UV Regularization Schemes

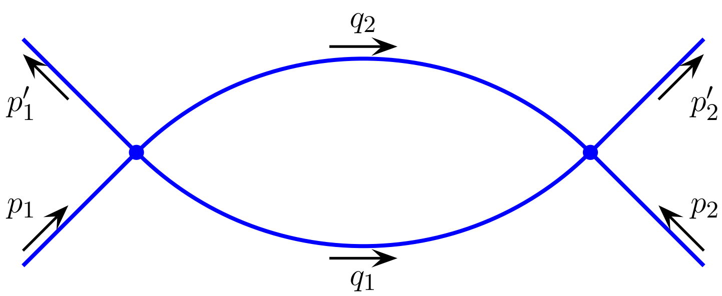
Many Feynman diagrams suffer from the ultraviolet divergences — the integral over the loop momentum k^μ (or several loop momenta $k_1^\mu, k_2^\mu, \ldots$) diverges for $k^\mu \to \infty$. To make sense of such a diagram, we need to *regulate* the divergence by somehow cutting-off, suppressing, or canceling the very-high-k^μ regime of the diagram, which in QFT terms calls for cutting-off, suppressing, or canceling the effects of the very-high-momentum modes of the quantum fields. In these notes I shall explain several common *UV regularization schemes* — also called the *UV cutoffs* — which render the divergent Feynman diagrams finite:

1. Wilson's hard edge cutoff.
2. Pauli–Villars.
3. Higher derivatives (in the $\lambda\phi^4$ theory) or covariant higher derivatives (CHD) in QED and other gauge theories.
4. Lattice. (Very briefly, but will explain in more detail after the Spring break.)
5. Dimensional regularization $d = 4 - 2\epsilon$.

All these regularization schemes are physically equivalent to each other — *i.e.*, they yield the same finite amplitudes at energies $E \ll \Lambda_{\text{UV}}$ — but some schemes are more convenient to use than the other schemes.

Wilson's Hard Edge Cutoff

Kenneth Wilson's hard-edge cutoff was developed in condensed matter context. In an effective long-distance quantum field theory, this scheme abruptly cuts off all modes of the quantum fields with momenta larger than Λ_{UV}. For the relativistic QFTs, this scheme means that the Euclidean momenta of all propagators of any Feynman diagram should be less that the UV cutoff scale Λ. For example, in the one-loop diagram



(1)

we should have

$$\textbf{both} \quad |q_{1E}| \leq \Lambda \quad \textbf{and} \quad |q_{2E}| \leq \Lambda \quad \textbf{while} \quad q_1 + q_2 \equiv q_{\text{net}} \,. \tag{2}$$

In terms of the (Euclidean) shifted loop momentum

$$k_E^\mu = q_{1E}^\mu - \xi q_{\text{net,E}}^\mu = (1-\xi) q_{\text{net,E}}^\mu - q_{2E}^\mu, \tag{3}$$

the conditions (2) limit k_E^μ to a lens-shaped region which becomes approximately spherical for $\Lambda \gg |q_{\text{net,E}}$:

$$|k_E| \leq k_E^{\max} = \Lambda - |q_{\text{net,E}}| \times f(\text{angle between } k_E \text{ and } q_{\text{net,E}}). \tag{4}$$

For the logarithmically divergent integrals this cutoff is as good as $|k_E| \leq \Lambda$, for example

$$\int_0^{k_E^{\max}} \frac{2k_e^3\, dk_e}{[k_e^2+\Delta]^2} = \log\frac{(k_E^{\max})^2}{\Delta} - 1 + O\left(\frac{\Delta}{(k_E^{\max})^2}\right) = \log\frac{\Lambda^2}{\Delta} - 1 + O\left(\frac{q\text{net, E}}{\Lambda}\right), \tag{5}$$

hence

$$\int_{\text{reg}} \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k_E^2+\Delta]^2} = \frac{1}{16\pi^2}\left(\log\frac{\Lambda^2}{\Delta} - 1 + O\left(\frac{q\text{net, E}}{\Lambda}\right)\right). \tag{6}$$

