ANOMALIES AND DIFFERENTIAL FORMS

Introduction to Differential Forms

Mathematics of various antisymmetric tensor fields becomes much simpler in the language of differential forms. Students interested in string theory and related fields should master this language and then go ahead and learn as much differential geometry and topology as they can; take a class on the subject or at least read a book. Wikipedia has a quick and dirty introduction to differential forms at http://en.wikipedia.org/wiki/Differential_form and related web pages.

The differential form language is also very useful to describe axial and chiral anomalies of all kinds — which is what these notes are about. So let me start with a very basic introduction to differential forms.

Consider a space or spacetime of dimension D; it can be Euclidean or Minkowski, flat or curved; it might even be a differential manifold without any metric at all. A differential form of rank $p \leq D$ in such a space combines an antisymmetric tensor with p indices and a differential suitable for integration over a sub-manifold of dimension p (a line for $p = 1$, a surface for $p = 2$, *etc.*, *etc.*). For example,

$$A \;=\; A_\mu(x)\,dx^\mu, \qquad B \;=\; B_{\mu\nu}(x)\,dx^\mu \wedge dx^\nu, \qquad C \;=\; C_{\lambda\mu\nu}(x)\,dx^\lambda \wedge dx^\mu \wedge dx^\nu, \ldots. \tag{1}$$

For $p = 2$, a 2–form should be integrated over an oriented surface, so the order of dx^μ and dx^ν matters; in fact they anticommute, $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$, so the 2-index tensor $B_{\mu\nu}(x)$ should be antisymmetric. Likewise, the 3-volume differential $dx^\lambda \wedge dx^\mu \wedge dx^\nu$ is totally antisymmetric with respect to permutation of indices $\lambda\mu\nu$, so the 3-index tensor $C_{\lambda\mu\nu}(x)$ should also be totally antisymmetric in all of its 3 indices. And a general form of rank p

$$E \;=\; E_{\mu_1\mu_2\cdots\mu_p}(x)\,dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \tag{2}$$

involves a p-index *totally antisymmetric tensor* $E_{\mu_1\mu_2\cdots\mu_p}(x)$.

The *exterior derivative* of a rank-p form E is a form dE of rank $p+1$ defined as

$$dE \;=\; \bigl(dE_{\mu_1\mu_2\cdots\mu_p}(x)\bigr)\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \;=\; \partial_\lambda E_{\mu_1\mu_2\cdots\mu_p}(x)\, dx^\lambda\wedge dx^{\mu_1}\wedge\,\cdots\wedge dx^{\mu_p}, \tag{3}$$

but this compact formula hides the antisymmetrization due to anticommutativity of the dx^μ differentials. In the antisymmetric tensor form, $J=dE$ means

$$\begin{aligned} J_{\mu_1\cdots\mu_{p+1}}(x) \;&=\; \frac{1}{p!}\,\partial_{[\mu_1}E_{\mu_2\cdots\mu_{p+1}]}(x) \;=\; \sum_{j=1}^{p+1}(-1)^{j-1}\partial_{\mu_j}E_{\mu_1\cdots\not\mu_j\cdots\mu_{p+1}}(x) \\ &=\; \partial_{\mu_1}E_{\mu_2\cdots\mu_{p+1}} \;-\; \partial_{\mu_2}E_{\mu_1\mu_3\cdots\mu_{p+1}} \;\pm\;\cdots\;+\;(-1)^p\partial_{\mu_{p+1}}E_{\mu_1\cdots\mu_p}\,. \end{aligned} \tag{4}$$

The exterior derivative generalizes the 3D notions of *gradient*, *curl*, and *divergence*. Indeed, a scalar $\phi(x)$ is a 0–form and its gradient $\nabla\phi$ is a vector defining a 1-form $(\nabla\phi)_i\,dx^i=d\phi$. Likewise, for a vector $\vec{A}(x)$ and its curl $\vec{B}(x)=\nabla\times\vec{A}(x)$ we have a 1-form $A=A_i(x)dx^i$ and a 2-form $B=B_{ij}(x)dx^i\wedge dx^j=dA$ where $B_{ij}=\partial_iA_j-\partial_jA_i$; note that in 3D this antisymmetric tensor is equivalent to an axial vector, $B_{ij}=\epsilon_{ijk}B_k$. Finally, for a vector $\vec{E}(x)$ and its divergence $f(x)=\nabla\cdot\vec{E}$ we have an exterior derivative relation $f=dE$ for the 2-form $E=E_i(x)\epsilon_{ijk}dx^j\wedge dx^k$ equivalent to the vector $E_i(x)$ and a 3-form $f=f(x)\epsilon_{ijk}dx^i\wedge dx^j\wedge dx^k$ equivalent to the scalar $f(x)$.

There is a version of Leibniz rule for wedge products of forms: For any p-form $B_{(p)}$ and a q-form $C_{(q)}$,

$$d\bigl(B_{(p)}\wedge C_{(q)}\bigr) \;=\; (dB_{(p)})\wedge C_{(q)} \;+\; (-1)^pB_{(p)}\wedge(dC_{(q)}). \tag{5}$$

But the most important property of the exterior derivative is its nilpotency: for any differential form E, $ddE=0$. This rule corresponds to the differential identities for all kinds of antisymmetric tensors, in particular the vector calculus identities $\nabla\times(\nabla\phi)=0$ and $\nabla\cdot(\nabla\times\vec{A})=0$. The proof is very simple: If E is a form of rank p, $J=dE$ is a form of rank $p+1$, and $K = dJ$ is a form of rank $p+2$, then applying eq. (4) twice, we have

$$\begin{aligned} K_{\lambda\mu\nu_1\cdots\nu_p}(x) &= \frac{1}{(p+1)!}\,\partial_{[\lambda}J_{\mu\nu_1\cdots\nu_p]}(x) \\ &= \frac{1}{(p+1)!\,p!}\,\partial_{[\lambda}\partial_{[\mu}E_{\nu_1\cdots\nu_p]]}(x) \\ &= \frac{1}{p!}\,\partial_{[\lambda}\partial_{\mu}E_{\nu_1\cdots\nu_p]}(x) \\ &= 0 \end{aligned} \tag{6}$$

where the last equality follow from $\partial_{[\lambda}\partial_{\mu]} = 0$.

