BRST Symmetry of QCD

The BRST symmetry — named after its discoverers Carlo **B**ecchi, Alain **R**ouet, Raymond **S**tora, and Igor **T**yutin — relates the ghosts and the longitudinal gluons to each other and makes sure that they always cancel each other out from all physical processes.

Before we spell out the action of the BRST symmetry, let's write the QCD Lagrangian as

$$\mathcal{L} \;=\; \mathcal{L}_{\text{physical}} \;+\; \mathcal{L}^{\text{gauge fixing}}_{\text{and ghosts}}, \tag{1}$$

$$\mathcal{L}_{\text{physical}} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; \sum_f \overline{\Psi}_{fi}(i\not{D}+m_f)\Psi^{fi}, \tag{2}$$

$$\mathcal{L}^{\text{gauge fixing}}_{\text{and ghosts}} \;=\; \partial_\mu \bar{c}^c D_\mu c^a \;+\; \tfrac{1}{2}\xi(b^a)^2 \;-\; b^a\partial^\mu A^a_\mu\,. \tag{3}$$

The $b^a(x)$ here are auxiliary fields: For $\xi = 0$ they are Lagrange multipliers for the Landau gauge condition $\partial^\mu A^a_\mu = 0$, while for $\xi \neq 0$ we may eliminate the b^a by their non-derivative equations of motion $\xi b^a = \partial^\mu A^a_\mu$, which then brings the gauge-fixing terms to their standard form $-(1/2\xi)(\partial^\mu A^a_\mu)^2$. The $c^a(x)$ are the ghost fields and the $\bar{c}^a(x)$ are the anti-ghost fields; despite the names, their quanta are not antiparticles of each other, and there is no charge-conjugation-like symmetry exchanging $c^a \leftrightarrow \bar{c}^a$. In particular, the quantum $\hat{c}^a$ and $\hat{\bar{c}}^a$ fields are not hermitian conjugates of each other. However, the ghosts and the antighosts do have opposite charges under the global $U(1)_{\text{ghost}}$ symmetry

$$c^a(x) \;\to e^{+i\theta}\times c^a(x), \quad \bar{c}^a \;\to\; e^{-i\theta}\times\bar{c}^a(c), \qquad \text{other fields unchanged;} \tag{4}$$

the corresponding conserved charge $\mathcal{G}$ is called the *ghost number.*

In component-field notations, the BRST symmetry acts as

$$\begin{aligned}
\delta\Psi^{fi}(x)) &= \epsilon\{Q,\Psi^{fi}(x)\} &&= g\epsilon\, c^a(x)(t^a)^i{}_j\Psi^{fj}(x), && \text{(5.a)}\\
\delta\overline{\Psi}_{fi}(x)) &= \epsilon\{Q,\overline{\Psi}_{fi}(x)\} &&= g\epsilon\,\overline{\Psi}_{fj}(x)(t^a)^j{}_i c^a(c), && \text{(5.b)}\\
\delta A^a_\mu(x) &= \epsilon[Q,A^a_\mu(x)] &&= i\epsilon\, D_\mu c^a(x) \;=\; i\epsilon\,\partial_\mu c^a(x) \;-\; ig\epsilon f^{abc}A^b_\mu(x)c^c(x), && \text{(5.c)}\\
\delta c^a(x) &= \epsilon\{Q,c^a(x)\} &&= ig\epsilon\, f^{abc}\,c^b(x)c^c(x), && \text{(5.d)}\\
\delta\bar{c}^a(x) &= \epsilon\{Q,\bar{c}^a(x)\} &&= -i\epsilon\, b^a, && \text{(5.e)}\\
\delta b^a(x) &= \epsilon[Q,b^a(x)] &&= 0, && \text{(5.f)}
\end{aligned}$$

where ϵ is an 'infinitesimal' odd Grassmann number and Q is the fermionic operator generating the BRST symmetry. Under the $U(1)_{\text{ghost}}$, Q has ghost number $\mathcal{G}=+1$. Note that the ϵ parameter is x–independent, so the BRST symmetry is global rather than local. Also, despite having a fermionic generator Q, the BRST symmetry is completely unrelated to supersymmetry.

