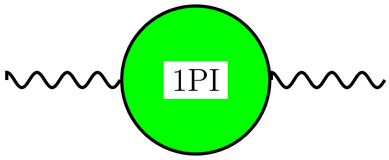
EM Field Renormalization

Calculating the $\Sigma_\gamma^{\mu\nu}(k)$ at One Loop Level

In QED, the two-photon 1PI bubble

$$i\Sigma^{\mu\nu}(k) \;=\; \tag{1}$$



should satisfy Ward identities

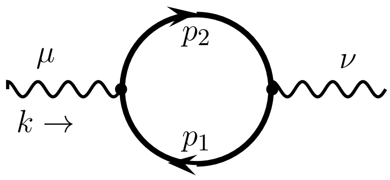
$$k_\mu \times \Sigma^{\mu\nu}(k) \;=\; 0, \quad k_\nu \times \Sigma^{\mu\nu}(k) \;=\; 0. \tag{2}$$

Between these identities and the Lorentz symmetry, the k–dependence of the $\Sigma^{\mu\nu}(k)$ should have form

$$\Sigma^{\mu\nu}(k) \;=\; \Pi(k^2) \times \left(g^{\mu\nu} \times k^2 \,-\, k^\mu k^\nu\right). \tag{3}$$

In this section, we calculate the $\Sigma^{\mu\nu}(k)$ to one-loop order and verify that it indeed has this form.

In pure QED, there is only one 1–loop diagram with 2 photonic external legs, namely



(4)

which gives us

$$i\Sigma^{\mu\nu}_{1\,\mathrm{loop}}(k) \;=\; -\int\!\frac{d^4p_1}{(2\pi)^4}\,\mathrm{tr}\left[(ie\gamma^\nu)\,\frac{i}{\not{p}_2 - m + i0}\,(ie\gamma^\mu)\,\frac{i}{\not{p}_1 - m + i0}\right] \tag{5}$$

where $p_2 \equiv p_1 + k$ and the overall minus sign comes from the closed electron loop. To evaluate the trace here, we use

$$\frac{1}{\not p - m + i0} = \frac{\not p + m}{p^2 - m^2 + i0}, \tag{6}$$

hence

$$\mathrm{tr}[\cdots] = e^2 \times \mathrm{tr}\left[\gamma^\nu \times \frac{\not p_2 + m}{p_2^2 - m^2 + i0} \times \gamma^\mu \times \frac{\not p_1 + m}{p_1^2 - m^2 + i0}\right] = \frac{e^2 \mathcal{N}^{\mu\nu}}{\mathcal{D}} \tag{7}$$

where

$$\mathcal{D} = (p_2^2 - m^2 + i0) \times (p_1^2 - m^2 + i0) \tag{8}$$

and

$$\begin{aligned} \mathcal{N}^{\mu\nu} &= \mathrm{tr}\big(\gamma^\nu(\not p_2 + m)\gamma^\mu(\not p_1 + m)\big) \\ &= \mathrm{tr}\big(\gamma^\nu \not p_2 \gamma^\mu \not p_1\big) + m^2\,\mathrm{tr}\big(\gamma^\nu\gamma^\mu\big) \\ &= 4p_1^\mu p_2^\nu + 4p_1^\nu p_2^\mu - 4(p_1 p_2) g^{\mu\nu} + 4m^2 g^{\mu\nu}, \end{aligned} \tag{9}$$

cf. my traceology notes from the Fall semester.

In the denominator $\mathcal{D}$, we may combine the two factors using the Feynman's parameter trick, thus

$$\begin{aligned} \frac{1}{\mathcal{D}} &= \int_0^1 dx \, \frac{1}{[(1-x)\times(p_1^2 - m^2 + i0) + x \times (p_2^2 - m^2 + i0)]^2} \\ \text{where } [\cdots] &= (1-x)\times p_1^2 + x \times (p_2 \equiv p_1 + k)^2 - m^2 + i0 \\ &= p_1^2 + 2x \times (p_1 k) + x \times k^2 - m^2 + i0 \\ &= (p_1 + xk)^2 + (x - x^2) \times k^2 - m^2 + i0 \\ &\equiv p^2 - \Delta(x) + i0 \end{aligned} \tag{10}$$

in terms of

$$p \stackrel{\text{def}}{=} p_1 + xk \quad \text{and} \quad \Delta(x) \stackrel{\text{def}}{=} m^2 - x(1-x) \times k^2. \tag{11}$$

Altogether,

$$\text{the trace} = \int_0^1 dx \, \frac{e^2 \mathcal{N}^{\mu\nu}}{[p^2 - \Delta(x) + i0]^2} \tag{12}$$

and therefore

$$\Sigma^{\mu\nu}_{1\,\mathrm{loop}}(k) \;=\; ie^2 \int\frac{d^4p_1}{(2\pi)^4}\int\limits_0^1 dx\,\frac{\mathcal{N}^{\mu\nu}}{[(p_1+xk)^2-\Delta+i0]^2} \;=\; ie^2\int\limits_0^1 dx\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}}{[p^2-\Delta+i0]^2}\,. \tag{13}$$