Next, a couple of definitions. A differential p-form is called *exact* iff it happens to be the exterior derivative of some $(p-1)$-form Ω, $Q = d\Omega$. A form Q is called *closed* iff its own exterior derivative vanishes, $dQ = 0$. By nilpotency of the exterior derivative, *any exact form is closed:* $dQ = dd\Omega = 0$. But the converse relation depends on the topology of the manifold in which the differential form live.

Poincare Lemma: *in a topologically trivial manifold, all closed differential forms are exact.* That is, if a p-form Q happens to have $dQ = 0$ then there exists a $(p-1)$-form Ω such that $d\Omega = Q$. But *this is not true if the manifold has topologically non-trivial p-cycles* over which Q may be integrated. Instead, there is a not trivial linear space of closed p-forms modulo exact p-forms, *i.e.*

$$\mathcal{H}_p \;=\; \left\{\text{equivalence classes of } \{Q_{(p)} \text{ such that } dQ_{(p)}\} \textbf{ modulo } d\Omega_{(p-1)}\right\}. \tag{7}$$

This space $\mathcal{H}_p$ is called the *p-cohomology* of the manifold in question. It has a finite dimension equal to the number of independent topologically non-trivial p cycles of the manifold.

For example, consider a 2-torus $T^2 = S^1 \times S^1$. It has two independent 1-cycles corresponding to the two S^1 circles of the manifold. In terms of the 2 periodic coordinates of the torus, (x, y), x modulo $2\pi R_1$, y modulo $2\pi R_2$, one cycle corresponds to x running from 0 to $2\pi R_1 \equiv 0$ at a fixed y, while the other cycle has y running from 0 to $2\pi R_2 \equiv 0$ at a fixed x.

Now consider the 1-forms $A = A_\mu dx^\mu$ on the torus. An exact 1-form $A = d\Lambda$ is a gradient of a 0-form Λ, thus $A_\mu(x, y) = \partial_\mu\Lambda(x, y)$. However, to be a proper 0-form, the function $\Lambda(x,y)$ must be *single-valued on the torus*, thus

$$\Lambda(x,y) \;=\; \Lambda(x+2\pi R_1,y) \;=\; \Lambda(x,y+2\pi R_2). \tag{8}$$

Consequently,

$$\oint\limits_{x\,\text{cycle}} A \;=\; \oint\limits_{y\,\text{cycle}} A \;=\; 0. \tag{9}$$

OOH, a closed 1-form A has a zero curl $\partial_\mu A_\nu - \partial_\nu A_\mu = 0$, so *locally* A is a gradient of some scalar, $A_\mu(x,y) = \partial_\mu \Phi(x,y)$. However, the scalar Φ here does not have to be single valued, as long as its gradient is single valued. Consequently, we may have

$$\oint\limits_{x\,\text{cycle}} A \;=\; \Phi(x+2\pi R_1,y) \;-\; \Phi(x,y) \;\neq\; 0 \tag{10}$$

and likewise

$$\oint\limits_{y\,\text{cycle}} A \;=\; \Phi(x,y+2\pi R_2) \;-\; \Phi(x,y) \;\neq\; 0. \tag{11}$$

For example $A_1 = dx$ is locally the gradient of $\Phi_1(x,y) = x$, but since this Φ_1 is multi-valued on the torus, the A_1 form is not exact. However, A_1 itself is single valued and has a zero curl, so it is closed. Likewise, the $A_2 = dy$ is closed but not exact. Furthermore, it is easy to see that any multi-value $\Phi(x,y)$ with a single valued gradient $\partial_\mu \Phi$ has form

$$\Phi(x,y) \;=\; \alpha\times x \;+\; \beta\times y \;+\; \text{single-valued } \Lambda(x,y) \tag{12}$$

for some constant coefficients α and β related to the integrals (10) and (11). Consequently, any closed 1-form A can be written as

$$A \;=\; \alpha A_1 \;+\; \beta A_2 \;+\; \text{exact } d\Lambda. \tag{13}$$

Which means that the $\mathcal{H}_1$ cohomology of the torus is 2-dimensional, with the $A_1 = dx$ and the $A_y = dy$ serving as its basis.

Differential Forms for Gauge Fields

Let's start with the abelian gauge fields. The potentials $A_\mu(x)$ form a 1-form $A = A_\mu dx^\mu$ subject to gauge transforms $A' = A - d\Lambda$ for a 0-form Λ. The tension fields $F_{\mu\nu}(x)$ form a 2-form

$$F \;=\; F_{\mu\nu} dx^\mu \wedge dx^\nu \;=\; dA, \tag{14}$$

which is gauge invariant because $dd\Lambda = 0$ by nilpotency of the exterior derivative d.

The covariant exterior derivative of a p-form Ψ of electric charge q is

$$D\Psi \;=\; d\Psi \;+\; iqA \wedge \Psi. \tag{15}$$

Unlike the ordinary exterior derivative d, the covariant exterior derivative D is not nilpotent. Instead,

$$DD\Psi \;=\; iqF \wedge \Psi. \tag{16}$$

Next, consider the non-abelian gauge fields. This time, the potentials $A^a_\mu(x)$ form a matrix-valued 1-form $\mathcal{A} = gA^a_\mu t^a dx^\mu$. Likewise, the non-abelian tension fields form a matrix-valued 2-form

$$\mathcal{F} \;=\; gF^a_{\mu\nu} t^a \, dx^\mu \wedge dx^\nu \;=\; d\mathcal{A} \;+\; i\mathcal{A} \wedge \mathcal{A}. \tag{17}$$

Note the antisymmetry of the wedge product $\mathcal{A} \wedge \mathcal{A}$: in components, it turns the product of $\mathcal{A}_\mu$ and $\mathcal{A}_\nu$ into the matrix commutator $[\mathcal{A}_\mu, \mathcal{A}_\nu]$. The covariant derivative of a fundamental multiplet $\Phi^i(x)$ of scalar fields becomes in matrix notations $D_\mu\Phi = \partial_\mu\Phi + i\mathcal{A}_\mu\Phi$ where Φ is a column vector. Together, these derivatives comprise a fundamental multiplet of 1-forms $D\Phi = d\Phi + i\mathcal{A}\Phi$. Likewise, for a fundamental multiplet of p-forms $\Phi^i_{(p)}$, the covariant exterior derivative D yields a $(p+1)$-form

$$D\Phi_{(p)} \;=\; d\Phi_{(p)} \;+\; i\mathcal{A} \wedge \Phi_{(p)}\,. \tag{18}$$

