This homework has 3 problems. Problems 1 and 2 are about the stress-energy tensor of the electromagnetic fields, while problem 3 is about the dyons — particles with both magnetic and electric charges. Or rather, problem 3 is about QM of a charged spinless particle orbiting a dyon. Altogether, it's a pretty large homework set, so start working early.

1. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T^{\mu\nu}_{\text{Noether}} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi_a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{1}$$

For the scalar fields, real or complex, this Noether stress-energy tensor is properly symmetric, $T^{\mu\nu}_{\text{Noether}} = T^{\nu\mu}_{\text{Noether}}$. But for the vector, tensor, spinor, *etc.*, fields, the Noether stress-energy tensor (1) comes out asymmetric, so to make it properly symmetric one adds a total-divergence term of the form

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\text{Noether}} \;+\; \partial_\lambda\mathcal{K}^{\lambda\mu\nu}, \tag{2}$$

where $\mathcal{K}^{\lambda\mu\nu} \equiv -\mathcal{K}^{\mu\lambda\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{3}$$

where A_μ is a real vector field and $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Write down $T^{\mu\nu}_{\text{Noether}}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(b) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\rm EM} \;=\; -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\,g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \tag{4}$$

Show that this expression indeed has form (2) for

$$\mathcal{K}^{\lambda\mu,\nu} \;=\; -F^{\lambda\mu}A^{\nu} \;=\; -\mathcal{K}^{\mu\lambda,\nu}. \tag{5}$$

(c) Write down the components of the stress-energy tensor (4) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density, and stress.

Next, consider the electromagnetic fields coupled to the electric current J^{μ} of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate $P^{\mu}_{\rm EM}$ and $P^{\mu}_{\rm mat}$. Consequently, we should have

$$\partial_{\mu}T^{\mu\nu}_{\rm net} \;=\; 0 \qquad \text{for} \quad T^{\mu\nu}_{\rm net} \;=\; T^{\mu\nu}_{\rm EM} \;+\; T^{\mu\nu}_{\rm mat} \tag{6}$$

but generally $\partial_{\mu}T^{\mu\nu}_{\rm EM} \neq 0$ and $\partial_{\mu}T^{\mu\nu}_{\rm mat} \neq 0$.

(d) Use Maxwell's equations to show that

$$\partial_{\mu}T^{\mu\nu}_{\rm EM} \;=\; -F^{\nu\lambda}J_{\lambda} \tag{7}$$

(in $c = 1$ units), and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_{λ} according to

$$\partial_{\mu}T^{\mu\nu}_{\rm mat} \;=\; +F^{\nu\lambda}J_{\lambda}. \tag{8}$$

(e) Rewrite eq. (7) in non-relativistic notations and explain its physical meaning in terms of the electromagnetic energy, momentum, work, and forces.

2. Continuing problem 1, consider the EM fields coupled to a specific model of charged matter, namely a complex scalar field $\Phi(x) \neq \Phi^*(x)$ of electric charge $q \neq 0$. Altogether, the net Lagrangian for the A^μ, Φ, and Φ^* fields is

$$\mathcal{L}_{\rm net} \;=\; D^\mu\Phi^* D_\mu\Phi \;-\; m^2\Phi^*\Phi \;-\; \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{9}$$

where

$$D_\mu\Phi \;=\; (\partial_\mu \,+\, iqA_\mu)\Phi \quad\text{and}\quad D_\mu\Phi^* \;=\; (\partial_\mu \,-\, iqA_\mu)\Phi^* \tag{10}$$

are the *covariant* derivatives.

(a) Write down the equation of motion for all fields in a covariant from. Also, write down the electric current

$$J^\mu \;\stackrel{\rm def}{=}\; -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{11}$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole system and show that

$$T^{\mu\nu}_{\rm net} \;\equiv\; T^{\mu\nu}_{\rm EM} \,+\, T^{\mu\nu}_{\rm mat} \;=\; T^{\mu\nu}_{\rm Noether} \,+\, \partial_\lambda\mathcal{K}^{\lambda\mu\nu}, \tag{12}$$

where $T^{\mu\nu}_{\rm EM}$ is exactly as in eq. (4) for the free EM fields, the improvement tensor $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$ is also exactly as in eq. (5), and

$$T^{\mu\nu}_{\rm mat} \;=\; D^\mu\Phi^*\,D^\nu\Phi \;+\; D^\nu\Phi^*\,D^\mu\Phi \;-\; g^{\mu\nu}\big(D_\lambda\Phi^*\,D^\lambda\Phi \;-\; m^2\Phi^*\,\Phi\big). \tag{13}$$