The second equality here obtains from changing the order of integration over the loop momentum p_1 and over the Feynman parameter x, followed by shifting the loop momentum variable from p_1 to $p = p_1 + xk$. To make full use of this shift, we need to re-express the numerator $\mathcal{N}^{\mu\nu}$ in terms of p, k, and x rather than p_1 and p_2. Thus, using

$$p_1 \;=\; p \;-\; x\times k, \quad p_2 \;=\; p_1 \;+\; k \;=\; p \;+\; (1-x)\times k, \tag{14}$$

we obtain

$$\begin{aligned} p_1^\mu p_2^\nu \;+\; p_2^\mu p_1^\nu \;&=\; 2p^\mu p^\nu \;+\; (1-2x)\times\big(p^\mu k^\nu \;+\; k^\mu p^\nu\big) \;-\; 2x(1-x)\times k^\mu k^\nu,\\ (p_1p_2) \;&=\; p^2 \;+\; (1-2x)\times(pk) \;-\; x(1-x)\times k^2, \end{aligned} \tag{15}$$

and hence

$$\begin{aligned} \mathcal{N}^{\mu\nu} \;=\; & 8p^\mu p^\nu \;+\; 4(1-2x)\times\big(p^\mu k^\nu \;+\; k^\mu p^\nu\big) \;-\; 8x(1-x)\times k^\mu k^\nu\\ &+\; 4g^{\mu\nu}\times\Big(m^2 \;-\; p^2 \;-\; (1-2x)\times(pk) \;+\; x(1-x)\times k^2\Big). \end{aligned} \tag{16}$$

There are many terms in this expression, and it is convenient to re-organize them into 3 groups: *the good, the bad, and the odd,* thus

$$\mathcal{N}^{\mu\nu} \;=\; \mathcal{N}^{\mu\nu}_{\mathrm{good}} \;+\; \mathcal{N}^{\mu\nu}_{\mathrm{bad}} \;+\; \mathcal{N}^{\mu\nu}_{\mathrm{odd}}\,, \tag{17}$$

where

$$\mathcal{N}^{\mu\nu}_{\mathrm{good}} \;=\; 8x(1-x)\times\Big(g^{\mu\nu}\times k^2 \;-\; k^\mu k^\nu\Big), \tag{18}$$

$$\mathcal{N}^{\mu\nu}_{\mathrm{bad}} \;=\; 8p^\mu p^\nu \;+\; 4g^{\mu\nu}\times\Big(m^2 \;-\; p^2 \;-\; x(1-x)\times k^2 \;=\; \Delta(x) \;-\; p^2\Big), \tag{19}$$

$$\mathcal{N}^{\mu\nu}_{\mathrm{odd}} \;=\; 4(1-2x)\times\Big(p^\mu k^\nu \;+\; k^\mu p^\nu \;-\; g^{\mu\nu}\times(pk)\Big). \tag{20}$$

The purpose of this re-organization is to extract the *good* terms (18) — which clearly have the desirable form (3) — while the remaining *bad* and *odd* terms should not contribute to the momentum integral.

Indeed, consider the *odd* terms (20) which comprise all the odd powers of the independent momentum variable p^α. Consequently, under the symmetry $p^\alpha \to -p^\alpha$ (for all 4 components of the p^α) the $\mathcal{N}^{\mu\nu}_{\rm odd}$ changes its sign. One the other hand, the $\int d^4p$ (over the whole momentum space) is invariant under this symmetry, and so is the denominator $[p^2-\Delta+i0]^2$. Consequently,

$$\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\rm odd}}{[p^2-\Delta+i0]^2}\ \xrightarrow[p\to-p]{}\ -\,\text{itself}\quad\Longrightarrow\quad\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\rm odd}}{[p^2-\Delta+i0]^2}\ =\ 0. \tag{21}$$

The momentum integral over the *bad* terms (19) also vanishes, but proving that takes more effort. First, let's Wick rotate the momentum integral from the Minkowski to the Euclidean space,

$$p^0\ \to\ ip^4,\quad d^4p\ \to\ id^4p_E\,,\quad p^2\ \to\ -p_E^2\,, \tag{22}$$

and hence

$$\begin{aligned}(\text{bad})\ \equiv\ i\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\rm bad}\ =\ 8p^\mu p^\nu\ +\ 4g^{\mu\nu}\times(\Delta-p^2)}{[p^2-\Delta+i0]^2}\\ =\ -\int\frac{d^4p_E}{(2\pi)^4}\,\frac{8p_E^\mu p_E^\nu\ +\ 4g^{\mu\nu}\times(\Delta+p_E^2)}{[\Delta+p_E^2]^2}\,.\end{aligned} \tag{23}$$

Note: even in the Euclidean momentum space, μ and ν remain Minkowski-signature vector indices, so $g^{\mu\nu}$ remains the Minkowski metric tensor, and p_E^μ and p_E^ν should be understood as (ip^4,p^1,p^2,p^3).