But for the quadratically divergent integrals like

$$\int_{\text{reg}} \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2+\Delta}\,, \tag{7}$$

we have

$$\begin{aligned} \int_0^{k_E^{\max}} \frac{2k_e^3\, dk_e}{k_e^2+\Delta} &= (k_E^{\max})^2 - \Delta \times \log\frac{(k_E^{\max})^2+\Delta}{\Delta} \\ &= \Lambda^2 - 2f\Lambda|q_{\text{net,E}}| + f^2 q_{\text{net,E}}^2 - \Delta\times\log\frac{\Lambda^2}{\Delta} + O\left(\frac{q_{\text{net,E}}}{\Lambda}\right) \end{aligned} \tag{8}$$

and hence

$$\int_{\text{reg}} \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k_E^2+\Delta]^2} = \frac{1}{16\pi^2}\left(\Lambda^2 - c_1\Lambda|q_{\text{net,E}}| + c_2 q_{\text{net,E}}^2 - \Delta\times\log\frac{\Lambda^2}{\Delta} + \text{negligible}\right) \tag{9}$$

for some $O(1)$ numeric constants c_1 and c_2. Thus, besides the expected quadratic divergence we get a sub-leading linear divergence and an extra finite term. Worse, both the linearly-divergent and the finite terms depend on the external momenta, so they do not cancel out when we re-express the amplitudes in terms of λ_{phys} instead of λ_{bare}. For this reason, Wilson's hard-edge cutoff is rarely applied to the amplitudes with worse-than-logarithmic UV divergences.

Conceptually, Wilson's hard-edge cutoff is very clear (especially in the path-integral formulation of the QFT) — we literally cut off the ultraviolet modes of the quantum fields. But in practice it can be rather awkward to use — especially for the divergences that are worse than logarithmic. And even in theory it has a couple of serious problems: First, the amplitudes obtained using this cutoff are not exactly analytic functions of the particles' momenta. The non-analyticities may be rather small for $|p| \ll \Lambda$, but their very existence destroys the so-called *dispersion relations* based on exact complex analyticity of the scattering amplitudes. Second, in gauge theories like QED or QCD, the hard edge of the momentum space breaks the gauge invariance $\Psi(x) \to e^{i\theta(x)}\Psi(x)$ of the charged fields, and this leads to all kinds of troubles.

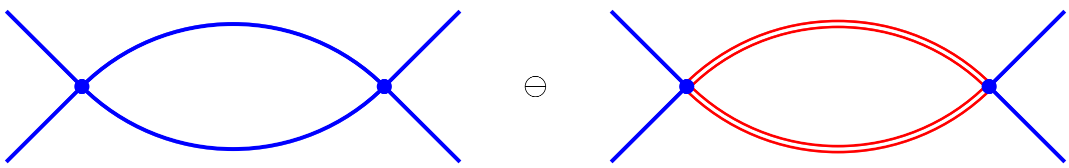
Pauli–Villars

In the Pauli–Villars regularization scheme — named after Wolfgang Pauli and Felix Villars who invented it in 1949 — one does not literally cut off the ultraviolet momenta $|k_e| > \Lambda$. Instead, their contribution to the loop integrals is canceled by the similar loops of very heavy compensating fields. For example, the PV-regulated $\lambda\Phi^4$ theory has two scalar fields — the physical field Φ and the regulator field χ — and the (bare) Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\Phi)^2 + \tfrac{1}{2}(\partial_\mu\chi)^2 - \frac{m^2}{2}\Phi^2 - \frac{\Lambda^2}{2}\chi^2 - \lambda\left(\frac{\Phi^4}{24} + \frac{\Phi^2\chi^2}{4} + \frac{\chi^4}{24}\right). \tag{10}$$

