Electric–Magnetic Duality and Magnetic Monopoles

Electric–Magnetic Duality

In vacuum, — and in the absence of any electric charges or currents, — the electric and the magnetic field behave in similar ways. Moreover, for $\rho = 0$ and $\mathbf{J} = 0$ there is an exact electric-magnetic symmetry $\mathbf{E} \leftrightarrow \mathbf{B}$, — or rather

$$\mathbf{B}' = +\mathbf{E}, \quad \mathbf{E}' = -\mathbf{B} \tag{1}$$

in Gauss units, — or more generally

$$\begin{aligned} \mathbf{B}' &= \cos\theta\,\mathbf{B} + \sin\theta\,\mathbf{E}, \\ \mathbf{E}' &= \cos\theta\,\mathbf{E} - \sin\theta\,\mathbf{B}. \end{aligned} \tag{2}$$

Indeed, any such *duality transform* preserves the Maxwell equations

$$\begin{aligned} \nabla\cdot\mathbf{E} &= 0 \\ \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\times\mathbf{B} &= +\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \end{aligned} \tag{3}$$

$$(\text{for } \rho = 0 \text{ and } \mathbf{J} = 0),$$

as well as the EM energy density

$$U = \frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \tag{4}$$

the Poynting vector

$$\mathbf{S} = \frac{1}{4\pi}\,\mathbf{E}\times\mathbf{B} \tag{5}$$

governing the momentum density and the energy flux, and the EM stress-tensor

$$T^{ij} = \frac{1}{4\pi}\left(E^iE^j + B^iB^j\right) - \delta^{ij}U. \tag{6}$$

As written, the electric-magnetic symmetry (2) works in empty space but not in presence of any electric charges or currents. However, we may repair this duality symmetry in a generalized electromagnetic theory which has both electric and magnetic charges and currents.

Both kinds of charges must be locally conserved, thus continuity equations

$$\begin{aligned} \nabla\cdot\mathbf{J}_{\rm el} \;+\; \frac{\partial}{\partial t}\rho_{\rm el} \;&=\; 0, \\ \nabla\cdot\mathbf{J}_{\rm mag} \;+\; \frac{\partial}{\partial t}\rho_{\rm mag} \;&=\; 0, \end{aligned} \tag{7}$$

which make for consistent generalized Maxwell equations

$$\begin{aligned} \nabla\cdot\mathbf{E} \;&=\; 4\pi\rho_{\rm el}\,, \\ \nabla\cdot\mathbf{B} \;&=\; 4\pi\rho_{\rm mag}\,, \\ \nabla\times\mathbf{E} \;&=\; -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \;-\; \frac{4\pi}{c}\mathbf{J}_{\rm mag}\,, \\ \nabla\times\mathbf{B} \;&=\; +\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \;+\; \frac{4\pi}{c}\mathbf{J}_{\rm el}\,. \end{aligned} \tag{8}$$

These equations are invariant under electric-magnetic dualities (2) provided they act on the charges and currents similarly to how they act on the fields: In matrix notations,

$$\begin{aligned} \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} \;&=\; \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \\ \begin{pmatrix} \rho'_{\rm el} \\ \rho'_{\rm mag} \end{pmatrix} \;&=\; \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \rho_{\rm el} \\ \rho_{\rm mag} \end{pmatrix}, \\ \begin{pmatrix} \mathbf{J}'_{\rm el} \\ \mathbf{J}'_{\rm mag} \end{pmatrix} \;&=\; \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{J}_{\rm el} \\ \mathbf{J}_{\rm mag} \end{pmatrix}, \end{aligned} \tag{9}$$

all for the same angle θ.

Microscopically, the electric charges and currents stem from the existence and motion of the electrically charged elementary particles such as electrons or protons. Likewise, the magnetic charges and currents should stem from the existence and motion of some not-yet-discovered magnetically charged elementary particles better known as *magnetic monopoles.*

A single monopole at rest generates a Coulomb-like magnetic field

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{M}{r^2}\,\mathbf{n} \tag{10}$$

(in Gauss units), which looks like the field of a single North or South pole of a long thin bar magnet, hence the name. The magnetic monopoles play important roles in theoretical high-energy physics, but the specifics are way too complicated for this class. Let me simply say that most unified theories of the fundamental forces predict that the monopoles do exist in Nature, but they are much too heavy to be made by the present-day accelerators, while the primordial monopoles made at the Big Bang was subsequently diluted by the cosmic inflation to such an extent that today the nearest monopole is a few good light years away from Earth. And that's why the monopoles have not been detected yet, and are unlikely to be detected any time soon.

