1. In the previous homework — set#6, problem#4(c) — we saw that the little group of Lorentz symmetries preserving the lightlike momentum p^μ or a massless particle is generated by the 3 components of the vector

$$\hat{\mathbf{I}} = \hat{\mathbf{J}} + \mathbf{v} \times \hat{\mathbf{K}}. \tag{1}$$

In particular, the component in the direction of the particle's velocity

$$\hat{I}_{\|} = \mathbf{v} \cdot \hat{\mathbf{I}} = \mathbf{v} \cdot \hat{\mathbf{J}} = \hat{\lambda} \tag{2}$$

is the helicity operator. As I explained in class, the finite unitary representations of this little group are singlets of definite helicity, specifically the states $|p, \lambda\rangle$ obeying

$$\hat{I}_{\|} |p, \lambda\rangle = \lambda |p, \lambda\rangle \quad \text{and} \quad \hat{\mathbf{I}}_{\perp} |p, \lambda\rangle = 0. \tag{3}$$

(a) Show that in 4D terms, the conditions (3) amount to

$$\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma} \hat{J}^{\alpha\beta} \hat{P}^{\gamma} |p, \lambda\rangle = \lambda \hat{P}_{\mu} |p, \lambda\rangle . \tag{4}$$

(b) Use eq. (4) to show that the *continuous* Lorentz transforms do not change helicities of *massless* particles,

$$\forall L \in \mathrm{SO}^{+}(3,1), \quad \widehat{\mathcal{D}}(L) |p, \lambda\rangle = |Lp, \text{same}\, \lambda\rangle \times e^{i\,\text{phase}}. \tag{5}$$

2. The Spin$(3,1)$ group — the double cover of the continuous Lorentz group $SO^+(3,1)$ — is isomorphic to $SL(2,\mathbf{C})$, the group of complex (but not necessarily unitary) 2×2 matrices of unit determinant. The relations between such matrices and the Lorentz symmetries are explained in my notes on Lorentz representations for the fields, but some technical details were left out as exercises for the students. This problem collects these exercises.

   (a) Show that the components of the two 3–vectors

$$\hat{\mathbf{J}}_+ \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} + i\hat{\mathbf{K}}\big) \quad\text{and}\quad \hat{\mathbf{J}}_- \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} - i\hat{\mathbf{K}}\big) \;=\; \hat{\mathbf{J}}_+^\dagger. \tag{6}$$

   obey commutation relations

$$\big[\hat{J}_+^i, \hat{J}_+^j\big] \;=\; i\epsilon^{ijk}\hat{J}_+^k, \quad \big[\hat{J}_-^i, \hat{J}_-^j\big] \;=\; i\epsilon^{ijk}\hat{J}_-^k, \quad\text{but}\quad \big[\hat{J}_+^i, \hat{J}_-^j\big] \;=\; 0. \tag{7}$$

Now consider the $\mathbf{2}$ ($j_+ = \frac{1}{2}, j_- = 0$) and the $\overline{\mathbf{2}}$ ($j_+ = 0, j_- = \frac{1}{2}$) representations of the Lorentz or rather Spin$(3,1)$ group. In the $\mathbf{2}$ representation $\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$ and $\mathbf{K} = -\frac{i}{2}\boldsymbol{\sigma}$ while in the $\overline{\mathbf{2}}$ representation $\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$ and $\mathbf{K} = +\frac{i}{2}\boldsymbol{\sigma}$, hence for a 3-space rotation R through angle ϕ around axis $\mathbf{n}$

$$M_{\mathbf{2}}(R) \;=\; M_{\overline{\mathbf{2}}} \;=\; \exp\big(-\tfrac{i}{2}\phi\mathbf{n}\cdot\boldsymbol{\sigma}\big) \tag{8}$$

while for a pure Lorentz boost B of rapidity r in the direction $\mathbf{n}$

$$M_{\mathbf{2}}(B) \;=\; \exp\big(-\tfrac{1}{2}r\mathbf{n}\cdot\boldsymbol{\sigma}\big), \qquad M_{\overline{\mathbf{2}}}(B) \;=\; \exp\big(+\tfrac{1}{2}r\mathbf{n}\cdot\boldsymbol{\sigma}\big). \tag{9}$$

   (b) The more familiar β and γ parameters of a boost are related to its rapidity r as

$$\beta \;=\; \tanh(r), \quad \gamma \;=\; \cosh(r), \quad \beta\gamma \;=\; \sinh(r). \tag{10}$$

   Show that in terms of these parameters, eqs. (9) translate to

$$M_{\mathbf{2}}(B) \;=\; \sqrt{\gamma}\times\sqrt{1 - \beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}, \qquad M_{\overline{\mathbf{2}}}(B) \;=\; \sqrt{\gamma}\times\sqrt{1 + \beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{11}$$