Note: although the improvement tensor $\mathcal{K}^{\lambda\mu\nu}$ for the EM+matter system is the same as for the free EM fields, in presence of an electric current J^μ its derivative $\partial_\lambda\mathcal{K}^{\lambda\mu\nu}$ contains an extra $J^\mu A^\nu$ term. Pay attention to this term — it is important for obtaining the gauge-invariant stress-energy tensor (13) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi \;=\; iqF_{\mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* \;=\; -iqF_{\mu\nu}\Phi^* \tag{14}$$

to show that

$$\partial_\mu T^{\mu\nu}_{\rm mat} \;=\; +F^{\nu\lambda}J_\lambda \tag{15}$$

exactly as in eq. (8), and therefore the *net* stress-energy tensor (12) is conserved, *cf.* problem **1**(d).

3. Some unified theories of fundamental interactions predict the existence of *dyons* — magnetic monopoles that also have electric charges. Dyons are usually very heavy compared to ordinary particles, so when an ordinary charged particle orbits a dyon, the latter can be treated as a static source of the electric and the magnetic fields: In Gauss units,

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{Q}{r^2}\mathbf{n}, \qquad \mathbf{B}(\mathbf{x}) \;=\; \frac{M}{r^2}\mathbf{n}. \tag{16}$$

In this problem, we consider the motion of a spinless non-relativistic particle of mass m and electric charge q in these static fields. Let's start with the classical motion of the particle in question. The net angular momentum of the dyon+particle system is

$$\mathbf{J} \;=\; \mathbf{L}_{\rm mech} \;+\; \mathbf{J}_{\rm EM} \;=\; \mathbf{x}\times\vec{\pi} \;-\; \frac{qM}{c}\mathbf{n} \tag{17}$$

where $\vec{\pi} = m\mathbf{v}$ is the kinematic momentum of the particle (rather that its canonical momentum $\mathbf{p}$), while $\mathbf{J}_{\rm EM}$ is the angular momentum of the EM fields in the dyon+particle system.

(a) Verify that it is this net angular momentum (17) that is conserved by the classical motion of the particle, $d\mathbf{J}/dt = 0$.

In quantum mechanics, we have a similar formula for the net angular momentum,

$$\hat{\mathbf{J}} \;=\; \hat{\mathbf{x}} \times \hat{\vec{\pi}} \;-\; \frac{qM}{c}\frac{\hat{\mathbf{x}}}{\hat{r}} \tag{18}$$

where

$$\hat{\vec{\pi}} \;=\; \hat{\mathbf{p}} \;-\; \frac{q}{c}\mathbf{A}(\hat{\mathbf{x}}). \tag{19}$$

In light of eq. (19), the (equal time) commutation relations for the position and *kinematic* momentum operators are

$$[\hat{x}_i, \hat{x}_j] \;=\; 0, \qquad [\hat{x}_i, \hat{\pi}_j] \;=\; i\hbar\delta_{ij}\,, \tag{20}$$

but

$$[\hat{\pi}_i, \hat{\pi}_j] \;=\; \frac{iq\hbar}{c}\epsilon_{ijk}B_k(\hat{\mathbf{x}}) \;\xrightarrow[\text{in the dyon field}]{}\; \frac{iqM\hbar}{c}\,\epsilon_{ijk}\frac{\hat{x}_k}{\hat{r}^3}\,. \tag{21}$$

(b) Use these commutation relation to show that the components of the angular momentum operator (18) indeed commute with each other — and with the other vectors — as legitimate angular momentum operators. Specifically,

$$[\hat{x}_i, \hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\,\hat{x}_k\,, \tag{22}$$

$$[\hat{\pi}_i, \hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\,\hat{\pi}_k\,, \tag{23}$$

$$[\hat{J}_i, \hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\,\hat{J}_k\,. \tag{24}$$

(c) Show that the operators $\hat{J}_i$ are conserved, *i.e.*, that they commute with the particle's Hamiltonian

$$\hat{H} \;=\; \frac{\hat{\vec{\pi}}^2}{2m} \;+\; \frac{Qq}{\hat{r}}\,. \tag{25}$$