Second, let's use the $SO(4)$ symmetry of the Euclidean momentum space. Thanks to this symmetry,

$$\int\frac{d^3\Omega_p}{2\pi^2}\,p_E^ip_E^j\ =\ \delta^{ij}\times\frac{p_E^2}{4} \tag{24}$$

and hence for any spherically symmetric function $f(p_E^2)$

$$\int d^4p_E\,f(p_E^2)\times p_E^ip_E^j\ =\ \delta^{ij}\times\int d^4p_E\,f(p_E^2)\times\frac{p_E^2}{4}\,. \tag{25}$$

Or in terms of the Minkowski indices μ and ν,

$$\int d^4p_E\,f(p_E^2)\times p_E^\mu p_E^\nu\ =\ -g^{\mu\nu}\times\int d^4p_E\,f(p_E^2)\times\frac{p_E^2}{4}\,. \tag{26}$$

More generally, in D spacetime dimensions,

$$\int d^D p_E\, f(p_E^2)\times p_E^\mu p_E^\nu \;=\; -g^{\mu\nu}\times\int d^D p_E\, f(p_E^2)\times\frac{p_E^2}{D}\,. \tag{27}$$

Applying this formula to the Euclidean momentum integral in eq (23) gives us

$$\begin{aligned}\text{(bad)} \;&=\; -\int\frac{d^4p_E}{(2\pi)^4}\,\frac{-g^{\mu\nu}\times(8p_E^2/D)\,+\,4g^{\mu\nu}\times(\Delta+p_E^2)}{[\Delta+p_E^2]^2}\\ &=\; -g^{\mu\nu}\times\int\frac{d^4p_E}{(2\pi)^4}\,\frac{4\Delta\,+\,(4-\frac{8}{D})\times p_E^2}{[\Delta+p_E^2]^2}\,.\end{aligned} \tag{28}$$

For the moment $D=4$, but we keep the dimension D as explicit parameter in order to allow for the dimensional regularization of the momentum integral. Indeed, this integral badly needs DR — or some other UV regulator — because it's quadratically divergent in $D=4$. Thus, we let

$$\text{(bad)} \;=\; -g^{\mu\nu}\times\mu^{4-D}\int\frac{d^Dp_E}{(2\pi)^D}\,\frac{4\Delta\,+\,(4-\frac{8}{D})\times p_E^2}{[\Delta+p_E^2]^2}\,, \tag{29}$$

for generic D, evaluate the integral for $D<2$ (which regulates the UV divergence), and then analytically continue the result back to $D=4$. As usual, to evaluate the integral for non-integer D we relate the integrand to an exponential $\exp(-tp_E^2)$. Specifically, we let

$$\begin{aligned}\frac{4\Delta\,+\,(4-\frac{8}{D})\times p_E^2}{[\Delta+p_E^2]^2} \;&=\; \frac{(4-\frac{8}{D})}{\Delta+p_E^2}\;+\;\frac{\frac{8}{D}\,\Delta}{[\Delta+p_E^2]^2}\\ &=\; \int_0^\infty dt\,\left(\left(4-\frac{8}{D}\right)\,+\,\frac{8}{D}\,\Delta\times t\right)\times\exp\bigl(-t(\Delta+p_E^2)\bigr),\end{aligned} \tag{30}$$

which leads to

$$\begin{aligned}\text{(bad)} \;&=\; -g^{\mu\nu}\times\mu^{4-D}\int\frac{d^Dp_E}{(2\pi)^D}\int_0^\infty dt\,\left(\left(4-\frac{8}{D}\right)\,+\,\frac{8}{D}\,\Delta\times t\right)\times\exp\bigl(-t(\Delta+p_E^2)\bigr),\\ &=\; -g^{\mu\nu}\times\int_0^\infty dt\,\left(\left(4-\frac{8}{D}\right)\,+\,\frac{8}{D}\,\Delta\times t\right)e^{-t\Delta}\times\mu^{4-D}\int\frac{d^Dp_E}{(2\pi)^D}\,e^{-tp_E^2}\\ &=\; -g^{\mu\nu}\times\int_0^\infty dt\,\left(\left(4-\frac{8}{D}\right)\,+\,\frac{8}{D}\,\Delta\times t\right)e^{-t\Delta}\times\mu^{4-D}\,(4\pi t)^{-D/2}.\end{aligned} \tag{31}$$