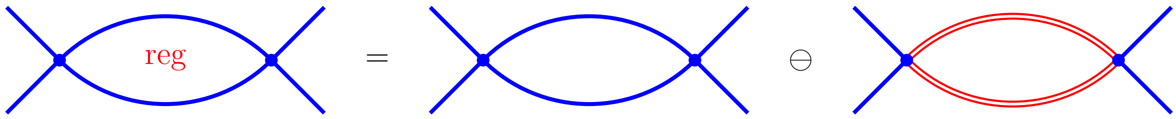
Also, every loop of the compensator field χ carries a minus sign!

Consequently, the one-loop amplitude for the elastic scattering of 2 physical particles is given by



(11)

— plus two similar diagram pairs for the s channel and the u channel — where double red lines denote the propagators of the superheavy compensator field χ. Together, the two diagrams (11) can be identified as the *regulated t-channel diagram*,



(12)

which yields

$$
\begin{aligned}
i\mathcal{F}(t) &= \frac{\lambda^2}{2}\int\frac{d^4q_1}{(2\pi)^4}\,\frac{1}{q_1^2-m^2+i\epsilon}\times\frac{1}{(q_2=q_{\rm net}-q_1)^2-m^2+i\epsilon}\\
&\quad -\frac{\lambda^2}{2}\int\frac{d^4q_1}{(2\pi)^4}\,\frac{1}{q_1^2-\Lambda^2+i\epsilon}\times\frac{1}{(q_2=q_{\rm net}-q_1)^2-\Lambda^2+i\epsilon}\\
&= \frac{\lambda^2}{2}\int\frac{d^4q_1}{(2\pi)^4}\left(\begin{aligned}&\frac{1}{q_1^2-m^2+i\epsilon}\times\frac{1}{(q_2=q_{\rm net}-q_1)^2-m^2+i\epsilon}\\ -&\frac{1}{q_1^2-\Lambda^2+i\epsilon}\times\frac{1}{(q_2=q_{\rm net}-q_1)^2-\Lambda^2+i\epsilon}\end{aligned}\right).
\end{aligned}
\tag{13}
$$

Note: the second equality here is a part of the Pauli–Villars regularization scheme: One should subtract the superheavy compensator loop from the loop of the physical fields before integrating over the loop momenta.

The momentum integral (13) has two essential feature common to all UV regularization schemes. On one hand, for $|q_1|\ll\Lambda$, the second term in the integrand is negligibly small compared to the first terms, so the regulated integrand is approximately equal to the un-regulated integrand. Thus, the UV regulator does not affect the integrand except at the ultraviolet momenta $|q_1|\gtrsim\Lambda$. On the other hand, for the very large momenta $|q_1|\gg\Lambda$, the second term approximates the first term and starts canceling it. Thus,

$$
\text{for } |q_1^2|\gg\Lambda^2,\quad \text{(the integrand)}\ \longrightarrow\ \frac{-2\Lambda^2}{(q_1^2)^3}\,, \tag{14}
$$

which leads to a convergent momentum integral (13). The actual calculation of this finite integral is a part of your current homework#13.

In general, a single PV compensator field χ — or in more complicated theories that $\lambda\Phi^4$, one PV compensator field for each physical field — suffices to regulate all the logarithmic UV divergences and most linear $O(\Lambda)$ divergences, but regulating the quadratic UV divergences requires multiple PV compensators for some physical fields. But let us skip the technical aspects of such multiple PV compensators until we actually have to use them in this class.

The bottom line is: The Pauli–Villars is a good UV regulator for the $\lambda\Phi^4$ theory or QED as long as we stick to the perturbation theory and all the external momenta are much smaller than the cutoff scale Λ. However, the PV-regulated theory should not be taken literally because the compensating field χ has unphysical quanta. Indeed, $\chi(x)$ is a scalar field, but the minus sign for each χ loop means χ is a fermion rather than a boson. By the spin-statistics theorem, it means that the quanta of the χ field have either negative energies or negative norms in the Hilbert space — or both, — and that would be quite impossible for physical particles.

