1. Back in homework set#7 we have worked out the $u(p,s)$ and $v(p,s)$ in the Weyl convention for the Dirac matrices and Dirac spinors. In this problem we are going to establish some convention-independent properties of these Dirac spinors, — although you may use the Weyl convention formulae from problem 5 of homework#7 to verify them. We shall use these properties in class when we get to the Quantum Electrodynamics (QED).

   (a) Dirac spinors $u(p,s)$ and $v(p,s)$ are normalized to

   $$u^\dagger(p,s)u(p,s') \;=\; 2E_p\delta_{s,s'}\,, \quad v^\dagger(p,s)v(p,s') \;=\; 2E_p\delta_{s,s'}\,. \tag{1}$$

   Show that the combinations $\bar{u}u$ and $\bar{v}v$ have a different normalization, namely

   $$\bar{u}(p,s)u(p,s') \;=\; +2m\delta_{s,s'}\,, \quad \bar{v}(p,s)v(p,s') \;=\; -2m\delta_{s,s'}\,. \tag{2}$$

   (b) There are only two independent $SO(3)$ spinors, hence $\sum_s \xi_s\xi_s^\dagger = \sum_s \eta_s\eta_s^\dagger = \mathbf{1}_{2\times2}$. Use this fact to show that

   $$\sum_{s=1,2} u_\alpha(p,s)\bar{u}_\beta(p,s) \;=\; (\not{p}+m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p,s)\bar{v}_\beta(p,s) \;=\; (\not{p}-m)_{\alpha\beta}\,. \tag{3}$$

2. In class we have studied the charge conjugation symmetry **C** in some detail, but we spent much less time on other discrete symmetries. In this problem, we focus on the *parity* **P**, the im-proper Lorentz symmetry which reflects the space but not the time, $(\mathbf{x},t)\to(-\mathbf{x},+t)$. This symmetry acts on the Dirac spinor fields according to

   $$\widehat{\Psi}'(-\mathbf{x},+t) \;=\; \pm\gamma^0\widehat{\Psi}(+\mathbf{x},+t) \tag{4}$$

   where the overall $\pm$ sign is the *intrinsic parity* of the fermion species described by the $\widehat{\Psi}$ field.

(a) Verify that the Dirac equation transforms covariantly under (4) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(\mathbf{x},t)\to\mathcal{L}(-\mathbf{x},t)$).

In the Fock space, eq. (4) becomes

$$\widehat{\mathbf{P}}\widehat{\Psi}(\mathbf{x},t)\widehat{\mathbf{P}} \;=\; \pm\gamma^0\widehat{\Psi}(-\mathbf{x},t) \tag{5}$$

for some unitary operator $\widehat{\mathbf{P}}$ that squares to one. Let's find how this operator acts on the particles and their states.

(b) First, check the plane-wave solutions $u(\mathbf{p},s)$ and $v(\mathbf{p},s)$ from homework#7 (problem 5) and show that

$$u(-\mathbf{p},s) \;=\; +\gamma^0 u(+\mathbf{p},s) \quad\text{but}\quad v(-\mathbf{p},s) \;=\; -\gamma^0 v(+\mathbf{p},s). \tag{6}$$

(c) Now expand the $\widehat{\Psi}(x)$ and the $\widehat{\overline{\Psi}}(x)$ fields into creation and annihilation operators and apply eqs. (5) and (6) to the expansions. Show that these equations imply that

$$\begin{aligned}\widehat{\mathbf{P}}\,\hat{a}_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \pm\hat{a}_{-\mathbf{p},+s}\,, &\quad \widehat{\mathbf{P}}\,\hat{a}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \pm\hat{a}^\dagger_{-\mathbf{p},+s}\,,\\ \widehat{\mathbf{P}}\,\hat{b}_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \mp\hat{b}_{-\mathbf{p},+s}\,, &\quad \widehat{\mathbf{P}}\,\hat{b}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \mp\hat{b}^\dagger_{-\mathbf{p},+s}\,,\end{aligned} \tag{7}$$

and hence

$$\widehat{\mathbf{P}}\,|F(\mathbf{p},s)\rangle \;=\; \pm\,|F(-\mathbf{p},+s)\rangle \quad\text{and}\quad \widehat{\mathbf{P}}\,\big|\overline{F}(\mathbf{p},s)\big\rangle \;=\; \mp\,\big|\overline{F}(-\mathbf{p},+s)\big\rangle\,. \tag{8}$$

Note that the fermion F and the antifermion $\overline{F}$ have opposite intrinsic parities!

