This exam has three problems, the first two problems about QED with charged scalars, and the third problem about spontaneous symmetry breaking.

Similar to the midterm exam, please do not waste time and paper by copying the homework solutions, or supplementary notes, or the textbook, or anything I have explicitly derived in class. Simply quote whichever formula you need and use it.

1. Let's extend the Quantum Electrodynamics theory to include not only the photons γ and the electrons $e^{\mp}$ but also the scalar particles $S^{\mp}$. The Lagrangian of this extended theory is

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D}-m)\Psi +\; D_\mu\Phi^* D^\mu\Phi \;-\; M^2\Phi^*\Phi \;=\; \mathcal{L}^{\rm free} + \mathcal{L}^{\rm int}, \qquad (1)$$

where the quadratic part $\mathcal{L}^{\rm free}$ describes the free fields while the cubic+quartic part $\mathcal{L}^{\rm int}$ describes their interactions. In the Feynman rules, the propagators and the external lines follow from the $\mathcal{L}^{\rm free}$ while the vertices follow from the $\mathcal{L}^{\rm int}$. Altogether, the Feynman rules for QED with both electrons and charged scalars are as follows:

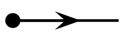
Photon propagator: A^μ  A^ν ($q \to$) $\;=\; \dfrac{-ig^{\mu\nu}}{q^2+i0}\,,$ (F.1)

Incoming photon:  $\;=\; \mathcal{E}_\mu(k,\lambda),$ (F.2)

Outgoing photon:  $\;=\; \mathcal{E}^*_\mu(k,\lambda),$ (F.3)

Electron propagator: $\overline{\Psi}$  Ψ ($q \to$) $\;=\; \dfrac{i}{\not{q}-m+i0}\,,$ (F.4)

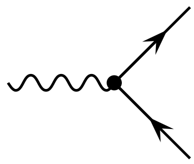
Incoming e^- or outgoing e^+:  $\;=\; u(p,s)$ or $v(p,s),$ (F.5)

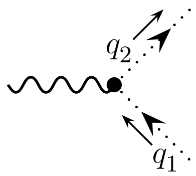
Outgoing e^- or incoming e^+:  $\;=\; \bar{u}(p,s)$ or $\bar{v}(p,s),$ (F.6)

Scalar propagator: Φ^*  Φ ($q \to$) $\;=\; \dfrac{i}{q^2-M^2+i0}\,,$ (F.7)

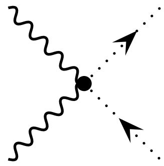
Incoming S^- or outgoing S^+:  $\;=\; 1,$ (F.8)

Outgoing S^- or incoming S^+:  $\;=\; 1,$ (F.9)

QED vertex $ee\gamma$:  $= +ie\gamma^\mu,$ (F.10)

Scalar QED vertex $SS\gamma$: 

$= +ie(q_1 + q_2)^\mu$ (F.11)

Seagull vertex $SS\gamma\gamma$:  $= +2ie^2\, g^{\mu\nu}.$ (F.12)

Note: the dotted lines (F.7–9) for the charged scalars have arrows. Also note that in the $S^-S^+\gamma$ vertex (F.11), the directions of momenta q_1 and q_2 must agree with the arrows of the scalar lines; otherwise, the vertex becomes $+ie(q_1 - q_2)^\mu$ or $+ie(-q_1 + q_2)^\mu$ or $+ie(-q_1 - q_2)^\mu$.

(a) The QED Feynman rules (F.1–6) and (F.10) were explained in class. Explain the remaining rules (F.7–9) and (F.11–12) in terms of the Lagrangian (1).
Note: don't re-derive the Feynman rules as such, just explain why the scalar propagators and external lines have arrows, why do those arrows point as in eqs. (F.7–9), and why the vertices are as in eqs. (F.11–12).

(b) Given the Feynman rules, draw the tree diagram(s) for the scalar pair production $e^-e^+ \to S^-S^+$ and calculate the tree-level amplitude $\left\langle S^-, S^+ \right| \mathcal{M} \left| e^-, e^+ \right\rangle$.
Hint: Mind the arrow directions on the dotted lines of scalars.

(c) Average $|\mathcal{M}|^2$ over the incoming particles' spins and calculate the partial cross-section for the scalar pair production.
For simplicity, neglect the electron's mass m. But don't neglect the scalar's mass M.

(d) Compare the angular dependence of the scalar pair production to the muon production we have studied in class. Also, calculate the total cross-section $\sigma_{\text{tot}}(e^-e^+ \to S^-S^+)$ and compare its energy dependence and its high-energy value to that of the $\sigma_{\rm tot}(e^-e^+ \to \mu^-\mu^+)$.

2. Continuing the first problem about QED with charged scalars, consider the annihilation of a scalar and an anti-scalar into two photons, $S^+S^- \to \gamma\gamma$.

   (a) Draw and evaluate **all** tree diagrams contributing to the $\langle\gamma\gamma|\,\mathcal{M}\,|S^+S^-\rangle$ amplitude. Make sure the amplitude respects the Bose symmetry between the two photons.

   (b) Write the tree amplitude as $\mathcal{M} = \mathcal{M}^{\mu\nu} \times \mathcal{E}^*_\mu(k_1,\lambda_1)\mathcal{E}^*_\nu(k_2,\lambda_2)$ and verify the Ward identities

   $$k_1^\mu \times \mathcal{M}_{\mu\nu} \;=\; 0, \qquad k_2^\nu \times \mathcal{M}_{\mu\nu} \;=\; 0. \tag{2}$$

   Hint: If these identities seem to be broken, go back to part (a) and make sure you have not missed a diagram. If this does not help, check your signs.