For an adjoint multiplet of scalars $\Xi(x) = \Xi^a(x)t^a$ or p-forms $\Xi_{(p)} = \Xi^a_{(p)} t^a$, the gauge field in the covariant derivative acts by matrix commutation rather than matrix multiplication. Thus, in the matrix-valued differential form language,

$$D\Xi_{(p)} \;=\; d\Xi_{(p)} \;+\; i[\mathcal{A},\Xi_{(p)}] \;=\; d\Xi_{(p)} \;+\; i\mathcal{A}\wedge\Xi_{(p)} \;-\; i(-1)^p\Xi_{(p)}\wedge\mathcal{A}. \tag{19}$$

In particular, the tension fields $\mathcal{F}^a_{\mu\nu}$ themselves form an adjoint multiplet of 2-forms $\mathcal{F}$, hence

$$D\mathcal{F} \;=\; d\mathcal{F} \;+\; i\mathcal{A}\mathcal{F} \;-\; i\mathcal{F}\mathcal{A}. \tag{20}$$

However, back in homework set#6 we saw that the non-abelian tension fields obey the homogeneous Yang–Mills equation

$$D_\lambda\mathcal{F}_{\mu\nu} \;+\; D_\mu\mathcal{F}_{\nu\lambda} \;+\; D_\nu\mathcal{F}_{\lambda\mu} \;=\; 0, \tag{21}$$

which in the differential form language becomes

$$D\mathcal{F} \;=\; 0. \tag{22}$$

Similar to the abelian case, the covariant exterior derivative is not nilpotent but squares to the tension 2-form, $DD = i\mathcal{F}$. That is, for a fundamental multiplet of p forms

$$DD\Phi_{(p)} \;=\; i\mathcal{F}\wedge\Phi_{(p)}\,, \tag{23}$$

while for an adjoint multiplet of p forms

$$DD\Xi_{(p)} \;=\; i[\mathcal{F},\Xi_{(p)}] \;=\; i\mathcal{F}\wedge\Xi_{(p)} \;-\; i\Xi_{(p)}\wedge\mathcal{F}. \tag{24}$$

Also, the covariant exterior derivative obeys Leibniz rules. For example, for an adjoint multiplet $B_{(p)}$ of p-forms and another adjoint multiplet $C_{(q)}$ of q forms,

$$D\big(B_{(p)}\wedge C_{(q)}\big) \;=\; (DB_{(p)})\wedge C_{(q)} \;+\; (-1)^p B_{(p)}\wedge (DC_{(q)}). \tag{25}$$

Moreover, since traces of commutators vanish, we have

$$\mathrm{tr}\big(D(B\wedge C)\big) \;=\; \mathrm{tr}\big(d(B\wedge C) + i[\mathcal{A},(B\wedge C)]\big) \;=\; d\,\mathrm{tr}(B\wedge C) \;+\; 0, \tag{26}$$

and therefore

$$d\,\mathrm{tr}(B\wedge C) \;=\; \mathrm{tr}\big(D(B\wedge C)\big) \;=\; \mathrm{tr}\big((DB)\wedge C\big) \;+\; (-1)^p\,\mathrm{tr}\big(B\wedge(DC)\big). \tag{27}$$

We shall find this Leibniz rule quite useful later in these notes.

Anomaly forms

In class, we have calculated the axial anomaly of QED or QCD with massless fermions as

$$\partial_\mu J^\mu_A \;=\; -\frac{N_f}{16\pi^2}\,\mathrm{tr}\big(\epsilon^{\alpha\beta\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu}\big). \tag{28}$$

But axial symmetries of massless Dirac fermions exist in all even spacetime dimensions $d=2n=2,4,6,\ldots$, and in all such dimensions they suffer from anomalies due to gauge fields that couple to the fermions. Specifically,

$$\partial_\mu J^\mu_A \;=\; -\frac{2N_f}{n!}\,\frac{(-1)^n}{(4\pi)^n}\,\mathrm{tr}\big(\epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\mathcal{F}_{\alpha_1\beta_1}\mathcal{F}_{\alpha_2\beta_2}\cdots\mathcal{F}_{\alpha_n\beta_n}\big), \tag{29}$$

as you should have derived in problem 2 of homework set#24a.

Now let's rewrite these formulae in the language of differential forms. In d spacetime dimensions, a current J^μ is dual to the $(d-1)$-form $*J$, which can be integrated over a hypersurface (of dimension $d-1$) to yield the net charge or charge variation. For a conserved current, $\partial_\mu J^\mu(x) = 0$ translates to $d*J = 0$. But for an anomalous axial current J_A^μ, eq. (29) translates to the differential form language as

$$d*J_A \;=\; -\frac{2N_f}{n!}\,\frac{(-1)^n}{(4\pi)^n}\,Q_{(2n)} \tag{30}$$

$$\text{where } Q_{(2n)} \;\stackrel{\text{def}}{=}\; \operatorname{tr}\Bigl(\mathcal{F}\wedge\mathcal{F}\wedge\cdots\wedge\mathcal{F}\Bigr), \qquad \text{product of } n \text{ tension 2-forms.} \tag{31}$$

Note: while the anomaly equations (30) apply only for $d = 2n$, the anomaly forms $Q_{(2n)}$ themselves can be constructed in spaces of any dimension $d \geq 2n$, and they turn out to be quite useful in many contexts besides the anomaly eqs. (30). In particular, they are quite useful in string theory.

