PHY–396 K. Problem set #1. Due September 4, 2024.

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0. First of all, refresh your memory of special relativity. Make sure you understand index summation conventions in Minkowski or Euclidean spaces. If you don't understand (or have hard time deciphering) expressions such as $B_i = \epsilon_{ijk}\partial_j A_k$ (in 3 space dimensions) or $\partial_\mu F^{\mu\nu} = J^\nu$ (in the Minkowski spacetime), *get up to speed ASAP* or you would not be able to follow the class.

1. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \tag{1}$$

where $c = \hbar = 1$, $F_{\mu\nu} \stackrel{\rm def}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (1) is *not* gauge invariant.

(a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.

(b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu \;=\; 0 \qquad \text{and} \qquad (\partial^2 + m^2)A^\mu = J^\mu. \tag{2}$$

2. Next, consider a complex (*i.e.*, complex-number valued) scalar field $\Phi(x)$ with the Lagrangian density

$$\mathcal{L} \;=\; (\partial_\mu\Phi^*)(\partial^\mu\Phi) \;-\; m^2\Phi^*\Phi \;-\; \frac{\lambda}{4}\,(\Phi^*\Phi)^2. \tag{3}$$

For the complex fields, the infinitesimal variations $\delta\Phi(x)$ and $\delta\Phi^*(x)$ are linearly independent from each other, hence linearly independent Euler–Lagrange field equations of motion

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right) \;-\; \frac{\partial\mathcal{L}}{\partial\Phi} \;=\; 0 \qquad \text{and} \qquad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\right) \;-\; \frac{\partial\mathcal{L}}{\partial\Phi^*} \;=\; 0. \tag{4}$$

(a) Spell out the Euler–Lagrange equation for the complex scalar field with the Lagrangian density (3).

(b) Show that when the fields $\Phi(x)$ and $\Phi^*(x)$ obey their Euler–Lagrange equations, the *current*

$$J^\mu(x) \stackrel{\text{def}}{=} 2\,\mathrm{Im}(\Phi^*\partial^\mu\Phi) \;=\; -i\Phi^*\partial^\mu\Phi \;+\; i\Phi\partial^\mu\Phi^* \tag{5}$$

is conserved, $\partial_\mu J^\mu = 0$.

Note: the current J^μ here is not the electric current and it does not couple to the EM fields. Instead, the corresponding global charge $\int d^3\mathbf{x}\, J^0$ is some kind of a conserved quantum number similar to the baryon number or the lepton number.

Later in class we shall learn that the current (5) and its conservation are related by the Noether theorem to the phase symmetry of the complex scalar field:
$\Phi(x) \to e^{i\theta}\Phi(x)$, $\Phi^*(x) \to e^{-i\theta}\Phi^*(x)$, but $\mathcal{L}$ remain invariant.

3. In spacetimes of higher dimensions $D > 4$, there are antisymmetric-tensor fields analogous to the EM-like vector fields; such tensor fields play important roles in supergravity and string theory.

For example, consider a free 2-index antisymmetric tensor field $B_{\mu\nu}(x) \equiv -B_{\nu\mu}(x)$, where μ and ν are D-dimensional Lorentz indices running from 0 to $D-1$. To be precise, $B_{\mu\nu}(x)$ is the *tensor potential*, analogous to the electromagnetic vector potential $A_\mu(x)$. The analog of the EM tension fields $F_{\mu\nu}(x)$ is the antisymmetric 3–index tension tensor

$$H_{\lambda\mu\nu}(x) \;=\; \partial_\lambda B_{\mu\nu} \;+\; \partial_\mu B_{\nu\lambda} \;+\; \partial_\nu B_{\lambda\mu}\,. \tag{6}$$

(a) Check that this tensor is totally antisymmetric in all 3 indices.

(b) Show that regardless of the Lagrangian, the H fields satisfy differential identities

$$\tfrac{1}{6}\partial_{[\kappa}H_{\lambda\mu\nu]} \;\equiv\; \partial_\kappa H_{\lambda\mu\nu} \;-\; \partial_\lambda H_{\mu\nu\kappa} \;+\; \partial_\mu H_{\nu\kappa\lambda} \;-\; \partial_\nu H_{\kappa\lambda\mu} \;=\; 0. \tag{7}$$

PS: By differential identities I mean higher-dimensional generalizations of the vector calculus identities $\mathbf{curl}(\mathbf{grad}\, s) = 0$ and $\mathbf{div}(\mathbf{curl}\,\mathbf{v}) = 0$. In the language of *differential forms,* all such identities are special cases of $d^2 = 0$ for the *exterior derivative* d.

(c) The Lagrangian for the free $B_{\mu\nu}(x)$ fields is simply

$$\mathcal{L}(B, \partial B) \;=\; \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \tag{8}$$

where $H_{\lambda\mu\nu}$ are as in eq. (6). Treating the $B_{\mu\nu}(x)$ as $\frac{1}{2}D(D-1)$ independent fields, derive their equations of motion.

Similar to the EM fields, the $B_{\mu\nu}$ fields are subject to *gauge transforms*

$$B'_{\mu\nu}(x) \;=\; B_{\mu\nu}(x) \;+\; \partial_\mu \Lambda_\nu(x) \;-\; \partial_\nu \Lambda_\mu(x) \tag{9}$$

where $\Lambda_\mu(x)$ is an arbitrary vector field.

(d) Show that the tension fields $H_{\lambda\mu\nu}(x)$ — and hence the Lagrangian (8) — are invariant under such gauge transforms.