Magnetic Monopoles in Quantum Mechanics

I wish I would teach you a fully quantum theory of EM fields coupled to both electrically and magnetically charged matter fields, but alas... Our current understanding of such theories is rather limited, and most of what's known is way beyond the scope of our introductory QFT class. So let's focus on a much simpler problem: take a classical static monopole generating the magnetic field (10) and consider the quantum motion of an electrically charged particle in this field. It turns out that such motion is inconsistent unless the magnetic charge M of the monopole and the electric charge Q of the quantum particle obey Dirac's quantization condition:

$$M\times Q \;=\; \tfrac{1}{2}\hbar c\times\text{an integer} \tag{11}$$

in Gauss units; in MKSA units this formula becomes

$$M\times Q \;=\; \frac{2\pi\hbar}{\mu_0}\times\text{an integer}, \tag{12}$$

while in the rationalized $\hbar = c = 1$ units we use in this class it becomes simply

$$M\times Q \;=\; 2\pi\times\text{an integer}. \tag{13}$$

A heuristic argument

Let me start with a heuristic argument for the Dirac's quantization condition based on the Aharonov–Bohm effect. The easiest way to visualize a magnetic monopole is by considering a pole of a long, thin magnet or an end point of a long, thin solenoid. Let us choose our coordinates such that the pole is at the origin and the magnet goes along the negative z semi-axis. Then, in spherical coordinates (r, θ, ϕ), everywhere outside the magnet

$$\mathbf{B}(r, \theta, \phi) \;=\; \frac{M}{r^2}\,\mathbf{n}_r \tag{14}$$

where $\mathbf{n}_r$ us the unit vector in the radial direction, while inside the magnet there is magnetic flux $\Phi = 4\pi M$ towards the pole.

Suppose the magnet is infinitely thin, infinitely long and does not interact with the rest of the universe except through the magnetic field it carries. Classically, all one can observe under such circumstances is the magnetic field (14), so for all intents and purposes we have a magnetic monopole of magnetic charge M. (In Gauss units; in rationalized units the magnetic charge is $4\pi M$.) In quantum mechanics however, one can also detect the Aharonov-Bohm effect due to the magnetic flux $\Phi = 4\pi M$ inside the magnet, and that would make the magnet itself detectable along its whole length. Moreover, in quantum field theory the AB effect would disturb the free-wave modes of the charged fields — instead of the plane waves we would get eigen-waves of some $\mathbf{x}$-dependent differential operator. This would give rise to a Casimir effect — a finite and detectable change of the net zero point energy. For a long thin magnet this Casimir energy would be proportional to the magnet's length, so the magnet would behave as a string with finite tension force T. Consequently, the two poles of the magnet would not be able to separate from each other to infinite distance and acts as independent magnetic monopoles. Instead, the North pole and the South pole would pull each other with a finite force T no matter how far they get from each other.

However, the Aharonov–Bohm effect disappears when the magnetic flux $\Phi = 4\pi M$ is an integral multiplet of $2\pi\hbar c/Q$. Consequently, for an infinitely thin magnet there would not be any Casimir effect, hence no string tension, and the poles would be allowed to move independently from each other as if they were separate magnetic monopoles. Since this can happen only when the magnetic flux is not detectable by the AB effect, this gives rise to the Dirac's quantization condition: *For all magnetic monopoles in the universe and for all electrically-charged particles in the universe,*

$$M \times Q \;=\; \tfrac{1}{2}\hbar c \times \text{an integer} \tag{11}$$

(in Gauss units). Hence, *if there is a magnetic monopole anywhere in the universe, then all the electrical charges must be quantized.*

Dirac Monopole

A more rigorous argument was made by P. A. M. Dirac himself years before the discovery of the Aharonov-Bohm effect. Instead of using just one vector potential $\mathbf{A}(\mathbf{x})$ to describe the magnetic field of a monopole, Dirac have used two potentials $\mathbf{A}_N(\mathbf{x})$ and $\mathbf{A}_S(\mathbf{x})$ related by a gauge transform. From the mathematical point of view, Dirac monopole is a *gauge bundle*, a construction that generalizes multiple coordinate patches in Riemannian geometry.