   (c) Let $M = M_{\mathbf{2}}(L)$ and $\overline{M} = M_{\overline{\mathbf{2}}}(L)$ be matrices representing the same continuous Lorentz symmetry $L \in SO^+(3,1)$ in the $\mathbf{2}$ and the $\overline{\mathbf{2}}$ spinor representations. Use eqs. (8) and (9) to show that for any such L,

$$\overline{M} = \sigma_2 M^* \sigma_2 \quad \text{and} \quad M \;=\; \sigma_2 \overline{M}^* \sigma_2\,. \tag{12}$$

Hint: prove and use $\sigma_2 \boldsymbol{\sigma}^* \sigma_2 = -\boldsymbol{\sigma}$.

Next, consider the vector representation of the Lorentz symmetry and the equivalent bi-spinor representation of the $SL(2,\mathbf{C})$. In the matrix form, the $(j_+ = j_- = \frac{1}{2})$ bi-spinor multiplet of $SL(2,\mathbf{C})$ is a complex 2×2 matrix V which transforms according to

$$V \;\mapsto\; V' \;=\; M \times V \times M^\dagger \quad \text{for } M \in SL(2,\mathbf{C}). \tag{13}$$

Let's identify this bi-spinor with a Lorentz vector V^μ according to

$$V \;=\; V^\mu \overline{\sigma}_\mu \;=\; V^0\, \mathbf{1}_{2\times 2} \;+\; \mathbf{V}\cdot\boldsymbol{\sigma}, \tag{14}$$

where I follow the Peskin&Schroeder convention for the 2×2 matrices σ^μ and $\overline{\sigma}^\mu$:

$$\sigma^\mu \;=\; \overline{\sigma}_\mu = \bigl(1_{2\times 2}, +\boldsymbol{\sigma}\bigr), \qquad \overline{\sigma}^\mu \;=\; \sigma_\mu \;=\; \bigl(1_{2\times 2}, -\boldsymbol{\sigma}\bigr). \tag{15}$$

The bi-spinor transform (13) defines a linear transform

$$V^\mu \;\mapsto\; V'^\mu \;=\; L^\mu_{\ \nu} V^\nu \tag{16}$$

of the vector V^μ.

(d) Show that this transform is real (real V'^μ for real V^ν) and Lorentzian (preserves $V'^\mu V_\mu = V^\nu V_\nu$). Hint: show that $\det(V) = V_\mu V^\mu$.

(e) Show that the Lorentz transform (16) is orthochronous.
For extra challenge, show that it is also proper ($\det(L) = +1$) and therefore continuous, $L \in SO^+(3,1)$.

(f) Verify that this $SL(2,\mathbf{C}) \to SO^+(3,1)$ map respects the group law, $L^\mu_{\ \nu}(M_2 M_1) = L^\mu_{\ \lambda}(M_2) L^\lambda_{\ \nu}(M_1)$.

Finally, consider the tensor representations of the Lorentz symmetry.

(g) Show that the $(j_+ = 1, j_- = 1)$ representation is equivalent to a 2–index symmetric traceless tensor, $T^{\mu\nu} = T^{\nu\mu}$, $g_{\mu\nu}T^{\mu\nu} = 0$.
Also, show that the reducible $(j_+ = 1, j_- = 0) + (j_+ = 0, j_- = 1)$ representation is equivalent to a 2-index antisymmetric tensor, $F^{\mu\nu} = -F^{\nu\mu}$.

Hint: For any kind of angular momentum — Hermitian or not, — the tensor product of two doublets is a triplet plus a singlet, $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$.

3. Next, an exercise in Dirac matrices γ^μ. In this problem, you should not assume any explicit matrices for the γ^μ but simply use the anticommutation relations

$$\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu}. \tag{17}$$

When necessary, you may also assume that the Dirac matrices are 4×4, and the γ^0 matrix is hermitian while the $\gamma^1, \gamma^2, \gamma^3$ matrices are antihermitian, $(\gamma^0)^\dagger = +\gamma^0$ while $(\gamma^i)^\dagger = -\gamma^i$ for $i = 1, 2, 3$.

(a) The original Dirac equation used $\beta = \gamma^0$ and $\alpha^i = \gamma^0\gamma^i$ (for $i = 1, 2, 3$) instead of the γ^μ. Show that eqs. (17) are equivalent to requiring all 4 matrices β and α^i to anticommute with each other and to square to 1.