A very important property of all the anomaly forms $Q_{(2n)}$ is that all of them are closed,

$$dQ_{(2n)} \;=\; 0. \tag{32}$$

Indeed, by the Leibniz rule (27),

$$\begin{aligned} d\operatorname{tr}\bigl(\mathcal{F}\wedge\mathcal{F}\wedge\cdots\wedge\mathcal{F}\bigr) \;&=\; \operatorname{tr}\bigl((D\mathcal{F})\wedge\mathcal{F}\wedge\cdots\wedge\mathcal{F}\bigr) \;+\; \operatorname{tr}\bigl(\mathcal{F}\wedge(D\mathcal{F})\wedge\cdots\wedge\mathcal{F}\bigr)\\ &\qquad +\;\cdots\;+\;\operatorname{tr}\bigl(\mathcal{F}\wedge\cdots\wedge\mathcal{F}\wedge(D\mathcal{F})\bigr)\\ &=\; 0 \quad \text{because } D\mathcal{F} \;=\; 0. \end{aligned} \tag{33}$$

By the Poincare Lemma, it follows that in flat Minkowski or Euclidean spacetimes — which are topologically trivial — all these closed forms should be exact. That is, for any even $2n = 2,4,6,\ldots$, $Q_{(2n)} = d\Omega_{(2n-1)}$ for some $(2n-1)$-form $\Omega_{(2n-1)}$. Moreover, we may construct such forms — called the *Chern–Simons forms* — as traces of polynomials of $\mathcal{F}$ and $\mathcal{A}$ forms. For example:

$$\Omega_{(1)} \;=\; \operatorname{tr}\bigl(\mathcal{A}\bigr) \quad [\text{only for the abelian fields}], \tag{34}$$

$$d\Omega_{(1)} \;=\; \operatorname{tr}\bigl(\mathcal{F}\bigr) \;=\; Q_{(2)}\,, \tag{35}$$

$$\Omega_{(3)} \;=\; \operatorname{tr}\bigl(\mathcal{A}\wedge\mathcal{F} \,-\, \tfrac{i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\bigr), \tag{36}$$

$$d\Omega_{(3)} \;=\; \operatorname{tr}\bigl(\mathcal{F}\wedge\mathcal{F}\bigr) \;=\; Q_{(4)}\,, \tag{37}$$

$$\Omega_{(5)} = \mathrm{tr}\big(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F} - \tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F} - \tfrac{1}{10}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big), \tag{38}$$

$$d\Omega_{(5)} = \mathrm{tr}\big(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}\big) = Q_{(6)}\,, \tag{39}$$

etc., etc.

Verifying these formulae is a part of your homework set#24a (problem 3(a)).

Although the Chern–Simons forms $\Omega_{(2n-1)}$ are constructed from the anomaly forms $Q_{(2n)}$, they can be quite interesting in their own rights. They are particularly useful for gauge theories in odd spacetime dimensions $d = 2n - 1$. For example, in $d = 3$ dimensions, adding the Chern–Simons 3-form to the Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \frac{k}{8\pi}\,\Omega_{(3)}\,, \tag{40}$$

makes the gauge bosons massive without breaking the gauge invariance of the theory, *cf.* Fall 2024 midterm exam (problem 2). Similarly, in $d = 5$ dimensions one may add the Chern–Simons 5-form $\Omega_{(5)}$ to the Lagrangian, which gives rise to interesting parity-violating interactions between the gauge bosons. This is particularly important for the supersymmetric 5D gauge theories, where supersymmetry relates the $\Omega_{(5)}$ to the scalar-dependence of the gauge couplings.

There are also many uses of Chern–Simons form in string theory, in both odd and even spacetime dimensions. But that subject is beyond the scope of our QFT class.

Now consider the behavior of Chern–Simons forms under gauge transformations,

$$\mathcal{A}' = i(dU)U^{-1} + U\mathcal{A}U^{-1}, \qquad \mathcal{F}' = U\mathcal{F}U^{-1}, \tag{41}$$

for some x-dependent unitary matrix $U(x)$. (Or more general, an x-dependent element $U(x)$ of the gauge group G.) By the cyclic symmetry of the trace, all the anomaly forms $Q_{(2n)}$ are gauge invariant:

$$Q'_{(2n)} = \mathrm{tr}\big(\mathcal{F}'\wedge\cdots\wedge\mathcal{F}'\big) = \mathrm{tr}\big(U\mathcal{F}U^{-1}\wedge\cdots\wedge U\mathcal{F}U^{-1}\big) = \mathrm{tr}\big(\mathcal{F}\wedge\cdots\wedge\mathcal{F}\big) = Q_{(2n)}\,. \tag{42}$$

On the other hand, the Chern–Simons forms $\Omega_{(2n-1)}$ are not gauge invariant. However, since their exterior derivatives $d\Omega_{(2n-1)} = Q_{(2n)}$ happen to be gauge invariant, the gauge variations of the Chern–Simons forms must be closed,

$$d\big(\Omega'_{(2n-1)} - \Omega_{(2n-1)}\big) \;=\; 0. \tag{43}$$

Consequently, by the Poincare Lemma, in flat Minkowski or Euclidean spaces, the gauge variations of the Chern–Simons forms must be exact,

$$\Omega'_{(2n-1)} \;-\; \Omega_{(2n-1)} \;=\; -dH_{(2n-2)} \tag{44}$$

for some $(2n-2)$ form. In particular, for infinitesimal gauge transforms parametrized by a matrix-valued zero form $\Lambda = \Lambda^a(x)t^a$ — thus to the first order in Λ

$$U(x) \;=\; 1 \,+\, i\Lambda(x), \qquad \delta\mathcal{A} \;=\; -D\Lambda \;=\; -d\Lambda \,-\, i\mathcal{A}\Lambda \,+\, i\Lambda\mathcal{A}, \qquad \delta\mathcal{F} \;=\; -i\mathcal{F}\Lambda \,+\, i\Lambda\mathcal{F}, \tag{45}$$

— one may construct the $H_{(2n-2)}$ forms in eqs. (44) as

$$H_{(2n-2)} \;=\; \mathrm{tr}\big(\Lambda \times dK_{(2n-3)}\big) \tag{46}$$

for some $(2n-3)$-forms $K_{(2n-3)}$ that are polynomials in $\mathcal{A}$ and $\mathcal{F}$. (Or, equivalently, polynomials in $\mathcal{A}$ and $d\mathcal{A}$.) In particular,

$$H_{(0)} \;=\; \mathrm{tr}(\Lambda) \quad \text{[for the abelian theories only]}, \tag{47}$$

$$\delta_\Lambda\Omega_{(1)} \;=\; -dH_{(0)}\,, \tag{48}$$

$$H_{(2)} \;=\; \mathrm{tr}\big(\Lambda \times d\mathcal{A}\big), \tag{49}$$

$$\delta_\Lambda\Omega_{(3)} \;=\; -dH_{(2)}\,, \tag{50}$$

$$H_{(4)} \;=\; \mathrm{tr}\big(\Lambda \times d(\mathcal{A}\wedge d\mathcal{A} \,+\, \tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\big), \tag{51}$$

$$\delta_\Lambda\Omega_{(5)} \;=\; -dH_{(4)}\,, \tag{52}$$

etc., etc.