Most Riemannian manifolds cannot be covered by a single coordinate system without singularities or multi-valuedness. Instead, one covers the manifold with several overlapping patches and uses different coordinate systems for each patch. This is OK as long as: (1) the patches overlap their neighbors and collectively cover the whole manifold; (2) each patch has a single-valued non-singular coordinate system; (3) in the overlap regions, the coordinate systems of the overlapping patches map onto each other without singularities, *i.e.*, the derivatives $\partial x^\mu_{(1)}/\partial x^\nu_{(2)}$ are all finite and the matrix of those derivatives has a non-zero determinant (the Jacobian).

In a gauge bundle, different patches covering a manifold have not only different coordinate systems but also different gauges for the $A^\mu(x)$ and the charged fields. But in the overlap regions, all fields from the overlapping patches are related to each other by a gauge transform, *eg.*,

$$A^{(2)}_\mu(x) \;=\; A^{(1)}_\mu(x) \;-\; \partial_\mu \Lambda^{1,2}(x), \quad \text{each } \Psi^{(2)}_a(x) \;=\; \Psi^{(1)}_a(x) \times \exp\bigl(iq_a \Lambda^{1,2}(x)\bigr) \tag{15}$$

in the overlap between patches 1 and 2. Eq. (15) is written for the abelian gauge symmetry, but there are suitable generalizations to the non-abelian symmetry groups. In string theory, abelian and non-abelian gauge bundles on curved 6D manifolds play a very important role in obtaining effective four-dimensional theories from the ten-dimensional superstring.

Dirac himself did not use the gauge bundle language, he simply divided the space outside the monopole itself into two overlapping regions and wrote different but gauge-equivalent vector potentials for each region. In spherical coordinates (r, θ, ϕ), the Northern region (N) spans latitudes $0 \leq \theta < \pi - \epsilon$ while the Southern region (S) spans $\epsilon < \theta \leq \pi$; the two regions overlap in a broad band around the equator. The vector potentials for the two regions are

$$\begin{aligned} \mathbf{A}_N(r,\theta,\phi) &= M(+1-\cos\theta)\nabla\phi = M\,\frac{+1-\cos\theta}{r\,\sin\theta}\,\mathbf{n}_\phi\,, \\ \mathbf{A}_S(r,\theta,\phi) &= M(-1-\cos\theta)\nabla\phi = M\,\frac{-1-\cos\theta}{r\,\sin\theta}\,\mathbf{n}_\phi\,; \end{aligned} \tag{16}$$

The two potentials are gauge-equivalent:

$$\mathbf{A}_N - \mathbf{A}_S = \frac{2M}{r\,\sin\theta}\,\mathbf{n}_\phi = 2M\nabla\phi \tag{17}$$

so they lead to the same magnetic field, namely (14). Indeed,

$$\begin{aligned} \nabla\times\mathbf{A}_{\rm N\,or\,S} &= \nabla\times\left(M(\pm1-\cos\theta)\nabla\phi\right) \\ &= M\left(\nabla(\pm1-\cos\theta)\right)\times\nabla\phi \\ &= M\,\frac{\sin\theta\,\mathbf{n}_\theta}{r}\times\frac{\mathbf{n}_\phi}{r\sin\theta} \\ &= M\,\frac{\mathbf{n}_r}{r^2}\,. \end{aligned} \tag{18}$$

The vector potentials (16) may be analytically continued to the entire 3D space (except the monopole point $r = 0$ itself), but such continuations are singular: The $\mathbf{A}_N(r, \theta, \phi)$ has a so-called "Dirac string" of singularities along the negative z semi-axis ($\theta = \pi$), while the $\mathbf{A}_S(r, \theta, \phi)$ has a similar Dirac string of singularities along the positive z semi-axis ($\theta = 0$). To make a non-singular picture of the monopole field, Dirac used both vector potentials $\mathbf{A}_N$ and $\mathbf{A}_S$ but restricted each potential to the region of space where it is not singular. The two regions overlap, and in the overlap we may use either $\mathbf{A}_N$ or $\mathbf{A}_S$, whichever we like.