The vector potential due to the magnetic charge of the dyon can be written in spherical coordinates as

$$\mathbf{A}_{N,S}(r,\theta,\phi) \;=\; M\,\frac{\pm 1 - \cos\theta}{r\sin\theta}\cdot\mathbf{e}_\phi\,, \tag{26}$$

where $\mathbf{e}_\phi$ is the unit vector in the ϕ direction while the two signs correspond to the two different gauge choices for the Dirac monopole: '+' for the $\mathbf{A}_N$ potential on the Northern side of the dyon ($0 \le \theta < \pi - \epsilon$), and '$-$' for the $\mathbf{A}_S$ potential on the Southern side ($\epsilon < \theta \le \pi$).

(d) Show that for these gauge choices, the $\hat{J}_z$ operator acts in the spherical coordinate basis as

$$\hat{J}_z = -i\hbar\frac{\partial}{\partial\phi} \mp \frac{qM}{c}\psi. \tag{27}$$

Note that thanks to the Dirac's charge quantization rule, the $\mp(qM/c)$ factor in the second term here is always an integer or half-integer multiple of $\hbar$.

(e) Likewise, show that the other two components of the angular momentum have form

$$\begin{aligned} \hat{J}_+ &= \hat{J}_x + i\hat{J}_y = \hbar e^{+i\phi}\left[+\frac{\partial}{\partial\theta} + i\cot\theta\times\frac{\partial}{\partial\phi} - \frac{qM}{\hbar c}\frac{1\mp\cos\theta}{\sin\theta}\right], \\ \hat{J}_- &= \hat{J}_x - i\hat{J}_y = \hbar e^{-i\phi}\left[-\frac{\partial}{\partial\theta} + i\cot\theta\times\frac{\partial}{\partial\phi} - \frac{qM}{\hbar c}\frac{1\mp\cos\theta}{\sin\theta}\right], \end{aligned} \tag{28}$$

Now let's look for the simultaneous eigenstates $|n,j,m\rangle$ of the $\hat{\mathbf{J}}^2$ and $\hat{J}_z$ operators. By the usual rules of the angular momenta, for each given n and j, m runs from $-j$ to $+j$ by 1. However, in presence of the dyon, the spectrum of j is different from the spectrum of ℓ for the ordinary orbital angular momentum: Instead of $\ell = 0, 1, 2, 3, \ldots$, we now have

$$j = j_{\rm min}, j_{\rm min}+1, j_{\rm min}+2, \ldots \quad \text{where} \quad j_{\rm min} = \frac{|qM|}{\hbar c}. \tag{29}$$

In particular, for a half-integral $qM/\hbar c$, we have j running over half-integral rather than integral values.

(f) Use eqs. (27) and (28) to obtain this spectrum of allowed values of j.
Hint: use $\hat{J}_+|j, m = j\rangle = 0$ to argue that the wavefunction of this state is non-singular at both poles $\theta = 0$ and $\theta = \pi$ if and only if $j - j_{\rm min}$ is a non-negative integer.

Finally, let's diagonalize the Hamiltonian (25). As a first step, let's separate the radial and the angular directions of the operator $\vec{\pi}^2$.

(g) Use the commutation relations (20) through (24) to show that

$$\hat{\vec{\pi}}^2 = \hat{\pi}_r^2 + \frac{1}{\hat{r}^2}\left(\hat{\vec{J}}^2 - \left(\frac{qM}{c}\right)^2\right) \tag{30}$$

where

$$\hat{\pi}_r = \frac{1}{2}\{\hat{n}_i, \hat{\pi}_i\} \xrightarrow[\text{coordinate basis}]{} -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right). \tag{31}$$

(h) Now write down the radial Schrödinger equation for a given j and show that for $qQ < 0$ the bound state energies are

$$E(n_r, j) = -\frac{m(qQ)^2}{2\hbar^2} \times \frac{1}{(n_r + \lambda)^2} \tag{32}$$

where n_r is a positive integer $1, 2, 3, \ldots$, while λ is the positive root of

$$\lambda(\lambda + 1) = j(j + 1) - (qM/\hbar c)^2. \tag{33}$$

By comparison, in the absence of the magnetic charge j is $\ell = 0, 1, 2, 3, \ldots$, hence $\lambda = \ell$, and $n_r + \lambda = n_r + \ell$ is the principle quantum number N.