

Renormalizability and Dimensional Analysis

In these notes I shall explain the relation between energy dimensionalities of the coupling constants of a quantum field theory and between super-renormalizability, renormalizability, or non-renormalizability of the theory.

Let's start with the basic dimensional analysis. In the $\hbar = c = 1$ units, all quantities are measured in units of energy to some power. For example $[m] = [p^\mu] = E^{+1}$ while $[x^\mu] = E^{-1}$, where $[m]$ stands for the *dimensionality* of the mass rather than the mass itself, and ditto for the $[p^\mu]$, $[x^\mu]$, *etc.* The action

$$S = \int d^4x \mathcal{L}$$

is dimensionless (in $\hbar \neq 1$ units, $[S] = \hbar$), so the Lagrangian of a 4D field theory has dimensionality $[\mathcal{L}] = E^{+4}$.

Dimensionalities — also called the *canonical dimensions* — of the quantum fields follow from their free Lagrangians.

For example, a scalar field $\Phi(x)$ has

$$\mathcal{L}_{\text{free}} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2, \quad (1)$$

so $[\mathcal{L}] = E^{+4}$, $[m^2] = E^{+2}$, and $[\partial_\mu] = E^{+1}$ imply $[\Phi] = E^{+1}$. Likewise, the EM field has

$$\mathcal{L}_{\text{free}}^{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \implies [F_{\mu\nu}] = E^{+2}, \quad (2)$$

and since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the $A_\nu(x)$ field has dimension

$$[A_\nu] = [F_{\mu\nu}] / [\partial_\mu] = E^{+1}. \quad (3)$$

In fact, all the *bosonic* fields in 4D spacetime have canonical dimensions E^{+1} because their kinetic terms are quadratic in $\partial_\mu(\text{field})$. On the other hand, the fermionic fields like the Dirac field $\Psi(x)$ have dimensionality $[\Psi] = E^{+3/2}$. Indeed, the kinetic terms in the free

Dirac Lagrangian

$$\mathcal{L}_{\text{free}} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \quad (4)$$

involve two fermionic fields Ψ and $\bar{\Psi}$ but only one derivative ∂_μ . Consequently, $[\mathcal{L}] = E^{+4}$ implies $[\bar{\Psi}\Psi] = E^{+3}$ and hence $[\Psi] = [\bar{\Psi}] = E^{+3/2}$. Similarly, all other types of fermionic fields in 4D have canonical dimension $E^{+3/2}$.

In QFTs in other spacetime dimensions $d \neq 4$, similar arguments show that the bosonic fields such as scalars and vectors have canonical dimension

$$[\Phi] = [A_\nu] = E^{+(d-2)/2} \quad (5)$$

while the fermionic fields have canonical dimension

$$[\Psi] = E^{+(d-1)/2}. \quad (6)$$

In perturbation theory, dimensionality of coupling parameters such as λ in $\lambda\Phi^4$ theory or e in QED follows from the field's canonical dimensions. For example, in a 4D scalar theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 - \sum_{n \geq 3} \frac{C_n}{n!} \Phi^n, \quad (7)$$

the coupling C_n of the Φ^n term has dimensionality

$$[C_n] = [\mathcal{L}] / [\Phi]^n = E^{4-n}. \quad (8)$$

In particular, the cubic coupling C_3 has positive energy dimension E^{+1} , the quartic coupling $\lambda = C_4$ is dimensionless, while all the higher-power couplings have negative energy dimensions E^{negative} . Note how the sign of the coupling's energy dimension matches the renormalizability of the theory: the super-renormalizable coupling κ has a positive energy dimension, the renormalizable coupling λ is dimensionless, and the non-renormalizable couplings C_n for $n > 4$ have negative energy dimensions. This is an example of a general rule:

- All couplings of a renormalizable theory must have non-negative energy dimensions.
- If all the couplings of a theory have strictly positive energy dimensions, then the theory is super-renormalizable.
- But if any coupling of a theory has a negative energy dimension, then the theory is non-renormalizable, even if it also have other couplings of non-negative energy dimensions.

To see how this works, consider a generic interaction term in the Lagrangian of some QFT. In general such term is a product of some coupling constant g and several fields or their derivatives. Let n_b be the number of bosonic fields in this product, n_f the number of fermionic fields, and n_d the number of spacetime derivatives ∂_μ acting on all these fields. Consequently,

$$[\text{field product}] = E^{n_b + \frac{3}{2}n_f + n_d}, \quad (9)$$

and since the entire interaction term must have dimensionality E^{+4} — same as the entire Lagrangian — the coupling constant g must have dimensionality

$$[g] = E^\Delta \quad \text{for} \quad \Delta = 4 - n_b - \frac{3}{2}n_f - n_d. \quad (10)$$

In general, a QFT may have several coupling constants, and each coupling has its own energy dimension Δ according to eq. (10).