In QFT or even in quantum mechanics, a gauge transform of the vector potential should be accompanied by a phase transform of the charged fields or charged particles' wave functions. Consequently, for each charged species we must use different charged $\Psi^N(\mathbf{x})$ and $\Psi^S(\mathbf{x})$ in the Northern and Southern regions; in the overlap $\epsilon < \theta < \pi - \epsilon$, the two fields for the same species are related according to eq. (15). For the gauge transform (17) in question,

$$\Lambda(r,\theta,\phi) = 2M\phi \quad \Longrightarrow \quad \Psi_a^N(r,\theta,\phi) = \Psi_a^S(r,\theta,\phi)\cdot\exp\left(i\frac{Q}{\hbar c}\times 2M\phi\right). \tag{19}$$

Both $\Psi^N(\mathbf{x})$ and $\Psi^S(\mathbf{x})$ are single-valued functions of $\mathbf{x}$ everywhere they are defined. In the overlap region both functions are defined and both are single valued, so the phase factor $\exp(i\frac{2QM}{\hbar c}\phi)$ in eq. (19) must be single valued. This single-valuedness requires an integer value of $2QM/\hbar c$, hence the Dirac quantization condition

$$M \times Q = \tfrac{1}{2}\hbar c \times \text{an integer}. \tag{11}$$

Suggested Reading: J. J. Sakurai, *Modern Quantum Mechanics*, §2.7.

Implications for QFT and Electric–Magnetic Duality

In quantum field theory, for every existing *species* of a charged particle there are countless virtual particles of that species everywhere. Therefore, if as much as a single magnetic monopole exist anywhere in the Universe, then the electric charges of all particle species must be quantized,

$$Q = \frac{\hbar c}{2M} \times \text{an integer}. \tag{20}$$

Historically, Dirac discovered the magnetic monopole while trying to explain the rather small *value* of the electric charge quantum e — in Gauss units,

$$e^2 \approx \frac{\hbar c}{137}. \tag{21}$$

The monopole gives us an excellent reason for the charge quantization in the first place, but alas it does not explain the value (21) of the quantum, and Dirac was quite disappointed.

BTW, in Gauss units, the electric and the magnetic charges have the same dimensionality. But in light of eqs. (11) and (21), they are quantized in rather different units, e for the electric charges and

$$\frac{\hbar c}{2e} \approx \frac{137}{2} e \tag{22}$$

for the magnetic charges. Of course, as far as the Quantum ElectroDynamics is concerned, the monopoles do not have to exist at all. But if they do exist, their charges must be quantized in units of (22). Also, the very existence of a single monopole would explain the electric charge quantization.

Today, we have other explanations of the electric charge quantization; in particular the Grand Unification of strong, weak and electromagnetic interactions at extremely high energies produces quantized electrical charges. Curiously, the same Grand Unified Theories also predict that there **are** magnetic monopoles with charges (22). More recently, several attempts to unify all the fundamental interactions within the context of the String Theory also gave rise to magnetic monopoles, with charges quantized in units of $N\hbar c/2e$, where N is an integer such as 3 or 5. It was later found that in the same theories, there were superheavy particles with fractional electric charges e/N, so the monopoles in fact had the smallest non-zero charges allowed by the Dirac condition (11)! Nowadays, most theoretical physicists believe that any fundamental theory that provides for exact quantization of the electric charge should also provide for the existence of magnetic monopoles, but this conjecture has not been proved (yet).

In quantum field theories including EM fields coupled to both electrically and magnetically charged particles, the classical electric-magnetic symmetries

$$\begin{aligned} \begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} &= \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}, \\ \begin{pmatrix} Q' \\ M' \end{pmatrix} &= \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} Q \\ M \end{pmatrix} \end{aligned} \tag{23}$$

for the same θ,

are generally incompatible with the Dirac's charge quantization condition (11) and hence

$$Q = m_{\rm el} \times e, \quad M = n_{\rm mag} \times \frac{\hbar c}{2e}. \tag{24}$$

