The first problem of this set continues the story of the massive vector field. The other two problems (2) and (3) are about the $SO(N)$ symmetry of the quantum theory of N scalar fields.

1. Consider the massive vector field $\hat{A}^\mu(x)$ from all the previous homework sets 1–3. In particular, in problem 1 from the last homework#3 you (should have) expanded the free vector field into creation and annihilation operators multiplied by the plane-waves according to

$$\hat{A}^\mu(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_\lambda\Bigl(e^{-ikx}\times f^\mu_{\mathbf{k},\lambda}\times\hat{a}_{\mathbf{k},\lambda}\,+\,e^{+ikx}\times f^{*\mu}_{\mathbf{k},\lambda}\times\hat{a}^\dagger_{\mathbf{k},\lambda}\Bigr)^{k^0=+\omega_{\mathbf{k}}}. \tag{1}$$

The λ here labels the independent polarizations of a vector particle (for example, the helicities $\lambda=-1,0,+1$), while $f^\mu_{\mathbf{k},\lambda}$ are the polarization vectors obeying $k_\mu f^\mu(\mathbf{k},\lambda)=0$. Specifically, in the helicity basis

$$\begin{aligned} &\text{for } \lambda=\pm1: \quad f^0_{\mathbf{k},\lambda} \;=\; 0, \quad \mathbf{f}_{\mathbf{k},\lambda} \;=\; \mathbf{e}_\lambda(\mathbf{k}),\\ &\text{for } \lambda=0: \quad f^0_{\mathbf{k},\lambda} \;=\; \frac{|\mathbf{k}|}{m}, \quad \mathbf{f}_{\mathbf{k},\lambda} \;=\; \frac{\omega_{\mathbf{k}}}{m}\frac{\mathbf{k}}{|\mathbf{k}|}\,. \end{aligned} \tag{2}$$

In this problem, we are going to calculate the Feynman propagator for the massive vector field (1).

(a) First, a lemma about the polarization 4-vectors (2). Show that these 4-vectors obtain obtain by Lorentz boosting of the purely-spatial vectors $(0,\mathbf{e}_\lambda(\mathbf{k}))$ into the frame of the wave moving with the velocity $\mathbf{v}=\mathbf{k}/\omega_{\mathbf{k}}$,

$$f^\mu_{\mathbf{k},\lambda} \;=\; B^\mu_{\ \nu}(\mathbf{v})(0,\mathbf{e}_\lambda(\mathbf{k}))^\nu. \tag{3}$$

Also, verify that the $f^\mu_{\mathbf{k},\lambda}$ are normalized to

$$\langle\!\langle\,\text{for the same }\mathbf{k}\,\rangle\!\rangle: \quad g_{\mu\nu}f^\mu_{\mathbf{k},\lambda}f^{*\nu}_{\mathbf{k},\lambda'} \;=\; -\delta_{\lambda,\lambda'}\,. \tag{4}$$

(b) Next, another lemma: show that

$$\sum_\lambda f^\mu_{\mathbf{k},\lambda} f^{*\nu}_{\mathbf{k},\lambda} \;=\; -g^{\mu\nu} \;+\; \frac{k^\mu k^\nu}{m^2}\,. \tag{5}$$

(c) Now use these lemmas to calculate the "vacuum sandwich" of two vector fields (1) and show that

$$\begin{aligned} \langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\left[\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)e^{-ik(x-y)}\right]_{k^0=+\omega_\mathbf{k}} \\ &=\; \left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{m^2}\right)D(x-y) \end{aligned} \tag{6}$$

where

$$D(x-y) \;\stackrel{\text{def}}{=}\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\left[e^{-ik(x-y)}\right]^{k^0=+\omega_\mathbf{k}}. \tag{7}$$

Please note: no time ordering in the "vacuum sandwich" (6).