As written, the action of the BRST symmetry depends on the gauge coupling g, but we may eliminate this dependence by rescaling the gauge and ghost fields. At the same time, let us also switch to matrix notations for all the adjoint fields, thus

$$\mathcal{A}_\mu = g\sum_a A^a_\mu t^a, \qquad \mathcal{C} = g\sum_a c^a t^a, \qquad \overline{\mathcal{C}} = g\sum_a \bar{c}^a t^a, \qquad \mathcal{B} = g\sum_a b^a t^a, \tag{6}$$

Consequently,

$$\begin{aligned}
D_\mu\Psi(x) &= \partial_\mu\Psi(x) \;+\; i\mathcal{A}_\mu(x)\Psi(x),\\
D_\mu\mathcal{C}(c) &= \partial_\mu\mathcal{C}(x) \;+\; i[\mathcal{A}_\mu(x),\mathcal{C}(x)],\\
\mathcal{F}_{\mu,\nu}(x) &= \partial_\mu\mathcal{A}_\nu(x) \;-\; \partial_\nu\mathcal{A}_\mu(x) \;+\; i[\mathcal{A}_\mu(x),\mathcal{A}_\nu(x)],
\end{aligned} \tag{7}$$

the Lagrangian becomes

$$\mathcal{L}_{\text{phys}} = \frac{-1}{2g^2}\,\mathrm{tr}\,(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) \;+\; \sum_f \overline{\Psi}_f(i\not{D}+m)\Psi^f, \tag{8}$$

$$\mathcal{L}_{\text{gf+gh}} = \frac{2}{g^2}\,\mathrm{tr}\left(\partial_\mu\overline{\mathcal{C}}D^\mu\mathcal{C} \;+\; \frac{\xi}{2}\mathcal{B}^2 \;-\; \mathcal{B}\partial^\mu\mathcal{A}_\mu\right), \tag{9}$$

the infinitesimal gauge transforms $U(x) = 1 + i\Lambda(x)$ act as

$$\delta\Psi = i\Lambda\Psi, \quad \delta\overline{\Psi} = -i\overline{\Psi}\Lambda, \quad \delta\mathcal{A}_\mu = -D_\mu\Lambda, \quad \delta\mathcal{C} = i[\Lambda,\mathcal{C}], \quad \delta\overline{\mathcal{C}} = i[\Lambda,\overline{\mathcal{C}}], \quad \delta\mathcal{B} = i[\Lambda,\mathcal{B}], \tag{10}$$

and the BRST symmetry acts as

$$\begin{aligned}
\delta\Psi(x) &= \epsilon\big\{Q,\Psi(x)\big\} &&= \epsilon\,\mathcal{C}(x)\Psi(x), && (11.a)\\
\delta\overline{\Psi}(x) &= \epsilon\big\{Q,\overline{\Psi}(x)\big\} &&= \epsilon\,\overline{\Psi}(x)\mathcal{C}(x), && (11.b)\\
\delta\mathcal{A}_\mu(x) &= \epsilon\big[Q,\mathcal{A}_\mu(x)\big] &&= i\epsilon\,D_\mu\mathcal{C}(x), && (11.c)\\
\delta\mathcal{C}(x) &= \epsilon\big\{Q,\mathcal{C}(x)\big\} &&= \epsilon\,\mathcal{C}(x)\mathcal{C}(x), && (11.d)\\
\delta\overline{\mathcal{C}}(x) &= \epsilon\big\{Q,\overline{\mathcal{C}}(x)\big\} &&= -i\epsilon\,\mathcal{B}(x), && (11.e)\\
\delta\mathcal{B}(x) &= \epsilon\big[Q,\mathcal{B}(x)\big] &&= 0. && (11.f)
\end{aligned}$$

Note that thanks to the matrix structure of the fermionic $\mathcal{C}(x)$ field, its square on the RHS of eq. (11.d) does not vanish; instead, $\mathcal{C}(x)\mathcal{C}(x) = \frac{1}{2}\{\mathcal{C}(x),\mathcal{C}(x)\} = g^2[t^bt^c]c^bc^c = ig^2t^a \times f^{abc}c^b(x)c^c(x)$, which is properly antisymmetric in the fermionic ghost fields.