The best we can do is the electric-magnetic duality relating theories with different values of the electric charge quantum e:

$$\begin{aligned} \mathbf{E}' &= \mathbf{B}, \qquad \mathbf{B}' = -\mathbf{E}, \\ n'_{\rm el} &= m_{\rm mag}, \qquad n'_{\rm mag} = -n_{\rm el}, \\ e' &= \frac{\hbar c}{2e}, \end{aligned} \tag{25}$$

or in terms of the dimensionless QED coupling $\alpha = \frac{e^2}{\hbar c}$,

$$\alpha' = \frac{1}{4\alpha} \tag{26}$$

The real-life quantum electrodynamics has rather weak coupling $\alpha \approx 1/137$, so we may use perturbation theory expanding various amplitudes in powers of α. Alas, the QED-like theories with stronger couplings $\alpha \sim 1$ cannot use such perturbation theory, so they are much harder to understand and work with. Fortunately, in the really strong coupling limit $\alpha \gg 1$, we may use the electric-magnetic duality (25) to map the original theory to another theory with a weak coupling $\alpha' = \frac{1}{4\alpha} \ll 1$, and then use a perturbative expansion in powers of the dual coupling α'. This way, we can understand the original theory's behavior in both the weak-coupling and the strong-coupling regimes, and then try to interpolate between the two extremes to the $\alpha \sim 1$ regime.

Angular Momentum of Dyons

A *dyon* is an elementary particle — or a particle-like bound state of several elementary particles — which has both electric and magnetic charges. The net angular momentum of the dyon includes the spins of all the constituent particles, the orbital angular momentum of their relative motion, but there is also a contribution from the momentum density

$$\frac{1}{c}\mathbf{S} = \frac{1}{4\pi c}\mathbf{E} \times \mathbf{B} \tag{27}$$

of the EM fields generated by the dyon. To see how this works, let's consider a simple system of a single static magnetic monopole of charge M located at some point $\mathbf{x}_m$ and a single static electric charge Q located at some other point $\mathbf{x}_q \neq \mathbf{x}_m$. For this system: (1) The net linear momentum of the EM fields happens to vanish,

$$\mathbf{P}_{\text{EM}} = \frac{1}{4\pi c} \iiint \mathbf{E} \times \mathbf{B}\, d^3\mathbf{x} = 0. \tag{28}$$

Consequently, the net angular momentum of the EM fields

$$\mathbf{L}_{\text{EM}} = \frac{1}{4\pi c} \iiint \mathbf{x} \times (\mathbf{E} \times \mathbf{B})\, d^3\mathbf{x} \tag{29}$$

does not depend on our choice of the coordinate origin (acting as the pivot point). However, (2) this EM angular momentum does not vanish. Instead, it has magnitude MQ/c regardless of the distance between the electric and the magnetic charges, while it's direction points from the electric charge towards the monopole. (J. J. Thomson, 1904.)

In 1936, M. N Saha used this result to derive what's now called Dirac's quantization rule

$$M \times Q = \frac{\hbar c}{2} \times \text{an integer} \tag{30}$$

slightly before Dirac, albeit in a much less rigorous fashion. Basically, he argues that *semi-classically*, any kind of angular momentum of a quantum system should have magnitude $\frac{1}{2}\hbar \times$ an integer. Imposing this requirement on the Thomson's EM angular momentum $L_{\text{EM}} = |MQ|/c$ then immediately leads to the charge quantization rule (30). Soon after Saha, Dirac re-derived this rule in a much more rigorous fashion, and that's why it's called the Dirac's quantization rule rather than the Saha's rule.

Proofs: Let's start by verifying that the net linear momentum is zero. In eq. (28),

$$\mathbf{E} \times \mathbf{B} = -(\nabla\Phi) \times \mathbf{B} = -\nabla \times (\Phi\mathbf{B}) + \Phi(\nabla \times \mathbf{B}), \tag{31}$$

where the monopole's magnetic field

$$\mathbf{B}(\mathbf{x}) = M\frac{\mathbf{x} - \mathbf{x}_m}{|\mathbf{x} - \mathbf{x}_m|^3} = -\nabla\left(\frac{M}{|\mathbf{x} - \mathbf{x}_m|}\right) \tag{32}$$

has zero curl $\nabla\times\mathbf{B}=0$, even at the monopole's location $\mathbf{x}=\mathbf{x}_m$. Hence, eq. (31) becomes

$$\mathbf{E}\times\mathbf{B} \;=\; -\nabla\times(\Phi\mathbf{B}) \tag{33}$$

and therefore

$$P^i_{\rm EM} \;=\; -\frac{1}{4\pi c}\iiint\epsilon^{ijk}\nabla^j(\Phi B^k)\,d^3\mathbf{x} \;=\; -\frac{1}{4\pi c}\iint\epsilon^{ijk}\Phi B^k\,d^2{\rm area}^j, \tag{34}$$

or in vector notations

$$\mathbf{P}_{\rm EM} \;=\; +\frac{1}{4\pi c}\iint d^2\mathbf{area}\times\mathbf{B}\Phi. \tag{35}$$