(d) Next, consider a free scalar field (of the same mass m as the vector field) and its Feynman propagator $G_F^{\text{scalar}}(x-y)$. Show that

$$\left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{m^2}\right)G_F^{\text{scalar}}(x-y) \;=\; \langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{8}$$

To avoid the δ–function singularity in formulae like (8), the time-ordered product of the vector fields (or rather, just of their $\hat{A}^0$ components) is *modified** according to

$$\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y) \;=\; \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{9}$$

Consequently, the Feynman propagator for the massive vector field is defined using the

$\star$ See *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.

modified time-ordered product of the two fields,

$$G_F^{\mu\nu}(x-y) \stackrel{\text{def}}{=} \langle 0|\,\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \tag{10}$$

(e) Show that this propagator obtains as

$$G_F^{\mu\nu}(x-y) = \int\frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)\times\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\,. \tag{11}$$

The classical action for the free vector field can be written as

$$S = \tfrac{1}{2}\int d^4x\,A_\mu(x)\,\mathcal{D}^{\mu\nu}A_\nu(x) \tag{12}$$

where $\mathcal{D}^{\mu\nu}$ is a differential operator

$$\mathcal{D}^{\mu\nu} \stackrel{\text{def}}{=} (\partial^2+m^2)g^{\mu\nu} - \partial^\mu\partial^\nu. \tag{13}$$

(f) Check that the action (12) is correct, then show that the Feynman propagator (11) is a Green's function of the operator (13),

$$\mathcal{D}_x^{\mu\nu}G^F_{\nu\lambda}(x-y) = +i\delta^\mu_\lambda\delta^{(4)}(x-y). \tag{14}$$

2. Now let's change the subject and consider N interacting real scalar fields $\Phi_1,\ldots,\Phi_N$ with the $O(N)$ symmetric Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^N\bigl(\partial_\mu\Phi_a\bigr)^2 - \frac{m^2}{2}\sum_{a=1}^N\Phi_a^2 - \frac{\lambda}{24}\left(\sum_{a=1}^N\Phi_a^2\right)^2. \tag{15}$$

Next week, we shall learn the Noether theorem — here is the link for the impatient, — according to which the continuous $SO(N)$ subgroup of the $O(N)$ symmetry gives rise to $\frac{1}{2}N(N-1)$ conserved currents

$$J^\mu_{ab}(x) = -J^\mu_{ba}(x) = \Phi_a(x)\,\partial^\mu\Phi_b(x) - \Phi_b(x)\,\partial^\mu\Phi_a(x). \tag{16}$$

(a) Verify that all these currents are conserved, $\partial_\mu J^\mu_{ab}=0$, when the fields $\Phi^c(x)$ obey their classical equations of motion.

In the quantum field theory, the currents (16) become operators

$$\begin{aligned}\hat{\mathbf{J}}_{ab}(\mathbf{x},t) &= -\hat{\mathbf{J}}_{ba}(\mathbf{x},t) = -\hat{\Phi}_a(\mathbf{x},t)\nabla\hat{\Phi}_b(\mathbf{x},t) + \hat{\Phi}_b(\mathbf{x},t)\nabla\hat{\Phi}_a(\mathbf{x},t),\\ \hat{J}^0_{ab}(\mathbf{x},t) &= -\hat{J}^0_{ba}(\mathbf{x},t) = \hat{\Phi}_a(\mathbf{x},t)\hat{\Pi}_b(\mathbf{x},t) - \hat{\Phi}_b(\mathbf{x},t)\hat{\Pi}_a(\mathbf{x},t),\end{aligned} \tag{17}$$

and the rest of this problem is about the net charge operators

$$\hat{Q}_{ab}(t) = -\hat{Q}_{ba}(t) = \int d^3\mathbf{x}\,\hat{J}^0_{ab}(\mathbf{x},t) = \int d^3\mathbf{x}\left(\hat{\Phi}_a(\mathbf{x},t)\hat{\Pi}_b(\mathbf{x},t) - \hat{\Phi}_b(\mathbf{x},t)\hat{\Pi}_a(\mathbf{x},t)\right). \tag{18}$$

(b) Write down the equal-time commutation relations for the quantum $\hat{\Phi}_a$ and $\hat{\Pi}_a$ fields. Also, write down the Hamiltonian operator for the interacting fields.

