1. Let's go back to the massive vector field $\hat{A}^\mu(x)$ from the previous homework set#2 (problems 2–4). In particular, in problem 4 you (should have) expanded the quantum fields $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ into the (box-normalized) creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}$.

   (a) Change the operator normalization to continuum (rather than box) and relativistic, and show that in this normalization

$$\hat{\mathbf{A}}(\mathbf{x}) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\left(e^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}\,+\,e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}\right) \tag{1}$$

   in the Schrödinger picture, and hence

$$\hat{\mathbf{A}}(\mathbf{x},t) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\left(e^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(t)\,+\,e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(t)\right) \tag{2}$$

   in the Heisenberg picture.

   (b) Solve the Heisenberg equations for the creation and annihilation operators, plug the solutions into eq. (2) and show that it becomes

$$\hat{\mathbf{A}}(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\left(e^{-ikx}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(0)\;+\;e^{+ikx}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}} \tag{3}$$

   where $kx \stackrel{\text{def}}{=} k^\mu x_\mu = k^0 t - \mathbf{k}\cdot\mathbf{x}$ for $k^0 = +\omega_{\mathbf{k}}$.

   (c) Write down similar expansion for the electric field $\hat{\mathbf{E}}(\mathbf{x},t)$ and hence for the scalar potential

$$\hat{A}^0(\mathbf{x},t) \;=\; -\frac{1}{m^2}\,\nabla\cdot\hat{\mathbf{E}}(\mathbf{x},t)\quad\big[\text{when } J^0=0\big]. \tag{4}$$

   (d) Combine the results of parts (b) and (c) into a relativistic formula for the 4–vector field $\hat{A}^\mu(x)$,

$$\hat{A}_\mu(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\sum_\lambda \left(e^{-ikx}f_\mu(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}(0)\;+\;e^{+ikx}f^*_\mu(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}}, \tag{5}$$

   and write down explicit formulae for the polarization 4-vectors $f^\mu(\mathbf{k},\lambda)$.

(e) Check that these polarization vectors obey $k_\mu f^\mu(\mathbf{k},\lambda)=0$. Use that to show that the free quantum vector field (5) obeys the classical equation of motion $\partial_\mu \hat{A}^\mu(x)=0$. Also, show that the $\hat{A}^\mu(x)$ obeys the other classical equation of motion, $(\partial^2+m^2)\hat{A}^\mu(x)=0$.

2. The ordinary quantum mechanics of a single relativistic particle — or any fixed number of relativistic particles — violates the relativistic causality by allowing particles to move faster than light. In this problem, we shall see how this works for the simplest case of a single free relativistic spinless particle with the Hamiltonian

$$\hat{H} \;=\; +\sqrt{m^2+\hat{\mathbf{P}}^2} \tag{6}$$

(in the $c=\hbar=1$ units). By general rules of quantum mechanics, the amplitude $U(x\to y)$ for this particle to propagate from point $\mathbf{x}$ at time x^0 to point $\mathbf{y}$ at time y^0 obtains from the Hamiltonian (6) as

$$U(x\to y) \;=\; \big\langle \mathbf{y},y^0 \big| \mathbf{x},x^0\big\rangle^{\text{Heisenberg}}_{\text{picture}} \;=\; \langle\mathbf{y}|\exp\!\big(-i(y^0-x^0)\hat{H}\big)\,|\mathbf{x}\rangle^{\text{Schroedinger}}_{\text{picture}}\,. \tag{7}$$

(a) Use momentum basis for the Hamiltonian (6) to evaluate the coordinate-basis evolution kernel (7) as

$$U(x\to y) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\exp\Big(i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})-i\omega(\mathbf{k})\times(y^0-x^0)\Big) \tag{8}$$

$$\text{for}\quad \omega(\mathbf{k}) \;\stackrel{\text{def}}{=}\; +\sqrt{m^2+\mathbf{k}^2}\,, \tag{9}$$

then reduce the 3D momentum integral to the one-dimensional integral

$$U(x\to y) \;=\; \frac{-i}{4\pi^2 r}\int\limits_{-\infty}^{+\infty}dk\,k\,\exp\big(irk-it\omega(k)\big) \tag{10}$$

where $r=|\mathbf{y}-\mathbf{x}|$ and $t=y^0-x^0$.