The remaining integral over t here converges for $D<2$, and — miracle of miracles — it happens to vanish identically for any $D < 2$. Indeed, up to the overall constant factor $-g^{\mu\nu}\mu^{4-D}(4\pi)^{-D/2}$,

$$
\begin{aligned}
(\text{bad}) \; \propto \; & \int_0^\infty dt\, e^{-\Delta\times t}\left(\left(4-\frac{8}{D}\right)\times t^{-D/2} \,+\, \frac{8}{D}\,\Delta\times t^{1-(D/2)}\right) \\
= \; & \left(4-\frac{8}{D}\right)\times\Delta^{(D/2)-1}\Gamma\left(1-\frac{D}{2}\right) \,+\frac{8}{D}\,\Delta\times\Delta^{(D/2)-2}\Gamma\left(2-\frac{D}{2}\right) \\
= \; & \frac{8}{D}\times\Delta^{(D/2)-1}\times\left[\left(\frac{D}{2}-1\right)\times\Gamma\left(1-\frac{D}{2}\right) \,+\, \Gamma\left(2-\frac{D}{2}\right)\right] \\
= \; & \frac{8}{D}\times\Delta^{(D/2)-1}\times\left[-y\Gamma(y) \,+\, \Gamma(y+1) \quad \text{for } y \,=\, 1-\frac{D}{2}\right] \\
= \; & 0 \quad \text{because} \quad \Gamma(y+1) \,-\, y\Gamma(y) \;=\; 0 \quad \text{for any } y.
\end{aligned}
\tag{32}
$$

Consequently, analytically continuing from $D < 2$ to $D = 4$, we find that the dimensionally-regulated integral (28) vanishes and hence

$$
\int\limits_{\text{DR}} \frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\text{bad}}}{[p^2-\Delta+i0]^2} \;=\; 0. \tag{33}
$$

In other words, the *bad* terms in the numerator $\mathcal{N}^{\mu\nu}$ do not contribute to the photon's $\Sigma^{\mu\nu}_{1\,\text{loop}}(k)$.

At this point, the only terms in the the numerator $\mathcal{N}^{\mu\nu}$ that do contribute to the integral (13) are the *good* terms (18), thus

$$
\begin{aligned}
\Sigma^{\mu\nu}_{1\,\text{loop}}(k) \;&=\; ie^2\int_0^1 dx \int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\text{good}} \,=\, 8x(1-x)\times\left(g^{\mu\nu}k^2-k^\mu k^\nu\right)}{[p^2-\Delta+i0]^2} \\
&=\; 8e^2\left(g^{\mu\nu}k^2-k^\mu k^\nu\right)\times\int_0^1 dx\, x(1-x)\times\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{[p^2-\Delta+i0]^2}\,.
\end{aligned}
\tag{34}
$$

In other words, the one-loop $\Sigma^{\mu\nu}(k)$ does have the requisite form

$$
\Sigma^{\mu\nu}_{1\,\text{loop}}(k) \;=\; \left(g^{\mu\nu}k^2-k^\mu k^\nu\right)\times\Pi_{1\,\text{loop}}(k^2) \tag{35}
$$

where

$$\Pi_{1\,\text{loop}}(k^2) \;=\; 8e^2 \int\limits_0^1 dx\, x(1-x) \times \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{[p^2-\Delta+i0]^2}\,. \tag{36}$$

It remains to evaluate the momentum integral in the last formula and then integrate over x. The momentum integral has the form we have seen before in this class, so we evaluate it in the usual way: Wick rotate p to the Euclidean momentum space, and then dimensionally regularize the logarithmic divergence. Thus,

$$\begin{aligned}
\int \frac{d^4p}{(2\pi)^4}\, \frac{i}{[p^2-\Delta+i0]^2} &= \int \frac{d^4p_E}{(2\pi)^4}\, \frac{-1}{[p_E^2+\Delta]^2} \\
&\longrightarrow\; -\mu^{4-D} \int \frac{d^Dp_E}{(2\pi)^D}\, \frac{1}{[p_E^2+\Delta]^2} \\
&= -\mu^{4-D} \int \frac{d^Dp_E}{(2\pi)^D} \int\limits_0^\infty dt\, t\, e^{-t(\Delta+p_E^2)} \\
&= -\int\limits_0^\infty dt\, t\, e^{-\Delta\times t} \times \mu^{4-D} \int \frac{d^Dp_E}{(2\pi)^D}\, e^{-t\times P_E^2} \\
&= -\int\limits_0^\infty dt\, t\, e^{-\Delta\times t} \times \mu^{4-D} (4\pi t)^{-D/2} \\
&= -\frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \int\limits_0^\infty dt\, t^{1-(D/2)=\epsilon-1} \times e^{-\Delta\times t} \\
&= -\frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \Gamma(\epsilon)\Delta^{-\epsilon} \\
&= -\frac{1}{16\pi^2} \times \left(\frac{1}{\epsilon} \,-\, \gamma_E \,+\, \log\frac{4\pi\mu^2}{\Delta} \,+\, O(\epsilon)\right) \\
&\longrightarrow\; -\frac{1}{16\pi^2} \times \left(\frac{1}{\epsilon} \,-\, \gamma_E \,+\, \log\frac{4\pi\mu^2}{\Delta}\right) \\
&= -\frac{1}{16\pi^2} \times \left(\frac{1}{\epsilon} \,-\, \gamma_E \,+\, \frac{4\pi\mu^2}{m^2} \,-\, \log\frac{\Delta}{m^2}\right).
\end{aligned} \tag{37}$$

