At the next order α^{L+1}, the complete amplitudes involve $L+1$ loop diagrams without counterterms, L loop diagrams with a single $O(\alpha)$ counterterm, $L-1$ loop diagrams with two $O(\alpha)$ counterterms or one $O(\alpha^2)$ counterterm, *etc.*, *etc.*, and ending up with tree diagrams involving a single counterterm of order α^{L+1}. By the induction assumption, all diagram types except the latter obey the Ward–Takahashi identities.

- Consider the net order–α^{L+1} amplitudes

$$\Sigma_{\text{net}}^{\text{order}\,\alpha^{L+1}}(\not{p}) = \Sigma_{\text{loops}}^{\text{order}\,\alpha^{L+1}}(\not{p}) + \delta_m^{\text{order}\,\alpha^{L+1}} - \delta_2^{\text{order}\,\alpha^{L+1}} \times \not{p}, \tag{50}$$

$$(\Gamma^\mu)_{\text{net}}^{\text{order}\,\alpha^{L+1}}(p',p) = (\Gamma^\mu)_{\text{loops}}^{\text{order}\,\alpha^{L+1}}(p',p) + \delta_1^{\text{order}\,\alpha^{L+1}} \times \gamma^\mu, \tag{51}$$

where Σ_{loops} and $\Gamma^\mu_{\text{loops}}$ include all the $O(\alpha^{L+1})$ diagrams — from $L+1$ loops with no counterterm vertices to a single loop with an $O(\alpha^L)$ counterterm — except a pure counterterm vertex without any loops at all. The order α^{L+1} counterterms follow from these loop amplitudes and the renormalization conditions (in the counterterm perturbation theory)

$$\text{for on-shell } p' = p, \quad \Gamma^\mu_{\text{net}} \;=\; \gamma^\mu, \quad \Sigma^e_{\text{net}} \;=\; 0, \quad \frac{d\Sigma^e_{\text{net}}}{d\,\not p} \;=\; 0. \tag{52}$$

- From to the WT identity for the $(S_1^\mu)^{\text{order}\,\alpha^{L+1}}_{\text{loops}}$ and $(S_0)^{\text{order}\,\alpha^{L+1}}_{\text{loops}}$, we obtain a relation between the on-shell $(\Gamma^\mu)^{\text{order}\,\alpha^{L+1}}_{\text{loops}}(p' = p)$ and $(d/d\,\not p)\Sigma^{\text{order}\,\alpha^{L+1}}_{\text{loops}}(\not p)$, which translates to $\delta_1^{\text{order}\,\alpha^{L+1}} = \delta_2^{\text{order}\,\alpha^{L+1}}$ for the counterterms.

⋆ And this completes the induction step and hence the proof.