As far as the quark and the gluon fields are concerned, their BRST transforms (11.a–c) look like infinitesimal gauge transforms for

$$\Lambda(x) = -i\epsilon\,\mathcal{C}(x). \tag{12}$$

Consequently, the gauge invariance of the physical QCD Lagrangian (8) immediately makes it BRST invariant,

$$\big[Q,\mathcal{L}_{\text{phys}}\big] = 0. \tag{13}$$

But the gauge fixing and the ghost terms (9) in the quantum Lagrangian are not gauge invariant, so proving their BRST symmetry is more complicated. I will do that in a moment, but first let me address another issue, the nilpotency of the BRST operator Q.

Theorem: *The BRST operator Q is nilpotent, $Q^2 = 0$.*

To prove this theorem, we first show that $Q^2 = \frac{1}{2}\{Q,Q\}$ commutes with all the fields; by the Leibniz rules for the (anti) commutator, this means

$$\{Q,[Q,\text{any bosonic field}]\} \;=\; 0 \qquad \text{and} \qquad [Q,\{Q,\text{any fermionic field}\}] \;=\; 0. \tag{14}$$

Let's verify these double-commutator formulae field by field. Obviously

$$\{Q,[Q,\mathcal{B}]\} \;=\; 0 \qquad \text{and} \qquad [Q,\{Q,\overline{\mathcal{C}}\}] \;=\; [Q,\mathcal{B}] \;=\; 0. \tag{15}$$

Less obviously but still rather simply

$$[Q,\{Q,\Psi\}] \;=\; [Q,\mathcal{C}\Psi] \;=\; \{Q,\mathcal{C}\}\Psi \;-\; \mathcal{C}\{Q,\Psi\} \;=\; +\mathcal{C}\mathcal{C}\Psi \;-\; \mathcal{C}\mathcal{C}\Psi \;=\; 0, \tag{16}$$

and similarly

$$[Q,\{Q,\overline{\Psi}\}] \;=\; [Q,\overline{\Psi}\mathcal{C}] \;=\; \{Q,\overline{\Psi}\}\mathcal{C} \;-\; \overline{\Psi}\{Q,\mathcal{C}\} \;=\; +\overline{\Psi}\mathcal{C}\mathcal{C} \;-\; \overline{\Psi}\mathcal{C}\mathcal{C} \;=\; 0. \tag{17}$$

Likewise

$$[Q,\{Q,\mathcal{C}\}] \;=\; [Q,\mathcal{C}\mathcal{C}] \;=\; \{Q,\mathcal{C}\}\mathcal{C} \;-\; \mathcal{C}\{Q,\mathcal{C}\} \;=\; \mathcal{C}\mathcal{C}\mathcal{C} \;-\; \mathcal{C}\mathcal{C}\mathcal{C} \;=\; 0. \tag{18}$$

Finally, the gauge field $\mathcal{A}_\mu$ takes a bit of algebra:

$$\begin{aligned}\{Q,[Q,\mathcal{A}_\mu]\} \;&=\; i\{Q,D_\mu\mathcal{C}\} \;=\; i[Q,D_\mu]\mathcal{C} \;+\; iD_\mu\{Q,\mathcal{C}\} \\ &=\; -\{[Q,\mathcal{A}_\mu],\mathcal{C}\} \;+\; iD_\mu(\mathcal{C}\mathcal{C}) \;=\; -i\{D_\mu\mathcal{C},\mathcal{C}\} \;+\; i\{D_\mu\mathcal{C},\mathcal{C}\} \;=\; 0.\end{aligned} \tag{19}$$

The bottom line of this exercise is that the operator Q^2 commutes with all the quantum fields of QCD. Consequently, in the Hilbert space of QCD, this operator either vanishes or acts as a c-number constant. But since Q^2 has a non-zero ghost number (namely +2), it cannot act as a constant, so it must vanish. *Quod erat demonstrandum.*