Plugging this result into eq. (36), we arrive at

$$\begin{aligned}\Pi_{1\,\mathrm{loop}}(k^2) \;&=\; -\frac{8e^2}{16\pi^2}\int_0^1 dx\, x(1-x)\times\left(\frac{1}{\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2}{m^2}\;-\;\log\frac{\Delta(x)}{m^2}\right)\\ &=\; -\frac{e^2}{2\pi^2}\left[\frac{1}{6}\times\left(\frac{1}{\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2}{m^2}\right)\;-\;\int_0^1 dx\, x(1-x)\log\frac{\Delta(x)}{m^2}\right],\end{aligned} \tag{38}$$

where $\frac{1}{6}$ factor comes from

$$\int_0^1 dx\, x(1-x) \;=\; \frac{1}{6}\,. \tag{39}$$

For convenience, let us define

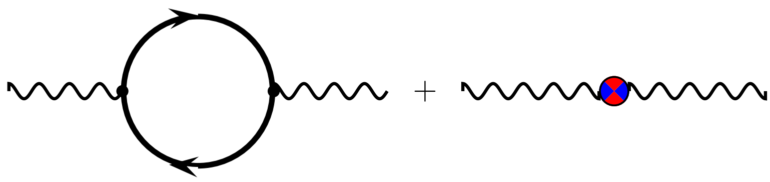
$$I(k^2/m^2) \;\stackrel{\mathrm{def}}{=}\; -6\int_0^1 dx\, x(1-x)\times\log\frac{\Delta\;=\;m^2-x(1-x)\times k^2}{m^2}\,, \tag{40}$$

then we may summarize the 1-loop photon propagator insertion as

$$\Pi_{1\,\mathrm{loop}}(k^2) \;=\; -\frac{e^2}{12\pi^2}\times\left(\frac{1}{\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2}{m^2}\;+\;I(k^2/m^2)\right). \tag{41}$$

Thus far, we have focused on the one-loop diagram (4) but ignored the counterterms. Adding the δ_3 counterterm to the picture gives us

$$i\Sigma^{\mu\nu}_{\mathrm{order}\,e^2}(k) \;=$$



$$=\; i\Sigma^{\mu\nu}_{1\,\mathrm{loop}}(k)\;-\;i\delta_3^{\mathrm{order}\,e^2}\times\left(g^{\mu\nu}k^2-k^\mu k^\nu\right), \tag{42}$$

or in terms of $\Pi(k^2)$,

$$\Pi^{\mathrm{order}\,e^2}(k^2) \;=\; \Pi^{1\,\mathrm{loop}}(k^2)\;-\;\delta_3^{\mathrm{order}\,e^2}. \tag{43}$$

Consequently, by setting

$$\delta_3^{\mathrm{order}\,e^2} \;=\; -\frac{e^2}{12\pi^2}\times\left(\frac{1}{\epsilon}\;+\;\text{a finite constant}\right) \tag{44}$$

we may cancel the ultraviolet divergence of the electron loop.