3. Consider a bound state of a charged Dirac fermion F and the corresponding antifermion, for example a $q\bar{q}$ meson or a positronium "atom" (a hydrogen-atom-like bound state of e^- and e^+). For simplicity, let this bound state have zero net momentum. In the Fock space of fermions and antifermions, such a bound state appears as

$$|B(\mathbf{p}_{\rm tot}=0)\rangle \;=\; \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\rm red},s_1,s_2)\times\hat{a}^\dagger(+\mathbf{p}_{\rm red},s_1)\,\hat{b}^\dagger(-\mathbf{p}_{\rm red},s_2)\,|0\rangle \tag{9}$$

for some wave-function ψ of the reduced momentum and of the two spins.

Suppose this bound state has a definite orbital angular momentum L — which controls the symmetry of the wave function ψ with respect to $\mathbf{p}_{\text{red}} \to -\mathbf{p}_{\text{red}}$ — and a definite net spin S — which controls the symmetry of ψ under $s_1 \leftrightarrow s_2$. Turns out that the L and the S of the bound state also determine its C-parity and P-parity.

(a) Show that $C = (-1)^{L+S}$.

(b) Show that $P = (-1)^{L+1}$.

Now let's apply these results to the positronium — a hydrogen-atom-like bound state of a positron e^+ and an electron e^-. The ground state of positronium is hydrogen-like 1S ($n = 1$, $L = 0$), with the net spin which could be either $S = 0$ or $S = 1$.

(c) Explain why the $S = 0$ state annihilates into photons much faster than the $S = 1$ state.
Hint#1: The annihilation rate of positronium into n photons happens in the n^{th} order of QED perturbation theory, so the rate $\propto \alpha^n$ (for $\alpha \approx 1/137$).
Hint#2: Since the EM fields couple linearly to the electric charges and currents (which are reversed by $\widehat{\mathbf{C}}$), each photon has $C = -1$.

4. Consider the bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where Γ is one of 16 matrices discussed in the previous homework set#7, problem 3(h). Altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\tfrac{i}{2}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^\mu\gamma^5\Psi, \quad P = \overline{\Psi}i\gamma^5\Psi. \tag{10}$$

(a) Show that all the bilinears (10) are Hermitian.
Hint: First, show that $\left(\overline{\Psi}\Gamma\Psi\right)^\dagger = \overline{\Psi}\overline{\Gamma}\Psi$.
Note: despite the Fermi statistics, $\left(\Psi^\dagger_\alpha\Psi_\beta\right)^\dagger = +\Psi^\dagger_\beta\Psi_\alpha$.

(b) Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(c) Find the transformation rules of the bilinears (10) under parity and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

Now consider the charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take the classical limit where $\Psi(x)$ and $\Psi^\dagger(x)$ *anticommute* with each other, $\Psi_\alpha \Psi^\dagger_\beta = -\Psi^\dagger_\beta \Psi_\alpha$.

(d) Show that in the Weyl convention, **C** turns $\overline{\Psi}\Gamma\Psi$ into $\overline{\Psi}\Gamma^c\Psi$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(e) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are C–even and which are C–odd.

5. Finally, a few *optional* — but strongly recommended — reading assignments.

(a) *Modern Quantum Mechanics* by J. J. Sakurai,[⋆] §4.4, about the time reversal symmetry in quantum mechanics.
If you have already read the Sakurai's book before but it has been a while, please read it again.

(b) Peskin & Schroeder textbook, §3.6, about the discrete symmetries of Dirac spinors. Focus on the subsections about the time reversal symmetry and about the combined **CPT** symmetry.

(c) My notes about spin-statistic theorem. In particular, read the proofs of the two lemmas I have used in class but did not have the time to prove.

⋆ The UT Math–Physics–Astronomy library has several hard copies but no electronic copies of the book. However, you can find several pirate scans of the book (in PDF format) all over the web; Google them up if you cannot find a legitimate copy.