Having proved the nilpotency of the BRST operator, the simplest way to establish that the ghost and gauge fixing terms in QCD Lagrangian are BRST symmetric is to show that

$$\mathcal{L}_{\mathrm{gf+gh}} \;=\; \{Q, Z\} \tag{20}$$

for some fermionic operator Z. Indeed, given eq. (20), we would immediately have

$$\big[Q, \mathcal{L}_{\mathrm{gf+gh}}\big] \;=\; [Q, \{Q, Z\}] \;=\; [Q^2, Z] \;=\; 0 \quad \text{by nilpotency of } Q. \tag{21}$$

To verify eq. (20), we take

$$Z \;=\; \frac{2i}{g^2}\,\mathrm{tr}\left(\overline{\mathcal{C}} \times \left(\frac{\xi}{2}\mathcal{B} \,-\, \partial^\mu \mathcal{A}_\mu\right)\right). \tag{22}$$

Anticommuting this operator with Q we obtain

$$\begin{aligned}
\{Q, Z\} \;&=\; \frac{2i}{g^2}\,\mathrm{tr}\left(\{Q, \overline{\mathcal{C}}\} \times \left(\frac{\xi}{2}\mathcal{B} \,-\, \partial^\mu \mathcal{A}_\mu\right) \,-\, \overline{\mathcal{C}} \times \left[Q, \left(\frac{\xi}{2}\mathcal{B} \,-\, \partial^\mu \mathcal{A}_\mu\right)\right]\right) \\
&=\; \frac{2i}{g^2}\,\mathrm{tr}\left(-i\mathcal{B} \times \left(\frac{\xi}{2}\mathcal{B} \,-\, \partial^\mu \mathcal{A}_\mu\right) \,-\, \overline{\mathcal{C}} \times (0 \,-\, i\,\partial^\mu D_\mu \mathcal{C})\right) \\
&=\; \frac{2}{g^2}\,\mathrm{tr}\left(\frac{\xi}{2}\mathcal{B}^2 \,-\, \mathcal{B}\partial^\mu \mathcal{A}_\mu \,-\overline{\mathcal{C}}\partial^\mu D_\mu \mathcal{C}\right) \\
&=\; \mathcal{L}_{\mathrm{gf+gh}} \qquad \text{(up to a total derivative)},
\end{aligned} \tag{23}$$

which proves the BRST symmetry of the gauge-fixing and ghost parts of the QCD action. And as I have argued a couple of pages above, the classical QCD action is BRST symmetric because of its gauge invariance.

BRST Symmetry in the Fock Space

Quantization of QCD via the path integral

$$Z[\text{sources}] \;=\; \iiiint\mathcal{D}[A^a_\mu(x)] \iiiint\mathcal{D}[c^a(x)] \iiiint\mathcal{D}[\bar{c}^a(x)] \iiiint\mathcal{D}[\Psi^{fi}(x)] \iiiint\mathcal{D}[\overline{\Psi}_{fj}(x)]\,\exp\big(-S_e\big) \tag{24}$$

leads to the Fock space $\mathcal{F}$ for quanta of all the fields appearing in the path integral, including both the physical and the un-physical particles.

- Physical particles: quarks, antiquarks, and transversely polarized gluons.
- Unphysical particles: ghosts, antighosts, and longitudinally polarized gluons.

Indeed, in perturbation theory, the interaction-picture quantum vector field decomposes as

$$\hat{A}^a_\mu(x) \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\sum_\lambda \Big(e^{-ipx}\,e_\mu(p,\lambda)\times\hat{a}(p,\lambda,a)\;+\;e^{+ipx}\,e^*_\mu(p,\lambda)\times\hat{a}^\dagger(p,\lambda,a)\Big)\,, \tag{25}$$

and since we do not want the $\hat{A}^a_\mu(x)$ to be restricted by any gauge conditions, we should include both the transverse and the longitudinal polarizations λ. Consequently, the Fock space of the theory includes the longitudinal gluons created by the $\hat{a}^\dagger(p,L,a)$. Likewise, the ghosts fields $c^a(x)$ and the antighost fields $\bar{c}^a(x)$ expand into fermionic creation and annihilation operators

$$\begin{aligned}
c^a(x) \;&=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\,\Big(e^{-ipx}\times\hat{a}(\mathrm{gh},p,a)\;+\;e^{+ipx}\times\hat{a}^\dagger(\mathrm{gh},p,a)\Big)\,,\\
\bar{c}^a(x) \;&=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}\,\Big(e^{-ipx}\times\hat{a}(\mathrm{agh},p,a)\;+\;e^{+ipx}\times\hat{a}^\dagger(\mathrm{gh},p,a)\Big)\,,
\end{aligned} \tag{26}$$

and the Fock space must include the ghost and the antighost states created by the $\hat{a}^\dagger(\mathrm{gh},p,a)$ and the $\hat{a}^\dagger(\mathrm{agh},p,a)$.