Next, consider a Feynman diagram for some QFT. Let the diagram have L loops, P_b bosonic propagators, P_f fermionic propagators, and V vertices of all kinds, so the diagram evaluates to

$$\int d^{4L}q \prod(\text{propagators}) \times \prod(\text{vertices}). \quad (11)$$

Consider the superficial degree of divergence \mathcal{D} of such a diagram. At large momenta q , each bosonic propagator behaves as $1/q^2$ while each fermionic propagator behaves as $1/q$. The vertices may also be momentum-dependent: if the interaction term in the Lagrangian involves n_d derivatives of fields, then the corresponding vertex includes n_d power of momenta,

so for large q it grows as q^{+n_d} . Altogether, the momentum integral (11) behaves as

$$\int d^{4L}q \frac{1}{q^{2P_b+P_f}} \times \prod_v^{\text{vertices}} q^{+n_d(v)}, \quad (12)$$

so its superficial degree of divergence is

$$\mathcal{D} = 4L - 2P_b - P_f + \sum_{v=1}^V n_d(v). \quad (13)$$

Now let's rework this formula using basic graph theory. By the Euler theorem

$$L - P_{\text{net}} + V = 1 \implies L = 1 + P_b + P_f - V, \quad (14)$$

hence

$$\mathcal{D} = 4 + (4 - 2 = 2) \times P_b + (4 - 1 = 3) \times P_f + \sum_{v=1}^V (n_d - 4). \quad (15)$$

Also, counting the line ends — bosonic or fermionic — we obtain

$$2P_b + E_b = \sum_v n_b(v), \quad (16)$$

$$2P_f + E_f = \sum_v n_f(v), \quad (17)$$

and hence

$$2P_b + 3P_f = \sum_{v=1}^V (n_b + \frac{3}{2}n_f) - E_b - \frac{3}{2}E_f. \quad (18)$$

Consequently, eq. (15) becomes

$$\mathcal{D} = 4 - E_b - \frac{3}{2}E_f + \sum_{v=1}^V (n_b + \frac{3}{2}n_f + n_d - 4). \quad (19)$$

Note that the combinations $(n_b + \frac{3}{2}n_f + n_d - 4)$ we sum over the vertices are precisely (minus) the energy dimensions of the corresponding couplings, *cf.* eq. (10). Thus, we arrive at the

key relation

$$\mathcal{D} = 4 - E_b - \frac{3}{2}E_f - \sum_{v=1}^V \Delta(g_v). \quad (20)$$

between the couplings' energy dimensions and the divergence degrees of the Feynman diagrams.

The rules relating couplings' dimensions Δ to the renormalizability of the QFT in question follow from eq. (20):

- If all the couplings of the theory have strictly positive dimensions Δ , then only a finite number of Feynman diagrams for the theory may have $\mathcal{D} \geq 0$ and hence suffer from the overall UV divergence. All the rest of the diagrams are either UV-finite or have divergent sub-diagrams — but once the subgraph divergence is canceled by an in-situ counterterm, the overall diagram becomes finite. And that's what makes the theory in question **super-renormalizable**.
- If some couplings of the theory are dimensionless ($\Delta = 0$) while other have $\Delta > 0$, then the theory has an infinite number of diagrams with $\mathcal{D} \geq 0$ and therefore divergent. But all such diagrams must have $E_b + \frac{3}{2}E_f \leq 4$, which means that there is only a finite number of divergent *amplitudes*. Consequently, all the UV divergences can be canceled by a finite set of counterterms, but the coefficients of such counterterms must be adjusted order-by-order in perturbation theory at all loop orders. And that's what makes the theory in question **renormalizable**.
- Finally, if a theory has a coupling with a negative dimension Δ , then the theory has an infinite number of divergent amplitudes. Indeed, for any given numbers of external bosonic and fermionic legs, eq. (20) allows for $\mathcal{D} \geq 0$ provided the diagram includes enough vertices with $\Delta < 0$. Consequently, the theory needs an infinite set of counterterms to cancel all such divergences, and that's what makes it **non-renormalizable**.