Again, verifying these formulae is a part of your homework set#24a (problem 3(b)).

The way we have constructed the $H_{(2n-2)}$ forms is known as the *anomaly descent:* Start with the $Q_{(2n)} = \mathrm{tr}(\mathcal{F}\wedge\cdots\wedge\mathcal{F})$ forms (known to mathematicians as *indices*), then construct the Chern–Simons forms so that $d\Omega_{(2n-1)} = Q_{(2n)}$, and then construct the H forms such that $\delta_\Lambda\Omega_{(2n-1)} = -dH_{(2n-2)}$. However, *once constructed, the $H_{(2n-2)}$ forms may be used in any dimension $d \geq 2n-2$,* including $d = 2n-2$ for which the original $Q_{(2n)}$ form does not exist. And as we shall see momentarily, the $H_{(2n-2)}$ form in $d = 2n-2$ dimensions governs the non-abelian gauge anomalies of chiral gauge theories: Under an infinitesimal gauge transform, the effective action for the gauge fields varies by

$$\delta S[A^a_\mu(x)] \;=\; (\mathrm{coeff}) \times \int\limits_{\substack{\text{whole}\\ \text{spacetime}}} H_{(2n-2=d)}\,. \tag{53}$$

Non Abelian Anomaly in 4 Dimensions

Let's start with the non-abelian anomaly in 4 dimensions, as they were already explained in class, *cf.* my notes on anomalies, pages 50–58. Let the theory in question have a general gauge group G and some chiral fermions, namely LH Weyl fermions in some generic multiplet (m_L) of G, and RH fermions in some different multiplet (m_R) of G. Both (m_L) and (m_R) multiplets may be reducible — *i.e.*, comprise several irreducible multiplets, — but we do not need their details for the general discussion here. All we need are the chiral traces

$$\mathcal{Z}^{abc} \;\stackrel{\text{def}}{=}\; \mathrm{tr}_{(m_L)}\bigl(t^a\{t^b,t^c\}\bigr) \;-\; \mathrm{tr}_{(m_R)}\bigl(t^a\{t^b,t^c\}\bigr) \tag{54}$$

for all sets of 3 gauge group generators t^a, t^b, t^b. (These traces were called $\mathcal{A}^{abc}$ in my notes on the gauge anomalies.) By the cyclic symmetry of the traces (54), the coefficients $\mathcal{Z}^{abc}$ are totally symmetric WRT permutations of a, b, c, and for any simple gauge group

$$\mathcal{Z}^{abc} \;=\; d^{abc} \times \bigl(\mathcal{Z}^{\text{net}} \;=\; R_3(m_L) - R_3(m_R)\bigr) \tag{55}$$

where $R_3(m)$ is the cubic index of the multiplet (m). Thus, if the gauge theory is chiral, $(m_L) \neq (m_R)$, but the multiplets of LH and RH fermions have the same cubic index, $R_3(m_L) = R_3(m_R)$, then the gauge anomaly happens to cancel and the functional integral over all the chiral fermions of the theory happens to be gauge invariant.

But suppose the anomalies do not cancel, $\mathcal{Z}^{\rm net}\neq 0$, or more generally $\mathcal{Z}^{abc}\neq 0$ for some combinations of the adjoint indices a,b,c. In this case, we have learned earlier in class that the effective action for the gauge fields stemming from the fermionic functional integrals is **not** gauge invariant. Instead, under the infinitesimal gauge transforms $\Lambda^a(x)$, the effective Euclidean action varies by

$$\Delta S_e[\Lambda] \;=\; \frac{g^2\mathcal{Z}^{abc}}{16\pi^2}\int d^4x_e\,\Lambda^a\times\epsilon^{\alpha\lambda\mu\nu}\partial_\alpha\left(A^b_\lambda\left(\partial_\mu A^c_\nu\,-\,\frac{g}{4}f^{cde}A^d_\mu A^e_\nu\right)\right). \tag{56}$$

Consequently, the gauge symmetry currents $J^{\mu,a}$ instead of being covariantly conserved, $D_\mu J^{\mu,a}=0$, obey anomalous equations

$$D_\mu J^{\mu a} \;=\; -\frac{g^2\mathcal{Z}^{abc}}{16\pi^2}\,\epsilon^{\alpha\lambda\mu\nu}\partial_\alpha\left(A^b_\lambda\left(\partial_\mu A^c_\nu\,-\,\frac{g}{4}f^{cde}A^d_\mu A^e_\nu\right)\right) \tag{57}$$

inconsistent with the Yang–Mills equations $D_\mu F^{\mu\nu,a}=J^{\mu,a}$.

In the differential form language, eq. (57) becomes

$$\begin{aligned} D*J^a \;&=\; -\frac{g^2\mathcal{Z}^{abe}}{32\pi^2}\,d\left(A^b\wedge\left(dA^e\,-\,\frac{g}{2}f^{ecd}A^c\wedge A^d\right)\right)\\ &=\; -\frac{\mathcal{Z}^{abe}}{32\pi^2}\,d\left(\mathcal{A}^b\wedge\left(d\mathcal{A}\,+\,\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\right)^e\right), \end{aligned} \tag{58}$$

while eq. (56) becomes

$$\Delta S_e[\Lambda] \;=\; -\frac{\mathcal{Z}^{abe}}{32\pi^2}\int\Lambda^a\times d\left(\mathcal{A}^b\wedge\left(d\mathcal{A}\,+\,\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\right)^e\right). \tag{59}$$

where the 4-form on the RHS is integrated over the whole Euclidean space. Furthermore, since the anomaly coefficients $\mathcal{Z}^{abe}$ obtain as generalized traces

$$\mathcal{Z}^{abe} \;=\; \mathrm{tr}_\chi\big(t^a\{t^b,t^e\}\big)\;\stackrel{\rm def}{=}\;\mathrm{tr}_{(m_L)}\big(t^a\{t^b,t^e\}\big)\;-\;\mathrm{tr}_{(m_R)}\big(t^a\{t^b,t^e\}\big), \tag{60}$$

cf. eq. (54), we may rewrite eq. (59) as

$$\Delta S_e[\Lambda] \;=\; -\frac{1}{32\pi^2}\int\mathrm{tr}_\chi\Big(\Lambda\times d\{\mathcal{A},(d\mathcal{A}\,+\,\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A})\}\Big). \tag{61}$$