The finite part of the δ_3 counterterm follows from the requirement

$$\Pi^{\text{net}}(k^2 = 0) \;=\; 0. \tag{45}$$

Since $I(0) = 0$, this means we need

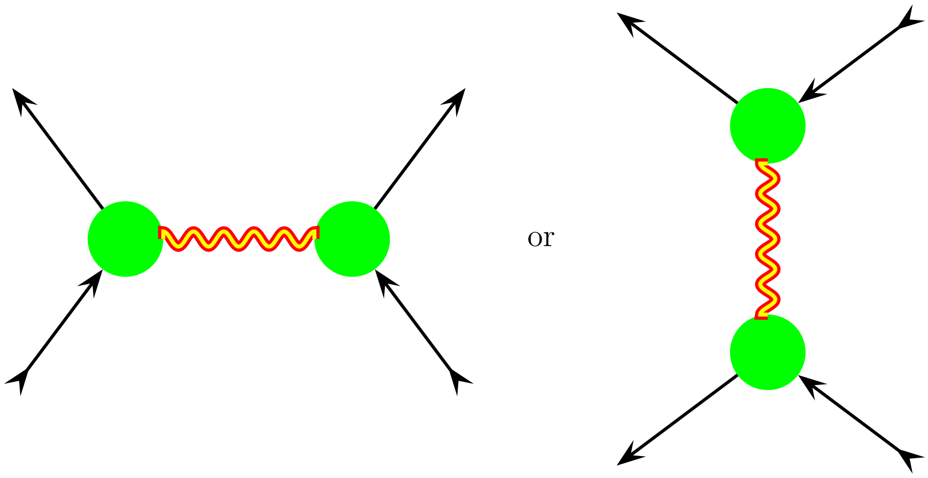
$$\delta_3^{\text{order}\,e^2} \;=\; -\frac{e^2}{12\pi^2} \times \left(\frac{1}{\epsilon} \,-\, \gamma_E \,+\, \log\frac{4\pi\mu^2}{m_e^2}\right). \tag{46}$$

which leaves us with

$$\Pi^{\text{order}\,e^2}(k^2) \;=\; -\frac{e^2}{12\pi^2} \times I(k^2/m_e^2). \tag{47}$$

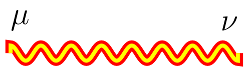
Energy Dependent Effective QED Coupling $\alpha(E)$

Consider a process involving a high-momentum virtual photon, for example pair-production (s-channel virtual photon) or Coulomb scattering (t-channel virtual photon). Beyond the tree level, the diagrams contributing to such processes have general form



(48)

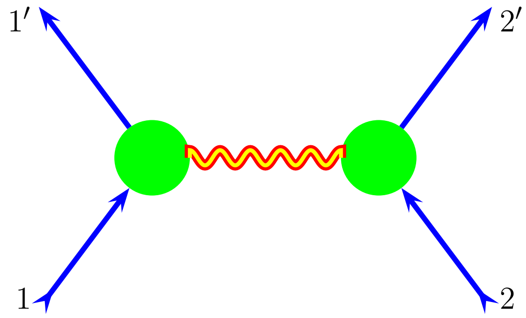
where green disks denote the dressed vertices while the double wavy line denotes the dressed photon propagator



$$= \frac{-i}{k^2 + i0} \times \frac{1}{1 - \Pi_{\text{net}}(k^2)} \times \left(g^{\mu\nu} \,+\, (\tilde{\xi} - 1)\frac{k^\mu k^\nu}{k^2 + i0}\right). \tag{49}$$

Note that while the dressed vertices depend on the specific charged particles involved in the process, the dressed photon propagator is universal for all processes (48), and this leads to the universal but energy dependent charge renormalization in QED.

To see how it works, consider Coulomb scattering of some very heavy but slow charged particles,

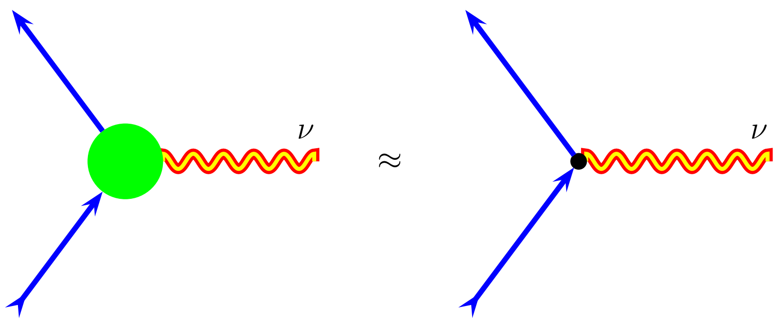


(50)

Specifically, we take the limit of the charged particles' 3-momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ — and hence the momentum transfer $\mathbf{q}$ — being much smaller than their masses but at the same time much larger than the electron's mass,

$$m_e \ll |\mathbf{p}_1|,\ |\mathbf{p}_2|,\ |\mathbf{q}| \ll M_1, M_2\,. \tag{51}$$

In this limit, the dressed vertex for a heavy but slow particle becomes similar to the tree-level vertex (I shall return to this issue in a later set of notes), which is in term dominated by the particle's charge rather than its 3-current or magnetic moment, thus



$$\approx\ (-iQ) \times 2M\delta^{\nu,0}. \tag{52}$$

where Q is the particle's electric charge Consequently, the whole scattering amplitude of the Coulomb scattering (50) becomes

$$\mathcal{M}\ =\ \frac{4M_1M_2Q_1Q_2}{t} \times \frac{1}{1-\Pi(t)}\,, \tag{53}$$

where the second factor on the RHS stems from the loop corrections to the photon's propagator. In terms of the effective Coulomb potential between the particles (1) and (2), the scattering amplitude (53) corresponds to

$$\widetilde{V}_{\rm eff}(\mathbf{q}) \stackrel{\rm def}{=} \int d^3\mathbf{x} e^{-i\mathbf{q}\mathbf{x}}\, V_{\rm eff}(\mathbf{x}) \;=\; \left(\frac{Q_1 Q_2}{4\pi\mathbf{q}^2}\right)_{\rm tree} \times \frac{1}{1-\Pi(-\mathbf{q}^2)}\,. \tag{54}$$