The surface integral here is over the surface of the integration volume in eq. (28), and since that volume was over the whole 3D space, its surface can be thought as as $R\to\infty$ limit of the sphere of radius R. In the $R\to\infty$ limit, the electrostatic potential of the $\mathbf{E}$ field behaves as $1/R$, the magnetic field of the monopole as $1/R^2$, the area as R^2, so the whole integral behaves as $1/R$ — if not smaller — and vanishes for $R\to\infty$. Consequently, the net momentum of the EM field indeed vanishes.

Now consider the net angular momentum of the EM field. Since the net linear momentum happen to vanish, we are free to put the coordinate origin wherever we like, so let's put it at the monopole's location. Consequently, in our coordinate system

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{M\mathbf{n}}{r^2}\,, \tag{36}$$

hence

$$\begin{aligned}\mathbf{x}\times(\mathbf{E}\times\mathbf{B}) \;&=\; r\mathbf{n}\times\left(\mathbf{E}\times\frac{M\mathbf{n}}{r^2}\right)\\ &=\; \frac{M}{r}\big(\mathbf{n}\times(\mathbf{E}\times\mathbf{n})\big)\\ &=\; \frac{M}{r}\big(\mathbf{E}\,-\,(\mathbf{E}\cdot\mathbf{n})\mathbf{n}\big).\end{aligned} \tag{37}$$

In index notations, this formula becomes

$$\big[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})\big]^i \;=\; ME^j\,\frac{\delta^{ij}-n^in^j}{r} \;=\; ME^j\nabla^jn^i, \tag{38}$$

hence eq. (29) for the angular momentum yields

$$\begin{aligned} L^i_{\rm EM} &= \frac{1}{4\pi c}\iiint\big[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})\big]^i\,d^3\mathbf{x} \\ &= \frac{1}{4\pi c}\iiint ME^j\nabla^j n^i\,d^3\mathbf{x} \\ &= \frac{M}{4\pi c}\iint n^iE^j\,d^2{\rm area}^j \;-\; \frac{M}{4\pi c}\iiint(\nabla^jE^j)n^i\,d^3\mathbf{x}\,. \end{aligned} \tag{39}$$

Again, the surface integral on the bottom line here is over a sphere of radius $R\to\infty$. For R much larger that the distance between the electric charge and the monopole, we may approximate

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{Q(\mathbf{x}-\mathbf{x}_q)}{|\mathbf{x}-\mathbf{x}_q|^3} \;\approx\; \frac{Q\mathbf{x}}{|\mathbf{x}|^3} \;=\; \frac{Q\mathbf{n}}{r^2} \tag{40}$$

hence

$$E^j\,d^2{\rm area}^j \;=\; \mathbf{E}\cdot d^2\mathbf{area} \;=\; \frac{Q}{R^2}\,R^2d^2\Omega \;=\; Q\,d^2\Omega \tag{41}$$

and therefore

$$\iint n^iE^j\,d^2{\rm area}^j \;\approx\; \iint n^iQ\,d^2\Omega \;=\; 0. \tag{42}$$

As to the volume integral on the bottom line of eq. (39), for the Coulomb field of the electric charge Q located at $\mathbf{x}_q$,

$$\nabla^jE^j(\mathbf{x}) \;=\; \nabla\cdot\mathbf{E}(\mathbf{x}) \;=\; 4\pi Q\delta^{(3)}(\mathbf{x}-\mathbf{x}_q), \tag{43}$$

hence

$$-\frac{M}{4\pi c}\iiint(\nabla^jE^j)n^i\,d^3\mathbf{x} \;=\; -\frac{MQ}{c}\iiint n^i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_q) \;=\; -\frac{MQ}{c}\,n^i(\mathbf{x}_q). \tag{44}$$

Thus altogether,

$$\mathbf{L}_{\rm EM} \;=\; \mathbf{0} \;-\; \frac{MQ}{c}\,\mathbf{n}(\mathbf{x}_q) \tag{45}$$

where $\mathbf{n}(\mathbf{x}_q)$ is the direction of the radius vector of the electric charge in the coordinate system whose origin is at the magnetic monopole. In a more general coordinate system,