We are particularly interested in the asymptotic behavior of the integral (10) in the limit of $r \to \infty$, $t \to \infty$, fixed t/r ratio. The best method for obtaining the asymptotic behavior of such integrals — or more general integrals of the form

$$\int dx\, f(x) \times \exp\bigl(-Ag(x)\bigr), \qquad A \to \infty \tag{11}$$

is the *saddle-point method* (AKA the mountain-pass method).

(b) If you are not familiar with the saddle-point method, read my notes on it.
Those notes were originally written for a QM class, so they include the Airy function example and the relation of the Airy functions to the WKB approximation. You do not need the WKB or the Airy functions for this homework, just the saddle-point method itself, so focus on the first 6 pages of my notes, the rest is optional.

(c) Now use the saddle point method to evaluate the integral (10) in the limit of $r \to \infty$, $t \to \infty$, while the ratio r/t stays fixed. Specifically, let $(r/t) < 1$ so we stay inside the future light cone.
Show that in this limit, the evolution kernel (10) becomes

$$U(x \to y) \;\approx\; \left(\frac{-iM}{2\pi}\right)^{3/2} \times \frac{t}{(t^2 - r^2)^{5/4}} \times \exp(-iM\sqrt{t^2 - r^2}). \tag{12}$$

(d) Finally, take a similar limit but go outside the light cone, thus fixed $(r/t) > 1$ while $r, t \to +\infty$. Show that in this limit, the kernel becomes

$$U(x \to y) \;\approx\; \frac{iM^{3/2}}{(2\pi)^{3/2}} \times \frac{t}{(r^2 - t^2)^{5/4}} \times \exp(-M\sqrt{r^2 - t^2}). \tag{13}$$

Hint: for $r > t$ the saddle point is at complex k.

Eq. (13) shows that the propagation amplitude $U(x \to y)$ diminishes exponentially outside the light cone, *but it does not vanish!* Thus, given a particle localized at point $\mathbf{x}$ at the time x^0, at a later time $y^0 = x^0 + t$ the wave function is *mostly* limited to the future light cone $r < t$, *but there is an exponential tail outside the light cone.* In other words, the probability of superluminal motion is exponentially small but non-zero.

Obviously, such superluminal propagation cannot be allowed in a consistently relativistic theory. And that's why relativistic quantum mechanics of a single particle is inconsistent. Likewise, relativistic quantum mechanics of any fixed number of particles does not work, except as an approximation.

In the quantum field theory, this paradox is resolved by allowing for creation and annihilation of particles. Quantum field operators acting at points x and y outside each others' future light-cones can either create a particle at x and then annihilate it at y, or else annihilate it at y and then create it at x, and we shall see in class (*cf.* my notes) that the two effects *precisely* cancel each other. Altogether, there is no net propagation outside the light cone, and that's how the relativistic QFT is perfectly causal while the relativistic QM is not.

3. Finally, a reading assignment. To help you understand the relations between the continuous symmetries, their generators, the multiplets, and the representations of the generators and of the finite symmetries, read about the rotational symmetry and its generators in chapter 3 of the J. J. Sakurai's book *Modern Quantum Mechanics*.[⋆] Please focus on sections 1, 2, 3, second half of section 5 (representations of the rotation operators), and section 9; the other sections 4, 6, 7, 8, 10, and 11 are not relevant to the present class material.

   PS: If you have already read the Sakurai's book before but it has been a while, please read it again.

⋆ The UT Math–Physics–Astronomy library has several hard copies but no electronic copies of the book. However, you can find several pirate scans of the book (in PDF format) all over the web; Google them up if you cannot find a legitimate copy.