In spacetimes of sufficiently high dimensions D, one may have similar tensor fields with more indices. Generally, the potentials form a p-index totally antisymmetric tensor $C_{\mu_1\mu_2\cdots\mu_p}(x)$, the tensions form a $p+1$ index tensor

$$\begin{aligned} G_{\mu_1\mu_2\cdots\mu_{p+1}} \;&=\; \frac{1}{p!}\,\partial_{[\mu_1} C_{\mu_2\cdots\mu_p\mu_{p+1}]} \\ &\equiv\; \partial_{\mu_1} C_{\mu_2\cdots\mu_{p+1}} \;-\; \partial_{\mu_2} C_{\mu_1\mu_3\cdots\mu_{p+1}} \;+\; \cdots \;+\; (-1)^p \partial_{\mu_{p+1}} C_{\mu_1\cdots\mu_p}\,, \end{aligned} \tag{10}$$

also totally antisymmetric in all its indices, and the Lagrangian is

$$\mathcal{L}(C, \partial C) \;=\; \frac{(-1)^p}{2(p+1)!}\, G_{\mu_1\mu_2\cdots\mu_{p+1}} G^{\mu_1\mu_2\cdots\mu_{p+1}}. \tag{11}$$

(e) Derive the differential identities and the equations of motion for the G fields.

(f) Show that the tension fields $G_{\mu_1\mu_2\cdots\mu_{p+1}}(x)$ — and hence the Lagrangian (11) — are invariant under gauge transforms of the potentials $C_{\mu_1\mu_2\cdots\mu_p}(x)$ which act as

$$C'_{\mu_1\mu_2\cdots\mu_p}(x) \;=\; C_{\mu_1\mu_2\cdots\mu_p}(x) \;+\; \frac{1}{(p-1)!}\,\partial_{[\mu_1} \Lambda_{\mu_2\cdots\mu_p]}(x) \tag{12}$$

where $\Lambda_{\mu_2\cdots\mu_p}(x)$ is an arbitrary $(p-1)$-index tensor field (totally antisymmetric).

4. Finally, consider the electric-magnetic duality and its generalization to the antisymmetric tensor fields from the previous problem. Let's start with the EM field $F^{\mu\nu}$ in the ordinary $3+1$ dimensions. The dual field to the $F^{\mu\nu}(x)$ is the

$$\widetilde{F}_{\kappa\lambda}(x) \stackrel{\text{def}}{=} \tfrac{1}{2}\epsilon_{\kappa\lambda\mu\nu}F^{\mu\nu}(x), \tag{13}$$

where $\epsilon_{\kappa\lambda\mu\nu}$ is the Levi–Civita totally antisymmetric tensor, or in 3D components

$$\widetilde{\mathbf{E}}(x) \stackrel{\text{def}}{=} -\mathbf{B}(x), \quad \widetilde{\mathbf{B}}(x) \stackrel{\text{def}}{=} +\mathbf{E}(x). \tag{14}$$

(a) Suppose the $F^{\mu\nu}(x)$ field obeys Maxwell equations for zero current $J^\mu(x) \equiv 0$. Show that in this case, the dual field $\widetilde{F}^{\mu\nu}(x)$ also obeys Maxwell equations for zero dual current $\widetilde{J}^\mu(x) \equiv 0$.

Now let's generalize the duality (13) to other types of antisymmetric tensor fields. In a spacetime of D dimensions ($D-1$ space, 1 time), a p-index tensor field $C_{\mu_1,\ldots,\mu_p}(x)$ and a q-index tensor field $\widetilde{C}_{\mu_1,\ldots,\mu_q}(x)$ with $p+q=D-2$ are considered dual to each other when the corresponding tension tensors $G_{\mu_1,\ldots,\mu_{p+1}}(x)$ and $\widetilde{G}_{\mu_1,\ldots,\mu_{q+1}}(x)$ are related to each other as

$$\begin{aligned}
\widetilde{G}_{\mu_1,\ldots,\mu_{q+1}}(x) &= \frac{1}{(p+1)!}\epsilon_{\mu_1,\ldots,\mu_D}\, G^{\mu_{q+2},\ldots,\mu_D}(x) \\
&\Updownarrow \\
G_{\mu_1,\ldots,\mu_{p+1}}(x) &= \frac{\pm 1}{(q+1)!}\epsilon_{\mu_1,\ldots,\mu_D}\, \widetilde{G}^{\mu_{p+2},\ldots,\mu_D}(x) \\
\text{for } \pm 1 &= (-1)^{D-1}\times(-1)^{(p+1)(q+1)} = \begin{cases} (-1)^p = (-1)^q & \text{for even } D, \\ +1 & \text{for odd } D. \end{cases}
\end{aligned} \tag{15}$$

(b) Show that the field equations — which include both the differential identities and the Lagrangian equation of motion — for the G and $\widetilde{G}$ tension fields are equivalent to each other. Specifically, the differential identity for the G field is equivalent to the free equation of motion for the $\widetilde{G}$ field while the free equation of motion for the G field is equivalent to the differential identity for the $\widetilde{G}$ field.

(c) For an example of such duality between the tension fields and the equations they obey, consider the two-index potential field $B^{\mu\nu}(x) = -B^{\nu\mu}(x)$ in $D = 3 + 1$ dimensions and show that it's dual to a *massless* scalar field $\phi(x)$.

Note: the ϕ field could be a true scalar or a pseudoscalar, depending on the parity of the B tensor, but the parity issue is beyond the scope of this homework.