(b) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$,
and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$.
Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

(c) The electron field in the EM background obeys the *covariant Dirac equation* $\big(i\gamma^\mu D_\mu \;-\; m\big)\Psi(x) \;=\; 0$ where $D_\mu\Psi = \partial_\mu\Psi - ieA_\mu\Psi$. Show that this equation implies

$$\Big(D_\mu D^\mu \;+\; m^2 \;-\; eF_{\mu\nu}S^{\mu\nu}\Big)\,\Psi(x) \;=\; 0. \tag{18}$$

Besides the 4 Dirac matrices γ^μ, there is another useful matrix $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$.

(d) Show that the γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ — and commutes with all the spin matrices, $\gamma^5 S^{\mu\nu} = +S^{\mu\nu}\gamma^5$.

(e) Show that the γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

(f) Show that $\gamma^5 = (i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and that $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = +24i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$.

(g) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = +6i\epsilon^{\kappa\lambda\mu\nu}\gamma_\kappa\gamma^5$.

(h) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} = -2iS^{\mu\nu}$, $\gamma^5\gamma^\mu$, and γ^5.

$*$ My conventions here are: $\epsilon^{0123} = -1$, $\epsilon_{0123} = +1$, $\gamma^{[\mu}\gamma^{\nu]} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$, $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda$, *etc.*

4. Since all the spin matrices $S^{\mu\nu}$ commute with the γ^5, for all *continuous* Lorentz symmetries $L^\mu_{\ \nu}$ their Dirac-spinor representations $M_D(L) = \exp\bigl(-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta}\bigr)$ are block-diagonal in the eigenbasis of the γ^5. This makes the Dirac spinor Ψ a *reducible* multiplet of the continuous Lorentz group $SO^+(3,1)$ — it comprises two different irreducible 2-component spinor multiplets, called the left-handed Weyl spinor ψ_L and the right-handed Weyl spinor ψ_R.

This decomposition becomes clear in the Weyl convention for the Dirac matrices where in 2×2 block notations

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{19}$$

and σ^μ and $\overline{\sigma}^\mu$ as in the Peskin & Schroeder convention (15).

(a) Show that in the Weyl convention (19), the γ^5 matrix is diagonal, specifically

$$\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}. \tag{20}$$

(b) Write down explicitly matrices for the $S^{\mu\nu}$ matrices in the Weyl convention and show that

$$S^{\mu\nu} = \begin{pmatrix} S_L^{\mu\nu} & 0 \\ 0 & S_R^{\mu\nu} \end{pmatrix} \tag{21}$$

where $S_L^{\mu\nu} = S_{\mathbf{2}}^{\mu\nu}$ and $S_R^{\mu\nu} = S_{\overline{\mathbf{2}}}^{\mu\nu}$ are respectively the $\mathbf{2}$ and $\overline{\mathbf{2}}$ representations of the Lorentz generators.

In light of eqs. (21), the Dirac spinor is a reducible $\mathbf{2}+\overline{\mathbf{2}}$ multiplet of the Spin(3, 1) Lorentz group, and for any continuous Lorentz transform L we have

$$M_D(L) \;=\; \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix} \quad \text{for } M_L(L) \;=\; M_{\mathbf{2}}(L) \text{ and } M_R(L) \;=\; M_{\overline{\mathbf{2}}}(L). \qquad (22)$$

Consequently, in the Weyl convention the 4-components Dirac spinor field $\Psi(x)$ splits into two 2-component Weyl spinor fields — the left-handed Weyl spinor field $\psi_L(x)$ and the right-handed Weyl spinor field $\psi_R(x)$ — which transform independently (from each other) under the continuous Lorentz symmetries,

$$\Psi_{\text{Dirac}}(x) \;=\; \begin{pmatrix} \psi_L(x), \\ \psi_R(x) \end{pmatrix} \quad \text{where} \quad \begin{aligned} \psi'_L(x') \;&=\; M_L(L)\psi_L(x), \\ \psi'_R(x') \;&=\; M_R(L)\psi_R(x). \end{aligned} \qquad (23)$$

(c) Use eqs. (12) to show that the hermitian conjugate of each Weyl spinor transforms equivalently to the other spinor. Specifically, the $\sigma_2 \times \psi^*_L(x)$ transforms under continuous Lorentz symmetries like the $\psi_R(x)$, while the $\sigma_2 \times \psi^*_R(x)$ transforms like the $\psi_L(x)$.

Note: the * superscript on a multi-component quantum field means hermitian conjugation of each component field but without transposing the components, thus

$$\psi_L \;=\; \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix}, \quad \psi^*_L \;=\; \begin{pmatrix} \psi^\dagger_{L1} \\ \psi^\dagger_{L2} \end{pmatrix}, \quad \text{while} \quad \psi^\dagger_L \;=\; (\,\psi^\dagger_{L1} \quad \psi^\dagger_{L2}\,), \qquad (24)$$

and likewise for the ψ_R and its conjugates.