   (c) Sum $|\mathcal{M}|^2$ over the outgoing photon polarizations and calculate the partial cross-section of the $S^+S^- \to \gamma\gamma$ annihilation.

   - For simplicity, assume $E \gg M$ and neglect the scalar mass M in your calculations.

   $\star$ Extra credit if you do take M into account, and do it right. But beware: the kinematics is much messier for $M \neq 0$, and you might need several hours just to work through the algebra. If you use Mathematica, make sure to liberally comment your code. Also, if you sum over photon polarizations using a different method from what I have explained in class, you must prove that your method works.

3. Finally, a problem on spontaneous symmetry breaking and Higgs mechanism. Consider an $SU(3)$ gauge theory coupled to chiral fermions — two triplets ψ^i_L and χ^i_L of left-handed Weyl fermions but without any right-handed Weyl fermions — and to a triplet ϕ^i of complex scalars. The Lagrangian of the theory is

$$\begin{aligned}\mathcal{L} \;=\; & -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; D_\mu\phi^\dagger D^\mu\phi \;+\; i\psi^\dagger_L\bar\sigma^\mu D_\mu\psi_L \;+\; i\chi^\dagger_L\bar\sigma^\mu D_\mu\chi_L \\ & +\; y\epsilon_{ijk}\phi^i(\chi^j_L)^\top\sigma_2\psi^k_L \;+\; y\epsilon^{ijk}\phi^*_i\chi^\dagger_{Lj}\sigma_2\psi^*_{Lk} \;-\; \frac{\lambda}{2}\left(\phi^\dagger\phi \,-\, \frac{v^2}{2}\right)^2,\end{aligned} \tag{3}$$

where $y \times \epsilon_{ijk}$ is the $SU(3)$–invariant set of Yukawa couplings, in the scalar potential $\lambda > 0$ and $v^2 > 0$, and the gauge fields A^a_μ and the gauge coupling g are hiding inside

$$\begin{aligned} F^a_{\mu\nu}(x) &= \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - g f^{abc} A^b_\mu(x) A^c_\nu(x), && (4)\\ D_\mu \phi^i(x) &= \partial_\mu \phi^i(x) + ig A^a_\mu(x) (\tfrac{1}{2}\lambda^a)^i{}_j \phi^j(x), && (5)\\ D_\mu \phi^*_i(x) &= \partial_\mu \phi^*_i(x) - ig A^a_\mu(x) \phi^*_j(x) (\tfrac{1}{2}\lambda^a)^j{}_i\,, && (6)\\ D_\mu \psi^i_L(x) &= \partial_\mu \psi^i_L(x) + ig A^a_\mu(x) (\tfrac{1}{2}\lambda^a)^i{}_j \psi^j_L(x), && (7)\\ D_\mu \chi^i_L(x) &= \partial_\mu \chi^i_L(x) + ig A^a_\mu(x) (\tfrac{1}{2}\lambda^a)^i{}_j \chi^j_L(x), && (8) \end{aligned}$$

where λ^a (for $a = 1, \ldots, 8$) are the $SU(3)$ Gell-Mann matrices — see this Wikipedia page for their explicit form, — and f^{abc} are the $SU(3)$ structure constants, $[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c$.

The scalar potential of the theory has a degenerate family of minima, namely an S^5 sphere in the $\mathbf{C}^3 = \mathbf{R}^6$ scalar field space. All these minima are related by the $SU(3)$ gauge symmetries to

$$\langle\phi\rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \qquad (9)$$

(a) The scalar VEV (9) spontaneously breaks some of the $SU(3)$ gauge symmetries but not all of them. Which gauge bosons become massive due to this Higgs mechanism and which remain massless? What is the unbroken gauge symmetry of the theory?

(b) Besides the local $SU(3)$ symmetry, the Lagrangian (3) has global $SU(2) \times U(1)$ symmetries. Spell out how these global symmetries act on the fermionic and the scalar fields of the theory, and verify that the Lagrangian is indeed invariant under these symmetries.

(c) How does the scalar VEV (9) affect the global $SU(2) \times U(1)$ symmetries of the theory? Do they break down spontaneously, survive unbroken, or mix up with the gauge symmetries? Should there be any massless Goldstone bosons? If yes, what would be their quantum numbers WRT the unbroken symmetries?

(d) Fix the unitary gauge (for the broken gauge symmetries only!), and find the masses of the remaining scalars and of all the gauge fields.

(e) Write down the quantum numbers of all the bosons with respect to all the unbroken continuous symmetries of the theory — local, global, or mixed, — and check that the fields belonging to the same multiplet have the same mass.

(f) Now find the fermion masses arising from the Yukawa couplings to the Higgs VEV. Which fermions become massive and which remain massless?

(g) Also, write down the quantum numbers of all the fermions with respect to all the unbroken continuous symmetries of the theory — local, global, or mixed, — and check that the fields belonging to the same multiplet have the same mass.

(h) Finally, turn the conjugates of the LH Weyl fermions ψ_L^i (but not the χ_L^i) into RH Weyl fermions

$$\chi_{R,i} = \sigma_2 \psi^*_{L,i}, \tag{10}$$

and then combine these RH Weyl fermions with the remaining LH Weyl fermions χ_L^i into Dirac fermions $\Psi^{1,2,3}$ of definite mass.

Also, write down how the unbroken gauge symmetries symmetries and the global or mixed $U(1)$ symmetry act on these Dirac fermions.

($\star$) For extra credit, spell out how the $SU(2)_{\text{global}}$ acts on the Dirac fermions $\Psi^{1,2,3}(x)$. Hint: this symmetry mixes the $\Psi^i(x)$ with their charge conjugates $\Psi^c_i(x) = \gamma^2\Psi^*_i(x)$.