Note that the loop correction to the effective Coulomb potential does not depend on the specific particles in question but only on the momentum transfer $\mathbf{q}$ between them, and even the particles' electric charges $Q_1 = n_1 \times e_{\rm phys}$ and $Q_2 = n_2 \times e_{\rm phys}$ enter only as overall factors. Consequently, the effective potential (54) can be interpreted as the momentum-dependent renormalization of the electric charge unit e:

$$e^2_{\rm eff}(\mathbf{q}^2) \;=\; \frac{e_0^2}{1-\Pi(-\mathbf{q^2})}\,, \tag{55}$$

where e_0 is the usual electric charge unit measured in electrostatic (or low-frequency) experiments.

Now, suppose the only particles in our theory that are much lighter than the heavy particles we scatter are the electrons and the photons. In this case, the $1-\Pi(-\mathbf{q}^2)$ factor in eq. (55) obtains from the electron and electron-photon loops in pure QED. In particular, at the one-loop level of analysis

$$\Pi^{\rm 1\,loop}_{\rm net}(-\mathbf{q}^2) \;=\; \frac{-e^2}{12\pi^2}\, I(-\mathbf{q}^2/m_e^2) \tag{47}$$

where

$$I(-\mathbf{q}^2/m_e^2) \;=\; -6\int_0^1 dx\, x(1-x) \times \log\frac{m_e^2 + \mathbf{q}^2 x(1-x)}{m_e^2}\,, \tag{40}$$

cf. the first section of these notes. For $\mathbf{q}^2 \gg m_e^2$, we may approximate

$$\log\frac{m_e^2 + \mathbf{q}^2 x(1-x)}{m_e^2} \;\approx\; \log\frac{\mathbf{q}^2 x(1-x)}{m_e^2} \;=\; \log\frac{\mathbf{q}^2}{m_e^2} \;+\; \log(x(1-x)), \tag{56}$$

and hence

$$I(-\mathbf{q}^2/m_e^2) \;\approx\; -\log\frac{\mathbf{q}^2}{m_e^2} \;-\; 6\int_0^1 dx\, x(1-x) \times \log((x(1-x)) \;=\; -\log\frac{\mathbf{q}^2}{m_e^2} \;+\; \frac{5}{3}\,. \tag{57}$$

Consequently, for $\mathbf{q}^2 \gg m_e^2$

$$\Pi_{\text{net}}(-\mathbf{q}^2) \;=\; \frac{e_0^2}{12\pi^2}\left(\log\frac{\mathbf{q}^2}{m_e^2} - \frac{5}{3}\right) \;+\; O(e_0^4) \tag{58}$$

and therefore

$$\frac{1}{e_{\text{eff}}^2(\mathbf{q}^2)} \;=\; \frac{1-\Pi_{\text{net}}(-\mathbf{q}^2)}{e_0^2} \;=\; \frac{1}{e_0^2} \;-\; \frac{1}{12\pi^2}\left(\log\frac{\mathbf{q}^2}{m_e^2} - \frac{5}{3}\right) \;+\; O(e_0^2). \tag{59}$$

Or in terms of $\alpha = e^2/4\pi$,

$$\frac{1}{\alpha_{\text{eff}}(\mathbf{q}^2)} \;=\; \frac{1}{\alpha_0} \;-\; \frac{1}{3\pi}\left(\log\frac{\mathbf{q}^2}{m_e^2} - \frac{5}{3}\right) \;+\; O(\alpha). \tag{60}$$