Unlike the physical Fock spaces we have studies earlier in class, the QCD Fock space $\mathcal{F}$ has a Hilbert norm of mixed signature:

$$\langle\mathrm{phys}|\mathrm{phys}\rangle\;>\;0,\qquad \langle g_L|g_L\rangle\;>\;0,\quad\text{but}\quad\langle\mathrm{gh}|\mathrm{gh}\rangle\;<\;0,\quad\text{and}\quad\langle\mathrm{agh}|\mathrm{agh}\rangle\;<\;0. \tag{27}$$

Indeed, since the ghost and antighost fields violate the spin-statistics theorem — they are fermions despite spin $= 0$ — their quanta has negative Hilbert norms. Fortunately, we may use the BRST operator $\hat{Q}$ to reduce the formal Fock space $\mathcal{F}$ to the physical Hilbert space $\mathcal{H}$ which has a positive-definite norm.

Mathematically, a nilpotent operator like $\hat{Q}$ defines a *cohomology* — the kernel of $\hat{Q}$ modulo its image. Note that $\hat{Q}^2 = 0$ means that

$$\textbf{if}\quad |\psi_1\rangle\;=\;\hat{Q}\,|\psi_2\rangle\quad\textbf{then}\quad\hat{Q}\,|\psi_1\rangle\;=\;0, \tag{28}$$

but on the other hand, $\hat{Q}\,|\psi_1\rangle = 0$ does **not** require $|\psi_1\rangle = \hat{Q}\,|\psi_2\rangle$ for some $|\psi_2\rangle\in\mathcal{F}$. Mathematically speaking, the kernel of $\hat{Q}$ in $\mathcal{F}$ includes the image of $\hat{Q}$, but there are more states in the kernel than just the image; it is those extra states — annihilated by $\hat{Q}$ but not obtaining from $\hat{Q}$ acting on other states — which form the cohomology

$$\mathcal{H}_Q \;=\; \bigl\{\, |\psi\rangle \in \mathcal{F} \text{ such that } Q\,|\psi\rangle = 0 \bigr\} \quad \textbf{modulo} \quad \text{states of the form } Q\,\bigl|\psi'\bigr\rangle . \tag{29}$$

The BRST operator $\hat{Q}$ is *pseudo-Hermitian* WRT the mixed-signature Hilbert norm (27) in the Fock space $\mathcal{F}$. Consequently, in that metric all states of the form $\hat{Q}\,|\psi'\rangle$ are orthogonal to all states ψ annihilated by $\hat{Q}$:

$$\textbf{if} \quad \hat{Q}\,|\psi\rangle \;=\; 0 \quad \textbf{then} \quad \forall\,\bigl|\psi'\bigr\rangle, \quad \langle\psi|\,\hat{Q}\,\bigl|\psi'\bigr\rangle \;=\; 0. \tag{30}$$

However, the mixed-signature Hilbert norm in $\mathcal{F}$ allows for null states with $\Bigl\langle \tilde{\psi}|\tilde{\psi} \Bigr\rangle = 0$, and such states cannot be eliminated by an orthogonal projection. In fact, by nilpotency of $\hat{Q}$ all states of the form $\hat{Q}\,|\psi'\rangle$ are null — $\langle\psi'|\,\hat{Q}\hat{Q}\,|\psi'\rangle = 0$ — and that's why we need the second condition in the definition (29). Technically, the states of $\mathcal{H}_Q$ are the *equivalence classes*

$$\text{In } \mathcal{H}_Q\,, \quad |\psi\rangle \;\cong\; |\psi\rangle \;+\; Q\,\bigl|\psi'\bigr\rangle \quad \forall\,\bigl|\psi'\bigr\rangle \in \mathcal{F}. \tag{31}$$