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Now that we know the significance of the couplings' dimensions

$$\Delta = 4 - n_b - \frac{3}{2}n_f - n_d, \quad (10)$$

let's classify the renormalizable ($\Delta = 0$) and the super-renormalizable ($\Delta > 0$) couplings of 4D field theories. Since any physical interaction term involves at least 3 fields (otherwise, it would be a part of the free Lagrangian), it follows that the only way to get $\Delta > 0$ is to have $n_b = 3$, $n_f = 0$, and $n_d = 0$, — in other words, **boson³** without ∂_μ derivatives. Likewise, there are only 3 ways to get a renormalizable coupling with $\Delta = 0$, namely **boson⁴**, **boson² \times ∂ boson**, and **boson \times fermion²**. All other combinations of fields lead to non-renormalizable couplings with $\Delta < 0$.

In terms of more specific types of fields and couplings, *there is only one kind of a super-renormalizable coupling*, namely the 3-scalar coupling

$$-\frac{\kappa}{6}\Phi^3, \quad \text{or for multiple fields} \quad -\sum_{i,j,k} \frac{\kappa_{ijk}}{6}\Phi_i\Phi_j\Phi_k. \quad (21)$$

Also, there are only 5 kinds of renormalizable couplings:

1. The 4-scalar coupling

$$-\frac{\lambda}{24}\Phi^4, \quad \text{or for multiple fields} \quad -\sum_{i,j,k,\ell} \frac{\lambda_{ijkl}}{24}\Phi_i\Phi_j\Phi_k\Phi_\ell. \quad (22)$$

2. Gauge couplings of vectors to charged scalars

$$-iqA^\mu \times (\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) + q^2 A_\mu A^\mu \times \Phi^* \Phi \subset D_\mu \Phi^* D^\mu \Phi, \quad (23)$$

or for non-abelian gauge symmetries

$$-igA^{a\mu} \times (\Phi^\dagger T^a \partial_\mu \Phi - \partial_\mu \Phi^\dagger T^a \Phi) + g^2 A_\mu^a A^{b\mu} \times \Phi^\dagger T^a T^b \Phi \subset D_\mu \Phi^\dagger D^\mu \Phi. \quad (24)$$

3. Non-abelian gauge couplings between the vector fields

$$-gf^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c} - \frac{g^2}{4}f^{abc}f^{ade}A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} \subset -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}. \quad (25)$$

4. Gauge couplings of vectors to charged fermions,

$$-qA^\mu \times \bar{\Psi}\gamma_\mu\Psi \quad \text{or} \quad -gA^{a\mu} \times \bar{\Psi}\gamma_\mu T^a\Psi \quad \subset \quad \bar{\Psi}(i\gamma_\mu D^\mu)\Psi. \quad (26)$$

If the fermions are massless and chiral, we may also have

$$-gA_\mu^a \times \bar{\Psi}\gamma^\mu \frac{1 \mp \gamma^5}{2} T^a\Psi, \quad (27)$$

or in the Weyl fermion language

$$-gA_\mu^a \times \psi_L^\dagger \bar{\sigma}_\mu T^a \psi_L \quad \text{or} \quad -gA_\mu^a \times \psi_R^\dagger \sigma_\mu T^a \psi_R.$$

5. Yukawa couplings of scalars to fermions,

$$-y\Phi_1 \times \bar{\Psi}\Psi \quad \text{or} \quad -iy\Phi_2 \times \bar{\Psi}\gamma^5\Psi, \quad (28)$$

depending on the scalar's parity — the Φ_1 should be a true scalar and the Φ_2 a pseudoscalar, — of for the chiral fermions

$$-y\Phi\psi^\top\sigma_2\psi - y^*\Phi^*\psi^\dagger\sigma_2\psi^*. \quad (29)$$

— And this is it! All other coupling types are non-renormalizable in 4 spacetime dimensions.

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In other spacetime dimensions $d \neq 3 + 1$, a coupling involving n_b bosonic fields, n_f fermionic fields, and n_d derivatives has dimensionality

$$\begin{aligned}\Delta(d) &= d - n_b \times \frac{d-2}{2} - n_f \times \frac{d-1}{2} - n_d \\ &= \left(4 - n_b - \frac{3}{2}n_f - n_d\right) - \frac{n_b + n_f - 2}{2} \times (d-4) \\ &= \Delta(d=4) - \frac{n_b + n_f - 2}{2} \times (d-4).\end{aligned}\tag{30}$$