Actually, the anticommutator here is unnecessary since the two factors in it commute with each other up to a total derivative,

$$\big[\mathcal{A},\big(d\mathcal{A}+\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\big)\big]\ =\ \mathcal{A}\wedge d\mathcal{A}\ -\ d\mathcal{A}\wedge\mathcal{A}\ =\ d(-\mathcal{A}\wedge\mathcal{A}),\tag{62}$$

hence

$$d\big[\mathcal{A},\big(d\mathcal{A}+\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\big)\big]\ =\ 0$$

and therefore

$$d\{\mathcal{A},(\cdots)\}\ =\ 2d\big(\mathcal{A}\wedge(\cdots)\big).\tag{63}$$

Thus,

$$\Delta S_e[\Lambda]\ =\ -\frac{1}{16\pi^2}\int\mathrm{tr}_\chi\Big(\Lambda\times d\big(\mathcal{A}\wedge(d\mathcal{A}+\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A})\big)\Big).\tag{64}$$

By inspection, the differential form under the integral here looks just like the $H_{(4)}$ form

$$H_{(4)}\ =\ \mathrm{tr}\Big(\Lambda\times d\big(\mathcal{A}\wedge(dA+\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A})\big)\Big)\tag{51}$$

which obtains by anomaly descent from the 6-form

$$Q_{(6)}\ =\ \mathrm{tr}\big(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}\big).\tag{65}$$

Indeed, the only differences between the 4-forms in eqs. (64) and (51) is the generalized trace over all fermionic species in eq. (64) *versus* the abstract trace over gauge indices only in eq. (51).

To repair this difference, let's rework the anomaly descent procedure. Take a most general gauge theory in 4D with some gauge group G — which may be simple or a product of several factors. Let's treat all fermions of the theory as Weyl fermions, left-handed or right-handed. The LH Weyl fermions form some kind of multiplet (m_L) of G; this multiplet may be irreducible, or it may have several irreducible components, we do not care as long as it is a complete multiplet of G. Likewise, the RH Weyl fermions form another multiplet (m_R) of G; again, we do not care if it's reducible or irreducible as long as it's complete. If both (m_L) and (m_R) multiplets are equivalent, the gauge theory in question is non-chiral so it's automatically free from any gauge anomaly. So let us focus on the chiral theories with $(m_L) \not\cong (m_R)$.

Despite the theory in question being 4-dimensional, let's formally define a 6-forms — presumably living in 6 or more dimensions —

$$\widehat{Q}_{(6)} \;=\; \mathrm{tr}_{(m_L)}\big(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}\big) \;-\; \mathrm{tr}_{(m_R)}\big(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}\big) \;=\; \mathrm{tr}_\chi\big(\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}\big). \tag{66}$$

In terms of the $\mathcal{F}^a$ components of the adjoint multiplet of tension fields,

$$\widehat{Q}_{(6)} \;=\; \mathcal{F}^a\wedge\mathcal{F}^b\wedge\mathcal{F}^c\times\mathrm{tr}_\chi(t^at^bt^c). \tag{67}$$

Moreover, the wedge product of component 2-forms is totally symmetric,

$$\mathcal{F}^a\wedge\mathcal{F}^b\wedge\mathcal{F}^c \;=\; \mathcal{F}^c\wedge\mathcal{F}^b\wedge\mathcal{F}^a \;=\; \cdots, \tag{68}$$

we we may just as well symmetrize their adjoint indices, thus

$$\begin{aligned}\widehat{Q}_{(6)} \;&=\; \mathcal{F}^a\wedge\mathcal{F}^b\wedge\mathcal{F}^c\times\tfrac{1}{6}\,\mathrm{tr}_\chi\big(t^at^bt^c \;+\; (5\text{ other permutations of }t^a,t^b,t^c)\big)\\ &=\; \mathcal{F}^a\wedge\mathcal{F}^b\wedge\mathcal{F}^c\times\tfrac{1}{2}\,\mathrm{tr}\big(t^at^bt^c \;+\; t^at^ct^b\big)\\ &=\; \mathcal{F}^a\wedge\mathcal{F}^b\wedge\mathcal{F}^c\times\tfrac{1}{2}\mathcal{Z}^{abc}\end{aligned} \tag{69}$$

for exactly the same anomaly coefficients $\mathcal{Z}^{abc}$ as in eq. (60). Thus, if all these coefficients happen to vanish — and hence the theory in question is anomaly free — then $\widehat{Q}_{(6)} = 0$, and we may stop at this point.

But if some of the $\mathcal{Z}^{abc}$ do not vanish, then the gauge theory is anomalous, and we may work out the details of its anomaly by descent from the formal 6-form $\widehat{Q}_{(6)}$. That is, given the $\widehat{Q}_{(6)}$, we work out the corresponding Chern–Simons 5-form

$$\widehat{\Omega}_{(5)} \;=\; \mathrm{tr}_\chi\Big(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F} \;-\; \tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F} \;-\; \tfrac{1}{10}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big), \tag{70}$$

$$d\widehat{\Omega}_{(5)} \;=\; \widehat{Q}_{(6)}\,. \tag{71}$$

Similar to the 6-form $\widehat{Q}_{(6)}$, the Chern–Simons 5-form $\widehat{\Omega}_{(5)}$ does not 'fit' into 4 spacetime dimensions of the gauge theory, but we may treat both of them as formal constructions in higher space dimensions.