(c) Show that

$$\begin{aligned}\left[\hat{Q}_{ab}(t),\hat{\Phi}_c(\mathbf{x},\text{same }t)\right] &= -i\delta_{bc}\hat{\Phi}_a(\mathbf{x},t) + i\delta_{ac}\hat{\Phi}_b(\mathbf{x},t),\\ \left[\hat{Q}_{ab}(t),\hat{\Pi}_c(\mathbf{x},\text{same }t)\right] &= -i\delta_{bc}\hat{\Pi}_a(\mathbf{x},t) + i\delta_{ac}\hat{\Pi}_b(\mathbf{x},t),\end{aligned} \tag{19}$$

(d) Show that the all the $\hat{Q}_{ab}$ commute with the Hamiltonian operator $\hat{H}$. In the Heisenberg picture, this makes all the charge operators $\hat{Q}_{ab}$ time independent.

(e) Verify that the $\hat{Q}_{ab}$ obey commutation relations of the $SO(N)$ generators,

$$\left[\hat{Q}_{ab},\hat{Q}_{cd}\right] = -i\delta_{[c[b}\hat{Q}_{a]d]} \equiv -i\delta_{bc}\hat{Q}_{ad} + i\delta_{ac}\hat{Q}_{bd} + i\delta_{bd}\hat{Q}_{ac} - i\delta_{ad}\hat{Q}_{bc}. \tag{20}$$

3. Continuing the previous problem, let's turn off the interactions (*i.e.*, take $\lambda = 0$) and focus on the free fields.

(a) Expand all the fields into linear combinations of the creation and annihilation operators $\hat{a}^\dagger_{\mathbf{p},a}$ and $\hat{a}_{\mathbf{p},a}$ ($a = 1,\ldots,N$), then show that in terms of these operators the charges (18) become

$$\hat{Q}_{ab} = \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\left(-i\hat{a}^\dagger_{\mathbf{p},a}\hat{a}_{\mathbf{p},b} + i\hat{a}^\dagger_{\mathbf{p},b}\hat{a}_{\mathbf{p},a}\right). \tag{21}$$

For $N = 2$, the two real scalar fields combine into one complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and its conjugate $\Phi^* = (\Phi_1 - i\Phi_2)/\sqrt{2}$, while the $SO(2)$ symmetry becomes the $U(1)$ phase symmetry

$$\Phi(x) \ \to\ e^{-i\theta}\Phi(x), \quad \Phi^*(x)\ \to\ e^{+i\theta}\Phi^*(x). \tag{22}$$

In the Fock space, the corresponding quantum fields $\hat{\Phi}(x)$ and $\hat{\Phi}^\dagger(x)$ give rise to particles and anti-particles of opposite charges; the creation and annihilation operators for such particles and antiparticles are

$$\begin{aligned}
\hat{a}_{\mathbf{p}} &= \frac{\hat{a}_{\mathbf{p},1} + i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} && \text{are particle annihilation operators,}\\
\hat{b}_{\mathbf{p}} &= \frac{\hat{a}_{\mathbf{p},1} - i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} && \text{are antiparticle annihilation operators,}\\
\hat{a}^\dagger_{\mathbf{p}} &= \frac{\hat{a}^\dagger_{\mathbf{p},1} - i\hat{a}^\dagger_{\mathbf{p},2}}{\sqrt{2}} && \text{are particle creation operators,}\\
\hat{b}^\dagger_{\mathbf{p}} &= \frac{\hat{a}^\dagger_{\mathbf{p},1} + i\hat{a}^\dagger_{\mathbf{p},2}}{\sqrt{2}} && \text{are antiparticle creation operators.}
\end{aligned} \tag{23}$$