Next, consider the Dirac Lagrangian $\mathcal{L} = \overline{\Psi}(i\gamma^\mu\partial_\mu \;-\; m)\Psi$.

(d) Express this Lagrangian in terms of the Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ (and their conjugates $\psi^\dagger_L(x)$ and $\psi^\dagger_R(x)$).

(e) Show that for $m = 0$ — and only for $m = 0$ — the two Weyl spinor fields become independent from each other.

5. Finally, consider the plane-wave solutions of the Dirac equation, $\Psi_\alpha(x) = u_\alpha \times e^{-ipx}$ and $\Psi_\alpha(x) = v_\alpha \times e^{+ipx}$ for some x-independent Dirac spinors $u_\alpha(p,s)$ and $v_\alpha(p,s)$.

   (a) Check that these waves indeed solve the Dirac equation, provided $p^2 = m^2$ while

$$(\not{p} - m)u(p,s) \;=\; 0, \quad (\not{p} + m)v(p,s) \;=\; 0 \tag{25}$$

   where $\not{p}$ is the Dirac slash notation for the $\gamma^\mu p_\mu$. Likewise, for any Lorentz vector a^μ, we may write $\not{a}$ to denote $\gamma^\mu a_\mu$.

   By convention, we always take $E = p^0 = +\sqrt{\mathbf{p}^2 + m^2}$ — that's why we have separate positive-frequency waves $e^{-ipx}u_\alpha$ and negative-frequency waves $e^{+ipx}v_\alpha$ — while the spinor coefficients $u(p,s)$ and $v(p,s)$ are normalized to

$$u^\dagger(p,s)u(p,s') \;=\; v^\dagger(p,s)v(p,s') \;=\; 2E\delta_{s,s'}\,. \tag{26}$$

   In this problem we shall write down explicit formulae for these spinors in the Weyl convention for the γ^μ matrices.

   (b) Show that for $\mathbf{p} = 0$,

$$u(\mathbf{p} = \mathbf{0}, s) \;=\; \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} \tag{27}$$

   where ξ_s is a two-component $SO(3)$ spinor encoding the electron's spin state. The ξ_s are normalized to $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$.

   (c) For other momenta, $u(p,s) = M_D(\text{boost}) \times u(\mathbf{p} = 0, s)$ for the boost that turns $(m, \vec{0})$ into p^μ. Use eq. (11) from problem 2 to show that

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\,\xi_s \\ \sqrt{p_\mu \overline{\sigma}^\mu}\,\xi_s \end{pmatrix}. \tag{28}$$

   (d) Use similar arguments to show that

$$v(p,s) \;=\; \begin{pmatrix} +\sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \;=\; \begin{pmatrix} +\sqrt{p_\mu \sigma^\mu}\,\eta_s \\ -\sqrt{p_\mu \overline{\sigma}^\mu}\,\eta_s \end{pmatrix} \tag{29}$$

   where η_s are two-component $SO(3)$ spinors normalized to $\eta_s^\dagger \eta_{s'} = \delta_{s,s'}$.

Physically, the η_s should have opposite spins from the ξ_s — the holes in the Dirac sea have opposite spins (as well as p^μ) from the missing negative-energy particles. Mathematically, this requires $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$; we may solve this condition by letting $\eta_s = \sigma_2 \xi_s^* = \pm i \xi_{-s}^*$.

(e) Check that $\eta_s = \sigma_2 \xi_s^* = \pm i \xi_{-s}^*$ indeed provides for the $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$, then show that this leads to

$$v(p,s) = \gamma^2 u^*(p,s) \quad \text{and} \quad u(p,s) = \gamma^2 v^*(p,s). \tag{30}$$

(f) Show that for the ultra-relativistic electrons or positrons of definite helicity $\lambda = \pm\frac{1}{2}$, the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components $\psi_L(x)$ or $\psi_R(x)$ of the Dirac spinor $\Psi(x)$, while the other component becomes negligible. Specifically,

$$\begin{aligned} u(p,-\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, & u(p,+\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix},\\ v(p,-\tfrac{1}{2}) &\approx -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, & v(p,+\tfrac{1}{2}) &\approx \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}. \end{aligned} \tag{31}$$

Note that for the electron waves the helicity agrees with the chirality — they are both left or both right, — but for the positron waves the chirality is opposite from the helicity.

In the previous problem we saw that for $m = 0$, the LH and the RH Weyl spinor fields decouple from each other. Now this exercise shows us which particle modes comprise each Weyl spinor: *The $\psi_L(x)$ and its hermitian conjugate $\psi_L^\dagger(x)$ contain the left-handed fermions and the right-handed antifermions, while the $\psi_R(x)$ and the $\psi_R^\dagger(x)$ contain the right-handed fermions and the left-handed antifermions.*