$\mathbf{n}(\mathbf{x}_q)$ becomes the unit vector pointing from the monopole to the electric charge, thus

$$\mathbf{n}(\mathbf{x}_q) \;\to\; \frac{\mathbf{x}_q - \mathbf{x}_m}{|\mathbf{x}_q - \mathbf{x}_m|} \tag{46}$$

and therefore

$$\mathbf{L}_{\text{net}} \;=\; \frac{MQ}{c}\,\frac{\mathbf{x}_m - \mathbf{x}_q}{|\mathbf{x}_q - \mathbf{x}_m|}\,. \tag{47}$$

Quod erat demonstrandum.

Although the angular momentum (47) stems from the EM fields, it combines with the particles' orbital and spinor angular momenta into a single conserved quantity. For example, problem 3 of your current homework#5 considers a spinless particle of electric charge q orbiting a static magnetic monopole of charge M. For this system, you should verify (among other things) that the conserved angular momentum is the combination

$$\hat{\mathbf{J}}_{\text{net}} \;=\; \hat{\mathbf{L}}_{\text{particle}} \;+\; \hat{\mathbf{L}}_{\text{EM}} \;=\; \hat{\mathbf{x}}\times\hat{\vec{\pi}} \;-\; \frac{Mq}{c}\,\frac{\hat{\mathbf{x}}}{\hat{r}}\,. \tag{48}$$

Also, the 3 components of this net angular momentum obey the usual commutation relation $[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$ and form the usual $|j,m\rangle$ multiplets where m runs from $-j$ to $+j$ by 1. However, the values of j are restricted to

$$\begin{aligned} j \;&=\; j_{\text{min}} \;+\; \text{a non-negative integer} \\ \text{where } j_{\text{min}} \;&=\; \frac{|Mq|}{\hbar c}\,. \end{aligned} \tag{49}$$

In particular, if $Mq/\hbar c$ happens to be a half-integer, then j must be half-integer rather than integer!

Now consider a dyon — a bound state of magnetic monopole(s) and electrically charged particle(s) with an overall magnetic charge $M \neq 0$ and electric charge $Q \neq 0$. Assuming none of the constituent particles have a spin, the net angular momentum of the dyon is

$$\hat{\mathbf{J}}_{\text{net}} \;=\; \hat{\mathbf{L}}^{\text{net}}_{\text{particles}} \;+\; \hat{\mathbf{L}}^{\text{net}}_{\text{EM}}\,, \tag{50}$$

and the allowed values of the net j are as in eq. (49): $j = (|MQ|/\hbar c)+$a non-negative integer. Moreover, when you move two dyons around each other, their combined wavefunction picks an extra phase due to electric charges moving through monopole magnetic fields. And when the similar dyons happen to end up exchanging their location, this extra phase turns out to be an overall sign

$$(-1)^{2MQ/\hbar c} \;=\; \begin{cases} +1 & \text{for integer } MQ/\hbar c, \\ -1 & \text{for half-integer } MQ/\hbar c. \end{cases} \tag{51}$$

This formula affirms the spin-statistics theorem for the dyons: If a dyon has integral $MQ/\hbar c$ then it has integer net angular momentum j and bosonic statistics, while if its $MQ/\hbar c$ is half-integral, then it has a half-integer j and fermionic statistics. Either way, the net j agrees with the statistics!

More generally, the monopoles and the electric charges comprising a dyon may have spins, in which case

$$\hat{\mathbf{J}}_{\text{net}} \;=\; \hat{\mathbf{L}}^{\text{net}}_{\text{particles}} \;+\; \hat{\mathbf{L}}^{\text{net}}_{\text{EM}} \;+\; \hat{\mathbf{S}}^{\text{net}}\,, \tag{52}$$

so the values of the net j being integer or half-integer according to

$$(-1)^{2j} \;=\; \prod_{\text{particles}} (-1)^{2s} \times (-1)^{2MQ/\hbar c}\,. \tag{53}$$

Again, the EM fields contribute non-trivially to the net angular momentum of the dyon, and also — via extra phases due to moving electric charges in monopole magnetic fields — to the statistics of multiple dyons. And in every case, the statistics matches other overall j: bosonic foe integer j and fermionic for half-integer j.