(b) Show that in terms of the operators (23),

$$\hat{Q}_{21} \ =\ -\hat{Q}_{12} \ =\ \hat{N}_{\text{particles}} \ -\ \hat{N}_{\text{antiparticles}} \ =\ \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \ -\ \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right). \tag{24}$$

(c) In terms of $\hat{\Phi}$ and $\hat{\Phi}^\dagger$, the commutation relations (19) become

$$[\hat{Q}_{21}, \hat{\Phi}(x)] \ =\ -\hat{\Phi}(x), \quad [\hat{Q}_{21}, \hat{\Phi}^\dagger(x)] \ =\ +\hat{\Phi}^\dagger(x). \tag{25}$$

Verify these commutators, then use the Campbell identity

$$\begin{aligned}
e^{\hat{A}}\hat{B}e^{-\hat{A}} &= \sum_{n=0}^{\infty} \frac{1}{n!}[\hat{A}, \ldots, [\hat{A}, \hat{B}]\cdots]_{n\,\text{times}}\\
&= B \ +\ [\hat{A}, \hat{B}] \ +\ \tfrac{1}{2}[\hat{A}, [\hat{A}, \hat{B}]] \ +\ \tfrac{1}{6}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] \ +\ \cdots
\end{aligned} \tag{26}$$

to show that the charge $\hat{Q}_{21}$ *generates* the phase symmetry (22) according to

$$\begin{aligned}
\exp(+i\theta\hat{Q}_{21})\hat{\Phi}(x)\exp(-i\theta\hat{Q}_{21}) &= e^{-i\theta}\hat{\Phi}(x),\\
\exp(+i\theta\hat{Q}_{21})\hat{\Phi}^\dagger(x)\exp(-i\theta\hat{Q}_{21}) &= e^{+i\theta}\hat{\Phi}^\dagger(x).
\end{aligned} \tag{27}$$

Now let's go back to $N > 2$ and show that the charges $\hat{Q}_{ab}$ generate the $SO(N)$ symmetry of the quantum fields. Any $SO(N)$ rotation matrix R can be written as a matrix exponential of an antisymmetric matrix, $R = \exp(A)$ for $A^\top = -A$. For this matrix A, let's define a unitary operator in the Fock space

$$\hat{U} \;=\; \exp\left(-\frac{i}{2}\sum_{ab} A_{ab}\hat{Q}_{ab}\right). \tag{28}$$

(d) Verify that this operator is indeed unitary for any real antisymmetric matrix A. Hint: check and use the hermiticity of the generators $\hat{Q}_{ab}$.

(e) Show that $\hat{U}$ implements the $SO(N)$ rotation R in the scalar field space,

$$\hat{U}\hat{\Phi}_a(x)\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{\Phi}_b\,. \tag{29}$$

Hint: use the commutation relations (19) and the Campbell identity (26).

(f) Argue that $[\hat{Q}_{ab}, \hat{H}] = 0$ and eq. (29) for the action of the $\hat{U}$ symmetries on the quantum fields together imply similar transformation laws for the creation and the annihilation operators

$$\hat{U}\hat{a}_{\mathbf{p},a}\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{a}_{\mathbf{p},b} \quad\text{and}\quad \hat{U}\hat{a}^\dagger_{\mathbf{p},a}\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{a}^\dagger_{\mathbf{p},b}\,. \tag{30}$$

(g) Finally, show that when $\hat{U}$ acts on a multi-particle state, it rotates the species index of each particle by R,

$$\hat{U}\,|n:(\mathbf{p}_1,a_1),\ldots,(\mathbf{p}_n,a_n)\rangle \;=\; \sum_{b_1,\ldots,b_n} R_{a_1,b_1}\cdots R_{a_n,b_n}\;|n:(\mathbf{p}_1,b_1),\ldots,(\mathbf{p}_n,b_n)\rangle\,. \tag{31}$$

Note: for simplicity assume that all particles have different momenta, $\mathbf{p}_1 \neq \mathbf{p}_2$, *etc.*, then use part (f).