By way of analogy, consider defining the subspace of transverse polarization vectors e^μ for a null (*i.e.*, light-like) momentum vector k^μ, $k^2 = 0$. In 3D terms, we may simply define the transverse e^μ as purely spatial vectors $(0, \vec{e})$ perpendicular to $\vec{k}$, but this presumes a particular Lorentz frame for the 4D Minkowski space. Without a specific rest frame, we need the more subtle equivalence-class construction:

$$V_{\perp k} \;=\; \bigl\{ \text{space of } e^\mu \text{ such that } e^\mu k_\mu = 0 \bigr\} \quad \textbf{modulo} \quad e^\mu \;\cong\; e^\mu \;+\; \text{number} \times k^\mu. \tag{32}$$

Note that the metric in this equivalence class is well defined:

$$e_1^\mu k_\mu \;=\; e_2^\mu k_\mu \;=\; 0 \quad \Longrightarrow \quad \forall c_1, c_2, \quad (e_1^\mu + c_1 k^\mu)(e_{2\mu} + c_2 k_\mu) \;=\; e_1^\mu e_{2\mu}\,. \tag{33}$$

Likewise, the $\hat{Q}$–cohomology (31) has a well-defined Hilbert norm:

$$\hat{Q}\,|\psi_1\rangle \;=\; \hat{Q}\,|\psi_2\rangle \;=\; 0 \quad \Longrightarrow \quad \forall\,\bigl|\psi_1'\bigr\rangle, \bigl|\psi_2'\bigr\rangle, \quad \bigl(\langle\psi_1| + \bigl\langle\psi_1'\bigr|\,\hat{Q}\bigr)\bigl(|\psi_2\rangle + \hat{Q}\,\bigl|\psi_2'\bigr\rangle\bigr) \;=\; \langle\psi_1|\psi_2\rangle\,. \tag{34}$$

Actually, this norm is positive definite in $\mathcal{H}_Q$, but I am not going to prove it here.

In perturbation theory, the BRST operator Q relates the longitudinal gluons to the ghosts and antighosts. Indeed, let's take the free-field limit $g \to 0$ in which the quantum fields expand into creation and annihilation operators as in eqs. (25) and (26). In this $g \to 0$ limit, the BRST symmetry action (5) reduces to

$$\left[\hat{Q}, \hat{A}^a_\mu(x)\right] \;=\; -\partial_\mu \hat{c}^a(x), \qquad \left\{\hat{Q}, \hat{\bar{c}}^a(x)\right\} \;=\; \hat{b}^a(x) \;=\; \frac{1}{\xi}\,\partial^\mu \hat{A}^a_\mu(x). \tag{35}$$

Expanding the fields into creation and annihilation operators, we obtain

$$\begin{aligned} \sum_\lambda e^*_\mu(p,\lambda) \times \left[\hat{Q}, \hat{a}^\dagger(\text{gluon}, p, \lambda, a)\right] \;&=\; -ip_\mu \times \hat{a}^\dagger(\text{ghost}, p, a), \\ \left\{\hat{Q}, \hat{a}^\dagger(\text{antighost}, p, a)\right\} \;&=\; -\frac{i}{\xi}\sum_\lambda p^\mu e_\mu(p,\lambda) \times \hat{a}^\dagger(\text{gluon}, p, \lambda, a), \end{aligned} \tag{36}$$

and likewise for the annihilation operators. In terms of single-particle states, these relations translate to

$$\begin{aligned} \hat{Q}\,|\text{gluon} : p, L_+, a\rangle \;&=\; -i\sqrt{2}p_0 \times |\text{ghost} : p, a\rangle\,, \\ \hat{Q}\,|\text{antighost} : p, a\rangle \;&=\; -\frac{ip_0}{\sqrt{2}\xi} \times |\text{gluon} : p, L_-, a\rangle\,, \end{aligned} \tag{37}$$

where L_+ and L_- denote two distinct longitudinal polarizations of a gluon,

$$e^\mu(L_+) \;=\; \frac{(p^0, +\mathbf{p})}{\sqrt{2}p^0}\,, \quad e^\mu(L_-) \;=\; \frac{(p^0, -\mathbf{p})}{\sqrt{2}p^0}\,. \tag{38}$$