Also, the Pauli–Villars scheme does not work in QCD or other non-abelian gauge theories because massive compensators for the gluon fields would break the gauge symmetry of the theory and hence the Slavnov–Taylor identities required for the theory's renormalizability.

Higher Derivatives

In the higher-derivative (HD) regularization scheme, we do not use any compensating fields but simply add small higher-derivative terms to the physical fields' Lagrangian. For example, in the $\lambda\Phi^4$ theory, the HD-regularized Lagrangian is

$$\mathcal{L}_{\text{reg}} \;=\; -\frac{1}{2\Lambda^2}(\partial^2\Phi)^2 \;+\; \frac{1}{2}(\partial_\mu\Phi)^2 \;-\; \frac{m^2}{2}\,\Phi^2 \;-\; \frac{\lambda^4}{24}\,\Phi^4. \tag{15}$$

In terms of the Feynman rules of the regularized theory, the extra term — being quadratic in Φ — modifies the propagators but leaves the vertices unchanged. Specifically, the propagator becomes the Green's function of the fourth-order derivative operator

$$\frac{1}{\Lambda^2}(\partial^2)^2 \;+\; \partial^2 \;+\; m^2, \tag{16}$$

hence

$$\text{———} \;=\; \frac{i}{-(q^4/\Lambda^2)+q^2-m^2+i\epsilon} \;\approx\; \frac{i}{q^2-m^2+i\epsilon}\times\frac{-\Lambda^2}{q^2-\Lambda^2+i\epsilon}\,. \tag{17}$$

Note that for the non-ultraviolet loop momenta $q^\mu \ll \Lambda$, the second factor in this propagator is very close to 1. Consequently, in the HD-regulated loop integral

$$\begin{aligned}
&\int\limits_{\rm reg}\frac{d^4q_1}{(2\pi)^4}\,\frac{1}{q_1^2-m^2+i\epsilon}\times\frac{1}{q_2^2-m^2+i\epsilon}\\
&=\int\frac{d^4q_1}{(2\pi)^4}\,\frac{1}{-(q_1^4/\Lambda^2)+q_1^2-m^2+i\epsilon}\times\frac{1}{-(q_2^4/\Lambda^2)+q_2^2-m^2+i\epsilon}\\
&\approx\int\frac{d^4q_1}{(2\pi)^4}\,\frac{1}{q_1^2-m^2+i\epsilon}\times\frac{1}{q_2^2-m^2+i\epsilon}\times\frac{-\Lambda^2}{q_1^2-\Lambda^2+i\epsilon}\times\frac{-\Lambda^2}{q_2^2-\Lambda^2+i\epsilon}\,,
\end{aligned}\tag{18}$$

the integrand is approximately equal to the un-regulated integrand (without the two red factors on the bottom line) at the non-ultraviolet loop momenta $q^\mu \ll \Lambda$. On the other hand, at the very large loop momenta $q^\mu \gg \Lambda$, the red factors become small so the whole integrand behaves as

$$\frac{1}{(q^2)^2}\times\frac{\Lambda^4}{(q^2)^2}\quad\text{rather than simply}\quad\frac{1}{(q^2)^2}\,,\tag{19}$$

and that's what makes the regulated integral (18) converge,

$$\int d^4q\,\frac{\Lambda^4}{(q^2)^4}\quad\text{is finite.}\tag{20}$$

Again, the actual evaluation of the integral (18) is a part of your current homework#13, but let me give you a hint: The red factors on the bottom line of eq. (18) are not sensitive to *small* shifts of the momentum q^μ by $O(q_{\rm net})\ll\Lambda$, so you may replace them with

$$\frac{-\Lambda^2}{q_1^2-\Lambda^2+i\epsilon}\times\frac{-\Lambda^2}{q_2^2-\Lambda^2+i\epsilon}\;\longrightarrow\;\frac{\Lambda^4}{[k^2-\Lambda^2+i\epsilon]^2}\tag{21}$$

where $k^\mu = q_1^\mu - \xi q_{\rm net}^\mu$ is the shifted loop momenta from the Feynman's parameter trick.