Going from the $\widehat{Q}_{(6)}$ 'down' to the $\widehat{\Omega}_{(5)}$ is the first stage of the anomaly descent procedure. The second stage is going from the Chern–Simons form $\widehat{\Omega}_{(5)}$ 'down' to the 4-form $\widehat{H}_{(4)}$ such that under infinitesimal gauge variations

$$\delta\widehat{\Omega}_{(5)} \;=\; -d\widehat{H}_{(4)}\,. \tag{72}$$

As we have already seen, this calls for

$$\begin{aligned}\widehat{H} \;&=\; \mathrm{tr}_\chi\Big(\Lambda\times d\big(\mathcal{A}\wedge(d\mathcal{A}\,+\,\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A})\big)\Big)\\ &=\; \tfrac{1}{2}\mathcal{Z}^{abc}\times\Lambda^a\times d\Big(\mathcal{A}^b\wedge\big(d\mathcal{A}\,+\,\tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\big)^c\Big).\end{aligned} \tag{73}$$

Unlike the higher-dimensional forms $\widehat{Q}_{(6)}$ and $\widehat{\Omega}_{(5)}$ we have used in the above anomaly descent procedure, the 4-form (73) is well defined in the 4 dimensions of the gauge theory. And it's the whole-spacetime integral of this 4-form that governs the anomalous variation of the effective action for the gauge fields,

$$\Delta S_e[\Lambda] \;=\; -\frac{1}{16\pi^2}\int\limits_{\mathbf{R}^4} H_{(4)}\,. \tag{74}$$

Gauge Anomalies in Other Even Dimensions

Weyl fermions — and hence chiral gauge theories — exist in all *even* spacetime dimensions. So let's adapt the anomaly descent procedure from the previous sections to other spacetime dimensions.

Take a chiral gauge theory in d dimensions with gauge group G, LH Weyl fermions in some multiplet (m_L) and RH Weyl fermions in another multiplet (m_R). Let

$$n \;=\; \frac{d}{2}\;+\;1, \tag{75}$$

and let's formally define a $(2n)$ form (living in $d+2$ or more dimensions)

$$\widehat{Q}_{(2n)} \;=\; \mathrm{tr}_{(m_L)}\big(\mathcal{F}\wedge\cdots\wedge\mathcal{F}\big) \;-\; \mathrm{tr}_{(m_R)}\big(\mathcal{F}\wedge\cdots\wedge\mathcal{F}\big) \tag{76}$$

where $\mathcal{F}\wedge\cdots\wedge\mathcal{F}$ has n matrix-valued factors of $\mathcal{F}=\mathcal{F}^at^a$. In terms of the component

2-forms $\mathcal{F}^a$,

$$\widehat{Q}_{(2n)} = \mathcal{F}^{a_1} \wedge \mathcal{F}^{a_2} \wedge \cdots \wedge \mathcal{F}^{a_n} \times \Bigl(\mathrm{tr}_{(m_L)} \bigl(t^{a_1} t^{a_2} \cdots t^{a_n}\bigr) - \mathrm{tr}_{(m_R)} \bigl(t^{a_1} t^{a_2} \cdots t^{a_n}\bigr) \Bigr), \quad (77)$$

and since the wedge product of 2 forms is totally symmetric, we may just as well totally symmetrize the adjoint indices $a_1, a_2, \ldots a_n$. Thus,

$$\widehat{Q}_{(2n)} = \frac{\mathcal{Z}^{a_1,a_2,\ldots,a_n}}{(n-1)!} \times \mathcal{F}^{a_1} \wedge \mathcal{F}^{a_2} \wedge \cdots \wedge \mathcal{F}^{a_n} \quad (78)$$

for

$$\begin{aligned} \mathcal{Z}^{a_1,a_2,\ldots,a_n} = {} & \mathrm{tr}_{(m_L)} \Bigl(t^{a_1} t^{a_2} \cdots t^{a_n} + \text{non-cyclic permutations} \Bigr) \\ & - \mathrm{tr}_{(m_R)} \Bigl(t^{a_1} t^{a_2} \cdots t^{a_n} + \text{non-cyclic permutations} \Bigr). \end{aligned} \quad (79)$$

Similar to 4D, if all the coefficients $\mathcal{Z}^{a_1,a_2,\ldots,a_n}$ happen to vanish, then the chiral gauge theory in question is anomaly free. Otherwise, the theory is anomalous and is inconsistent as a quantum theory.

For the anomalous theories, we may work out the gauge variance of the effective action by the two-step descent procedure: First, start with the formal $\widehat{Q}_{(2n)}$ form and construct the Chern–Simons $(2n-1)$ form $\widehat{\Omega}_{(2n-1)}$ such that

$$d\widehat{\Omega}_{(2n-1)} = \widehat{Q}_{(2n)} \,. \quad (80)$$

Second, consider infinitesimal gauge variance of this Chern–Simons form and construct a $(2n-2)$ form $\widehat{H}_{(2n-2)}$ such that

$$\delta_\Lambda \widehat{\Omega}_{(2n-1)} = -d\widehat{H}_{(2n-2)} \,. \quad (81)$$

This form has rank $2n - 2 = d$ so it may be integrated over a d-cycle such as the whole Euclidean space $\mathbf{R}^d$, and the integral gives the gauge variance of the effective action,

$$\Delta S_e[\Lambda] = (\text{coefficient}) \int\limits_{\mathbf{R}^d} H_{(2n-2)} \,. \quad (82)$$

Physically, we are mostly interested in the anomaly free theories, so let's take a closer look at the anomaly coefficients (79), or in short-hand notations

$$\mathcal{Z}^{a,b,\ldots,z} \;=\; \mathrm{tr}_{(m_L)-(m_R)}\Big(t^a t^b \cdots t^z \;+\; \text{permutations}\Big). \tag{83}$$

In $d = 4$ dimensions, $n = 3$ and the traces here are cubic polynomials of the gauge group generators. But in other dimensions, we get polynomials of other degrees — quadratic for $d = 2$, quartic for $d = 6$, degree $n = 6$ for $d = 10$, *etc.*, — so we end up with very different rules for the anomaly cancellation.