Altogether, out of 4 unphysical single-particle states (for given momentum p^μ and adjoint color a), two states — the L_+ gluon and the antighost — do not obey $\hat{Q}\,|\psi\rangle = 0$, while the other 2 states — the L_- gluon and the ghost — obtain as $|\psi\rangle = Q\,|\psi'\rangle$ for some $|ket\psi'$. Thus, none of these unphysical states belongs to the BRST cohomology $\mathcal{H}_Q$.

At the same time, in the $g \to 0$ limit the BRST transformations do not affect the quark and antiquark fields or the transverse modes of the vector fields, hence

$$\hat{Q}\,|\text{quark}\rangle \;=\; 0, \quad \hat{Q}\,|\text{antiquark}\rangle \;=\; 0, \quad \hat{Q}\,|\text{gluon} : T\rangle \;=\; 0. \tag{39}$$

In other words, all the physical one-particle states obey $\hat{Q}\,|\psi\rangle = 0$. Also, none of these states obtains from $\hat{Q}$ acting on some other quantum state $|\psi'\rangle$. Thus, the physical one-particle states do belong to the $\mathcal{H}_Q$.

Altogether, among the one-particle states of the free theory, the BRST cohomology comprises all the physical states and none of the unphysical states. Using the action of $\hat{Q}_0$ on the creation and annihilation operators, we may extend this result to all the multi-particle states in the Fock space $\mathcal{F}$: the physical states of the free theory comprise the $\mathcal{H}_Q$ cohomology. Or rather, they comprise the $\mathcal{H}_Q^{(0)}$ — the subspace of $\mathcal{H}_Q$ made from states of net ghost number $\mathcal{G} = 0$. In terms of the original Fock space $\mathcal{F}$, we first decompose it into eigenspaces of specific net ghost numbers

$$\mathcal{F} \;=\; \bigoplus_{\mathcal{G}=-\infty}^{+\infty} \mathcal{F}_{\mathcal{G}} \tag{40}$$

connected by the BRST operator $\hat{Q}$ — which always raises $\mathcal{G}$ by 1 —

$$\cdots \xrightarrow{\hat{Q}} \mathcal{F}_{-2} \xrightarrow{\hat{Q}} \mathcal{F}_{-1} \xrightarrow{\hat{Q}} \mathcal{F}_0 \xrightarrow{\hat{Q}} \mathcal{F}_{+1} \xrightarrow{\hat{Q}} \mathcal{F}_{+2} \xrightarrow{\hat{Q}} \cdots \tag{41}$$

In this decomposition, we define the $\mathcal{H}_Q^{(0)}$ cohomology as

$$\mathcal{H}_Q^{(0)} \;=\; \left\{ |\psi\rangle \in \mathcal{F}_0 \text{ such that } \hat{Q}\,|\psi\rangle = 0 \right\} \quad \textbf{modulo} \quad |\psi\rangle \;\cong\; |\psi\rangle \,+\, \hat{Q}\,\left|\psi'\right\rangle \quad \forall \left|\psi'\right\rangle \in \mathcal{F}_{-1}\,. \tag{42}$$

For the interacting theory with $g \neq 0$ the particle states become more complicated, and the $\hat{Q}$ operator itself also becomes more complicated. Nevertheless, the $\hat{Q}$–cohomology $\mathcal{H}_Q^{(0)}$ comprises all the physical states of the interacting theory and only the physical states. Note that BRST is an exact symmetry of QCD, so $\hat{Q}$ commutes with the Hamiltonian $\hat{H}$ and hence with the S-matrix (the scattering operator). Thus, if the asymptotic incoming state $|\text{in}\rangle$ is annihilated by the $\hat{Q}$, then the asymptotic outgoing state $\hat{S}\,|\text{in}\rangle$ is also annihilated by the $\hat{Q}$. Moreover, the S-matrix restricted to the $\mathcal{H}_Q^{(0)}$ is well defined and unitary: For any $|A\rangle\,, \langle B| \in \mathcal{H}_Q^{(0)}$,