Similar to the Pauli–Villars, the higher-derivative scheme is a good UV regulator for the perturbation theory — as long as all the external momenta are much smaller than the cutoff scale Λ — but it should not be taken literally because it contains unphysical superheavy particles. Indeed, a scalar field with a 4-derivative Lagrangian has two degrees of freedom rather than just one; looking at the poles of the propagator

$$\frac{i}{-(q^4/\Lambda^2)+q^2-m^2+i\epsilon}\;\approx\;\frac{i}{q^2-m^2+i\epsilon}\;-\;\frac{i}{q^2-\Lambda^2+i\epsilon}\tag{22}$$

we see 2 scalar particles, one of mass m and another of mass Λ. Moreover, the pole at $q^2=\Lambda^2$ has a negative residue, and we shall learn later in class that this means negative Hilbert-space norm for the superheavy particle. Thus, the HD-regulated theory should not be taken literally as a physical theory of 2 particle species but only as a UV regulator of the ordinary $\lambda\Phi^4$ theory.

Covariant Higher Derivatives

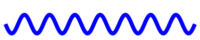
In gauge theories like QED or QCD, the higher-derivative regulating terms should be replaced with *covariant* higher derivative terms. For example, in QED all derivative acting on the electron field $\Psi(x)$ must be covariant derivatives $D_\mu = \partial_\mu - ieA_\mu(x)$ rather than the ordinary derivatives ∂_μ. Hence, the CHD-regulated QED Lagrangian looks like

$$\mathcal{L}_{\rm reg} \;=\; -\frac{1}{4}F_{\mu\nu}\left(1+\frac{\partial^2}{\Lambda^2}\right)F^{\mu\nu} \;+\; \overline{\Psi}\left(i\not{D} \;-\; m + \frac{i}{\Lambda^2}\not{D}\not{D}\not{D}\right)\Psi. \tag{23}$$

Similar to the ordinary higher-derivative terms, the CHD (covariant higher derivative) terms soften the photon's and the electron's propagators at ultraviolet momenta,

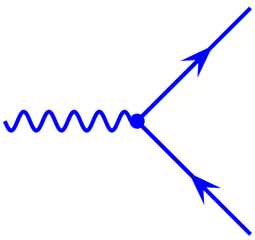
$$= \frac{i}{(1-(q^2/\Lambda^2))\,\not{q}-m+i\epsilon} \;\approx\; \frac{i}{\not{q}-m+i\epsilon}\times\frac{-\Lambda^2}{q^2-\Lambda^2+i\epsilon} \tag{24}$$

and



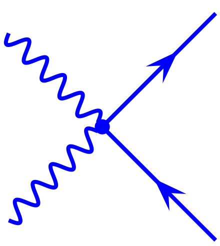
$$= \frac{-ig^{\mu\nu}}{-(q^4/\Lambda^2)+q^2+i\epsilon} \;\approx\; \frac{-ig^{\mu\nu}}{q^2+i\epsilon}\times\frac{-\Lambda^2}{q^2-\Lambda^2+i\epsilon} \tag{25}$$

(in the Feynman gauge). However, the covariant $\not{D}^3 = (\not{\partial} - ie\not{A})^3$ derivative acting on the electron field also modifies the electron-photon interaction. In terms of the Feynman rules, the single-photon vertex becomes larger at the ultraviolet electron momenta,