Since all interactions involve three or more fields, thus $n_b + n_f \geq 3$, the dimensionality of any particular coupling always decreases with spacetime dimension d . Consequently, there are more (super)renormalizable couplings with $\Delta \geq 0$ in lower dimensions $d = 2 + 1$ or $d = 1 + 1$ but fewer such couplings in higher dimensions $d > 3 + 1$. In particular,

- In $d \geq 6 + 1$ dimensions all couplings have $\Delta < 0$ and there are no renormalizable couplings at all!
- In $d = 5 + 1$ dimensions there is a unique $\Delta = 0$ coupling $(\kappa/6)\Phi^3$, while all the other couplings have $\Delta < 0$. Consequently, the only renormalizable theories are scalar theories with cubic potentials,

$$\mathcal{L} = \sum_i \left(\frac{1}{2}(\partial_\mu \Phi_a)^2 - \frac{1}{2}m_i^2 \Phi_a^2\right) - \frac{1}{6} \sum_{i,j,k} \mu_{ijk} \Phi_i \Phi_j \Phi_k.\tag{31}$$

However, while such theories are perturbatively OK, they do not have stable vacua since a cubic potential is always unbounded from below.

- In $d = 4 + 1$ dimensions, the $(\kappa/6)\Phi^3$ coupling has positive $\Delta = +\frac{1}{2}$ while all the other couplings have negative energy dimensions. Hence, the scalar theories (31) are super-renormalizable (but non-perturbatively sick), while all other interactive QFTs are non-renormalizable.
- ★ The bottom line is, *in $d > 3 + 1$ dimensions there are no renormalizable theories with stable vacua.*

On the other hand, in lower dimensions $d = 2 + 1$ or $d = 1 + 1$ there are many more (super)renormalizable $\Delta \geq 0$. In particular, in $d = 2 + 1$ dimensions such couplings include:

- Scalar couplings $(C_n/n!)\Phi^n$ up to $n = 6$;
- Gauge and Yukawa couplings like in 4D;
- Yukawa-like couplings $\tilde{y}\Phi^2 \times \bar{\Psi}\Psi$ involving 2 scalars;
- * Chern–Simons couplings of non-abelian gauge fields to each other, and some other exotic couplings, never mind the details.

Finally, *in $d = 1 + 1$ dimensions there are infinite numbers of renormalizable and even super-renormalizable couplings*. Indeed, for $d = 1+1$ the bosonic fields have energy dimension E^0 , so Δ of a coupling does not depend on the number n_b of bosonic fields it involves but only on the numbers of derivatives and fermionic fields,

$$\Delta = 2 - n_d - \frac{1}{2}n_f. \quad (32)$$

Consequently, all scalar potentials $V(\Phi)$ — including $C_n\Phi^n$ terms for any n , and even the non-polynomial potentials — have $\Delta = +2$, so any $V(\Phi)$ potential is super-renormalizable in 2D. Likewise, all Yukawa-like couplings $\Phi^n\bar{\Psi}\Psi$ have $\Delta = +1$, so we may have terms like $y_{IJ}(\Phi) \times \bar{\Psi}^I\Psi^J$ for any functions $y_{IJ}(\Phi)$.

At the $\Delta = 0$ level, we have renormalizable field-dependent kinetic terms

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}g_{ij}(\phi) \times \partial^\mu\phi^i \partial_\mu\phi^j \quad (33)$$

with any Riemannian metrics $g_{ij}(\phi)$ for the non-linear scalar field space, as well as a whole bunch of fermionic terms with arbitrary scalar-dependent coefficients,

$$\begin{aligned} \mathcal{L}_\Psi \supset \frac{1}{4}g_{IJ}(\Phi) \times \bar{\Psi}^I \gamma^\mu \left(i \overset{\rightarrow}{\partial}_\mu - i \overset{\leftarrow}{\partial}_\mu \right) \Psi^J + \Gamma_{IJK}(\Phi) \times \partial_\mu\Phi^k \times \bar{\Psi}^I \gamma^\mu \Psi^J \\ + \frac{1}{2}R_{IJKL}(\Phi) \times \bar{\Psi}^I \gamma^\mu \Psi^J \times \bar{\Psi}^K \gamma_\mu \Psi^L. \end{aligned} \quad (34)$$

In addition, there are gauge couplings with arbitrary scalar-dependent $g_{\text{gauge}}(\Phi)$, chiral couplings to Weyl or Majorana-Weyl fermions, *etc., etc.* In String Theory, many of these couplings show up the context of the 2D field theory on the world sheet of the string.