$$\langle B|\,\hat{S}\,|A\rangle \;=\; \left\langle B'\right|\hat{S}\left|A'\right\rangle \qquad \begin{array}{l} \forall\,|A'\rangle = |A\rangle + \hat{Q}\,|\text{whatever}\rangle\,, \\ \forall\,\langle B'| = \langle B| + \langle\text{whatever}|\,\hat{Q}, \end{array} \tag{43}$$

and

$$\sum_{|C\rangle \in \mathcal{H}_Q^{(0)}} \langle B|\,\hat{S}^\dagger\,|C\rangle \times \langle C|\,\hat{S}\,|A\rangle \;=\; \langle B|A\rangle\,. \tag{44}$$

One should be careful interpreting the $\mathcal{H}_Q^{(0)}$ as the physical Hilbert space of QCD in terms of the Feynman amplitudes $\mathcal{M}$. The processes turning physical incoming particles into unphysical outgoing particles are *not* forbidden. Indeed, in the previous set of notes we obtained non-zero amplitudes $\mathcal{M}(q+\bar{q}\to g_L g_L)$ and $\mathcal{M}(q+\bar{q}\to \mathrm{gh}+\mathrm{agh})$ for turning a quark and an antiquark into two longitudinal gluons or a ghost and an antighost. However, when we look at the net final state of the $q+\bar{q}$ annihilation,

$$\hat{S}\left|q+\bar{q}\right\rangle \;=\; \left|q+\bar{q}\right\rangle \;+\; \left|g_T+g_T\right\rangle \;+\; \left|g_L+g_L\right\rangle \;+\; \left|\mathrm{gh}+\mathrm{agh}\right\rangle \;+\; \left|3\text{ or more quanta}\right\rangle \tag{45}$$

the unphysical component of this state is of the form $\hat{Q}\left|\text{something}\right\rangle$,

$$\left|g_L+g_L\right\rangle \;+\; \left|\mathrm{gh}+\mathrm{agh}\right\rangle \;=\; \hat{Q}\left|g_L+\mathrm{agh}\right\rangle \tag{46}$$

and likewise for the unphysical states involving 3 or more quanta. Thus, in the $\mathcal{H}_Q^{(0)}$ space, the final state of annihilation *is equivalent to a purely physical state,*

$$\hat{S}\left|q+\bar{q}\right\rangle \;\cong\; \left|q+\bar{q}\right\rangle \;+\; \left|g_T+g_T\right\rangle \;+\; \left|3\text{ or more } \textit{physical} \text{ particles}\right\rangle . \tag{47}$$

As to the unitarity of the S-matrix in the $\mathcal{H}_Q^{(0)}$ space, it means that we may calculate the net cross-sections (or net partial cross-sections $d\sigma/d\Omega$) counting only the physical final particles — the contributions of the longitudinal gluons and the ghosts cancel each other. For example, in the previous set of notes we saw that

$$\mathcal{M}(q+\bar{q}\to g_L+g_L) \;=\; \mathcal{M}(q+\bar{q}\to \mathrm{gh}+\mathrm{agh}) \tag{48}$$

— and now we know that this relation follows from the BRST symmetry via eq. (46). Consequently, the two unphysical processes cancel each other from the net cross-section,

$$\begin{aligned}\frac{d\sigma(q+\bar{q}\to\cdots)}{d\Omega} \;=\; & \frac{\left|\left\langle g_T+g_T\right|\mathcal{M}\left|q+\bar{q}\right\rangle\right|^2}{64\pi^2 s} \\ & +\; \frac{\left|\left\langle g_L+g_L\right|\mathcal{M}\left|q+\bar{q}\right\rangle\right|^2}{64\pi^2 s} \;-\; \frac{\left|\left\langle \mathrm{gh}+\mathrm{agh}\right|\mathcal{M}\left|q+\bar{q}\right\rangle\right|^2}{64\pi^2 s}\,,\end{aligned} \tag{49}$$

and we are left with the cross-section for producing only the physical particles.