To see how this works, consider a simple gauge group G. In $d = 4$ we saw that this leads to

$$\mathrm{tr}_{\text{any }(m)}\big(t^a t^b t^c \;+\; t^a t^c t^b\big) \;=\; R_3(m) \times d^{abc} \tag{84}$$

for the same d^{abc} for all representations (m), with the only (m) dependence being the overall factor $R_3(m)$, the cubic index of the representation. Consequently, in 4 dimensions

$$\mathcal{Z}^{a,b,c} \;=\; d^{abc} \times \mathcal{Z}^{\text{net}} \tag{85}$$

for

$$\mathcal{Z}^{\text{net}} \;=\; R_3^{\text{net}}(m_L) \;-\; R_3^{\text{net}}(m_R) \;=\; \sum_{\text{LH Weyl multiplets}} R_3 \;-\; \sum_{\text{RH Weyl multiplets}} R_3\,. \tag{86}$$

Since the cubic index R_3 is odd WRT complex conjugation, it vanishes for any real or pseudoreal multiplet, so Weyl fermions in such multiplets of G do not contribute to the anomaly. Also, for the complex multiplets $R_3(\overline{m}) = -R_3(m)$, so from the anomaly point of view, a multiplet (m) of LH fermions is equivalent to the conjugate $(\overline{m})$ multiplet of RH fermions and vice verse. This is related to the fact that in 4 dimensions, the Hermitian conjugate $\psi_L^\dagger$ of a LH Weyl spinor is equivalent to a RH Weyl spinor ψ_R, and vice verse $\psi_R^\dagger \cong \psi_L$.

Finally, many kinds of gauge groups do not have cubic Casimirs and hence cubic invariants d^{abc}. For such groups, in 4D we have automagic $\mathcal{Z}^{abc} = 0$, and there is no gauge anomaly at all.

Similar rules work in other dimensions d divisible by 4, hence odd $n = (d/2) + 1$. But they do not work in dimensions $d \equiv 2 \pmod 4$ for which n is even. For example, in 2 dimensions $n = 2$, hence

$$\mathrm{tr}_{(m)}(t^a t^b) \;=\; R_2(m) \times \delta^{ab} \tag{87}$$

where $R_2(m)$ is the quadratic rather than cubic index of the multiplet (m). Consequently,

$$\mathcal{Z}^{ab} \;=\; \mathcal{Z}^{\mathrm{net}} \times \delta^{ab} \tag{88}$$

for

$$\mathcal{Z}^{\mathrm{net}} \;=\; R_2^{\mathrm{net}}(m_L) \;-\; R_2^{\mathrm{net}}(m_R) \;=\; \sum_{\text{LH Weyl multiplets}} R_2 \;-\; \sum_{\text{RH Weyl multiplets}} R_2\,. \tag{89}$$

But the quadratic index is even WRT the charge conjugation, so it does not vanish for real or pseudoreal multiplets. Thus all non-trivial fermion multiplets — complex, real, or pseudoreal — do contribute to the gauge anomaly in 2D. Also, for complex multiplets $R_2(\overline{m}) = +R_2(m)$, so we may not trade a multiplet (m) of LH Weyl fermions for a conjugate multiplet $(\overline{m})$ of RH Weyl fermions or vice verse. This is related to the fact that in 2D — or in any dimension $d \equiv 2 \pmod 4$ — the Hermitian conjugate $\psi_L^\dagger$ of the LH weyl spinor is equivalent to a LH Weyl spinor ψ_L rather than a RH Weyl spinor ψ_R, and likewise $\psi_R^\dagger \cong \psi_R$ rather than ψ_L.

Also, while many groups do not have cubic Casimirs, all groups have quadratic Casimirs and hence quadratic invariants δ^{ab}. Consequently, there are no automagically anomaly-free gauge groups in 2D.

In 6 dimensions, the situation is similar, and even more complicated. For $n = 4$, many groups have 2 independent quartic Casimirs, C_2^2 and a separate C_4. Consequently, there are 2 independent symmetric quartic invariants d_1^{abcd} and d_2^{abcd} while

$$\mathrm{tr}_{(m)}\big(t^a t^b t^c t^d \;+\; \text{permutations}\big) \;=\; R_{4,1}(m) \times d_1^{abcd} \;+\; R_{4,2}(m) \times d_2^{abcd} \tag{90}$$

for two independent quartic indices of multiplets, $R_{4,1}(m)$ and $R_{4,2}(m)$. Therefore, in 6D

$$\mathcal{Z}^{abcd} \;=\; \mathcal{Z}_1 \times d_1^{abcd} \;+\; \mathcal{Z}_2^{abcd} \tag{91}$$

for

$$
\begin{aligned}
\mathcal{Z}_1 &= R^{\text{net}}_{4,1}(m_L) - R^{\text{net}}_{4,1}(m_R), \\
\mathcal{Z}_2 &= R^{\text{net}}_{4,2}(m_L) - R^{\text{net}}_{4,2}(m_R).
\end{aligned}
\tag{92}
$$

Thus, an anomaly free chiral gauge theory in 6D needs

$$
\text{both} \quad R^{\text{net}}_{4,1}(m_L) = R^{\text{net}}_{4,1}(m_R) \quad \text{and} \quad R^{\text{net}}_{4,2}(m_L) = R^{\text{net}}_{4,2}(m_R). \tag{93}
$$

And similar to 2D, all multiplets — real or complex — do contribute to these indices, and there is no trading between between conjugate multiplets of LH and RH fermions. Consequently, canceling all the anomalies of a chiral 6D gauge theory is a non-trivial exercise.

In 10 dimensions — which are particularly relevant to the superstring theory — because for $n = 6$ most groups have 3 independent degree 6 Casimirs, hence 3 symmetric invariants and 3 separate indices $R_{6,1}(m)$, $R_{(6,2}(m)$, $R_{6,3}(m)$ which must cancel between the LH and the RH Weyl fermions. This constraint on the Weyl fermion spectrum of the theory is very difficult to satisfy.

In particular, the 10D super–Yang–Mills theory has LH Weyl fermions in the adjoint multiplet of the gauge group and no RH Weyl fermions at all. For any gauge group, the adjoint multiplet has at least one non-zero degree-6 index, so the SYM theory by itself is always anomalous.

Fortunately, in 1984 Michael Green and John Schwarz discovered that some of the 10D anomalies can be canceled by adding to the theory a 2-form field $B_{\mu\nu}(x)$ with certain interactions, and then the remaining anomalies cancel for $G = SO(32)$ or $G = E_8 \times E_8$. Moreover, they saw that the 2-form field with the required interactions already exists in the superstring theory, and verified the anomaly cancellation by a string-theoretic calculation. That result — and the explosion of the follow-up papers — signalled the birth of the modern string theory in 1984.