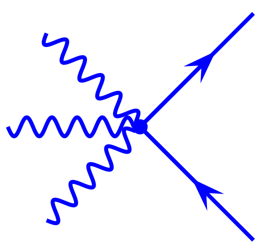


$$= ie\gamma^\mu \;-\; \frac{ie}{\Lambda^2}\left(p'^2\gamma^\mu + \not{p}'\gamma^\mu\not{p} + \gamma^\mu p^2\right), \tag{26}$$

and we also get two-photon and 3-photon vertices

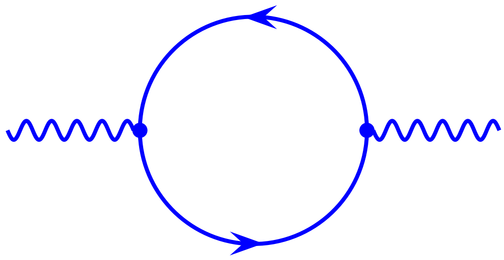


$$= O\left(\frac{e^2 p}{\Lambda^2}\right) \quad \text{and}$$



$$= O\left(\frac{e^3}{\Lambda^2}\right). \tag{27}$$

In a general loop diagram, at very large loop momenta $q^\mu \gg \Lambda$, the modified propagators reduce the integrand by powers of Λ^2/q^2 while the modifies vertices increase the integrand by opposite factors q^2/Λ^2. Altogether, in multi-loop diagrams the propagators have a stronger effect than the vertices, so the integrands become smaller for $q^\mu \gg \Lambda$ and the momentum integrals become convergent. But in the one-loop diagrams, the vertex corrections cancel the effect of the propagator corrections, so the UV-divergent one-loop diagrams like



(28)

remain divergent. To regulate such diagrams we need an additional UV regulator such as Pauli–Villars.

LATTICE

In this regularization scheme, we modify the very spacetime in which the QFT lives. First, we analytically continue from the $d = 3 + 1$ Minkowski spacetime to the $d = 4$ Euclidean spacetime, and then we discretize all 4 dimensions of the Euclidean spacetime. In other words, we turn it into a 4D crystalline lattice of very small lattice spacing $a = (\pi/\Lambda)$,

$$\forall\mu = 1,2,3,4: \quad x^\mu \;=\; n^\mu \times a \quad \text{for integer } n^\mu \text{ only.} \tag{29}$$

Constructing a quantum field theory in such a discrete spacetime involves the path integral formalism, which I shall explain in detail sometimes in March. For the moment, let me simply tell you that the lattice is the only known *non-perturbative* UV regulator of quantum field theories. In particular, since the QCD coupling becomes non-perturbatively strong at low energies $E \lesssim 1$ GeV, the only way to derive the low-energy hadronic physics — or even the mass spectrum of mesons and baryons — directly from QCD is to put QCD on a lattice.

On the other hand, in perturbation theory the lattice is a rather awkward cutoff due to breaking the $SO(4)$ symmetry of the continuous Euclidean spacetime. In particular, the lattice momentum space becomes a 4D torus — a direct product of 4 compact circles where each k^i is periodic modulo $(2\pi/a)$, — and the scalar propagator becomes

$$\frac{(a/2)^2}{\sum_{\mu=1}^{4}\sin^2(k^\mu a/2)+(ma/2)^2}\,, \tag{30}$$

— which asymptotes to $1/(k_E^2+m^2)$ for small momenta $k_e \ll (1/a)$, but is much uglier at larger momenta.

Dimensional Regularization

The dimensional regularization was invented by Gerard 't Hooft and Martinus Veltman 1972 as a way to cutoff the UV divergences of QCD without breaking its gauge symmetry, and soon became the preferred UV regularization scheme for most quantum field theories. Similar to the lattice cutoff, the dimensional regularization (DR) modifies the spacetime in which the QFT lives, but unlike the lattice the DR keeps the spacetime continuous. Instead, it analytically continues the spacetime dimension from $d=4$ to $d=4-2\epsilon$. Since we are going to use DR as a UV cutoff throughout this semester, I explain the technical aspects of dimensional regularization in a separate set of notes.