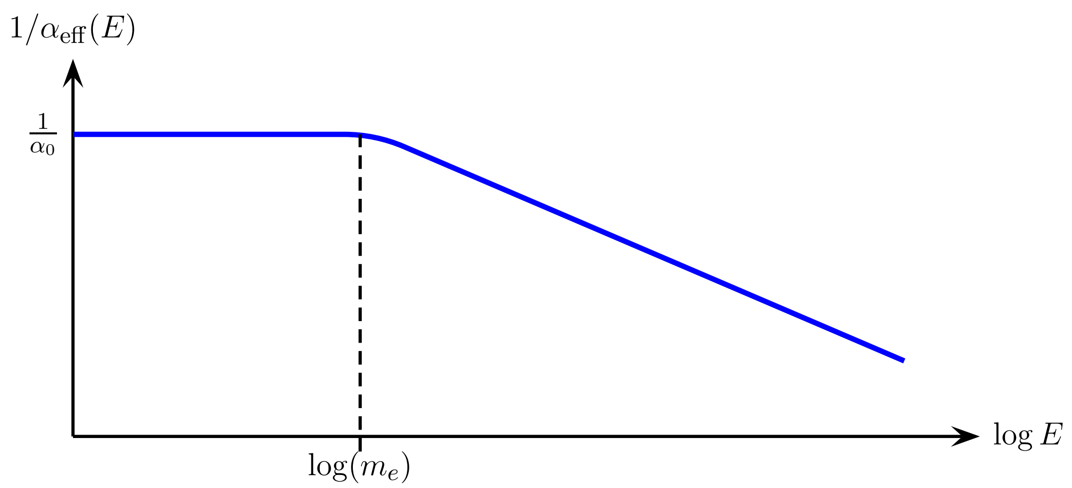
Although the above example focused on the Coulomb scattering of heavy but slow particles, similar arguments work for other high-energy QED processes. In general, any QED amplitude involving energy or momentum transfers $E \gg m_e$ involves an energy-dependent effective QED coupling — also called the *running* coupling $\alpha_{\text{eff}}(E)$,

$$\frac{1}{\alpha_{\text{eff}}(E)} \;=\; \frac{1}{\alpha_0} \;-\; \frac{1}{3\pi}\left(\log\frac{E^2}{m_e^2} - \frac{5}{3}\right) \;+\; O(\alpha). \tag{61}$$

Note however that eq. (61) applies only for $E \gg m_e$. For smaller energies $I(q^2/m_e^2) = O(q^2/m_e^2)$ and therefore

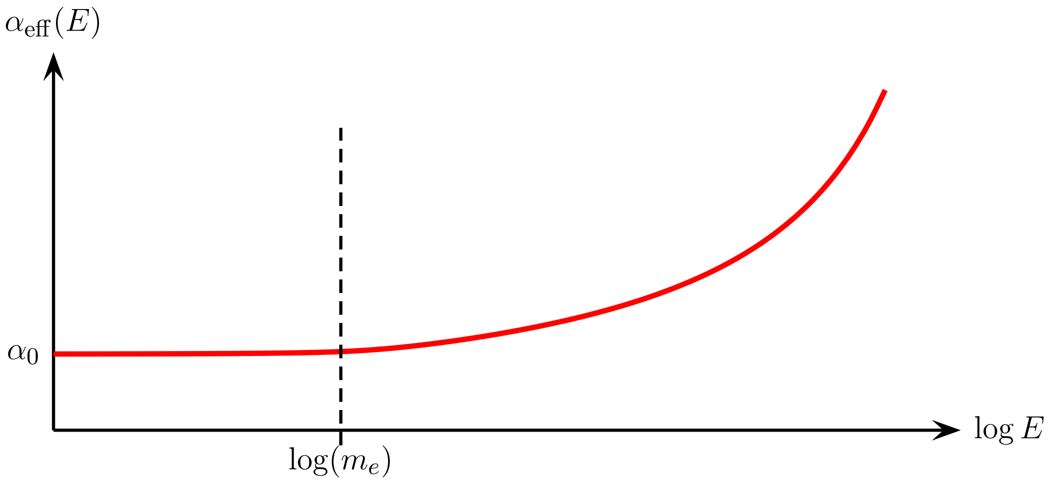
$$\frac{1}{\alpha_{\text{eff}}(E)} \;\approx\; \frac{1}{\alpha_0} \;-\; O(E^2/m_e^2) \;\approx\; \frac{1}{\alpha_0}\,. \tag{62}$$

Graphically,



(63)

which means that the effective coupling stays constant for $E \ll m_e$ and then starts *increasing* for $E \gg m_e$ according to eq. (61).



(64)

As written, eq. (61) applies to the simplest version of QED comprised of photons and electrons but no other fields. In real life, the EM fields couples to electrons, muons, taus, and several species of quarks. At the one-loop level, each of these charged fermions has a similar effect to the electron, except for a different mass and charge. Altogether, they yield

$$\begin{aligned} \frac{1}{\alpha_{\rm eff}(E)} &= \frac{1}{\alpha_0} + \frac{1}{3\pi} \sum^{\rm fermions} (Q/e)^2 \times N_{\rm colors} \times I(E^2/m^2) \ + \ O(\alpha) \\ \text{away from thresholds} \ &\longrightarrow \frac{1}{\alpha_0} - \frac{1}{3\pi} \sum_{m<E}^{\rm fermions} (Q/e)^2 \times N_{\rm colors} \times \left(\log \frac{E^2}{m^2} - \frac{5}{3} \right) \ + \ O(\alpha). \end{aligned} \tag{65}$$

In particular, at $E = M_Z = 91$ GeV, $\alpha_{\rm eff} \approx 1/128$ instead of the low-energy $\alpha_0 \approx 1/137$.