

1. Let's start with a bunch of reading assignments.

(a) First, read carefully [my notes on the Golden Rule and the phase space factors](#) .

Note: you will need the two-particle phase space factor in problems 2–4 of this homework, so please pay attention. Also, a later homework will involve a three-particle phase space, so do not forget the general formulae for n final-state particles.

(b) If you have trouble following my derivation of the Fermi's Golden Rule, read the [online notes of professor Wacker](#) on the subject. Alternatively, read §5.7 of J. J. Sakurai's *Modern Quantum Mechanics* on time-dependent perturbation theory, and focus on the constant perturbation and the Fermi golden rule.

(c) Finally, read §4.5 of the *Peskin and Schroeder* textbook for an alternative explanation of the phase space factor for the scattering cross-sections.

2. Next, a warm-up exercise. Consider two species of scalar fields, Φ and ϕ , with a cubic coupling to each other,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{M^2}{2}\Phi^2 + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\mu}{2}\Phi\phi^2. \quad (1)$$

(a) Write down the vertices and the propagators for the Feynman rules for this theory.

(b) Suppose $M > 2m$, so a single Φ particle may decay to two ϕ particles. Calculate the rate Γ of this decay (in the rest frame of the original Φ) to lowest order in perturbation theory.

3. Now consider N scalar fields ϕ_i of the same mass m and with $O(N)$ symmetric quartic couplings to each other,

$$\mathcal{L} = \frac{1}{2}\sum_i(\partial_\mu\phi_i)^2 - \frac{m^2}{2}\sum_i\phi_i^2 - \frac{\lambda}{8}\left(\sum_i\phi_i^2\right)^2. \quad (2)$$

(a) Write down the Feynman propagators and the vertices for this theory. Then write

down the tree-level scattering amplitude $\mathcal{M}(\phi_i + \phi_j \rightarrow \phi_k + \phi_\ell)$ for general $i, j, k, \ell = 1, \dots, N$.

- (b) Calculate the tree-level scattering amplitudes \mathcal{M} , the partial cross-sections $d\sigma/d\Omega_{\text{cm}}$ (in the center-of-mass frame), and the total cross-sections for the following 3 processes:

(i) $\phi_1 + \phi_2 \rightarrow \phi_1 + \phi_2$.

(ii) $\phi_1 + \phi_1 \rightarrow \phi_2 + \phi_2$.

(iii) $\phi_1 + \phi_1 \rightarrow \phi_1 + \phi_1$.

4. Finally, for a harder exercise consider the *linear sigma model*. This model comprises N massless scalar or pseudoscalar fields π_i and one massive scalar field σ with both quartic and cubic couplings to the pions, specifically

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \sum_i (\partial_\mu \pi_i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{\lambda}{8} \left(\sum_i \pi_i^2 + \sigma^2 + 2f\sigma \right)^2 \\ &= \frac{1}{2} \sum_i (\partial_\mu \pi_i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{\lambda f^2}{2} \times \sigma^2 \\ &\quad - \frac{\lambda f}{2} \times \left(\sigma^3 + \sigma \sum_i \pi_i^2 \right) - \frac{\lambda}{8} \left(\sum_i \pi_i^2 + \sigma^2 \right)^2 \end{aligned} \quad (3)$$

Both the masslessness of the π_i fields and the specific relations between the quartic couplings, the cubic couplings, and the sigma's mass $M_\sigma^2 = \lambda f^2$ in this model stem from the *spontaneous breaking down* of the $O(N+1)$ symmetry, which I shall explain in class later this semester. I shall also explain the relation of this model to the approximate chiral symmetry of QCD and hence to the real-life pi-mesons and their low-energy scattering amplitudes.

But in this homework, you should simply take the Lagrangian (3) as it is, and explore its implications for the scattering of π particles.

- (a) Write down all the vertices and all the propagators for the Feynman rules for this theory.

- (b) Draw *all* the tree diagrams and calculate the tree-level scattering amplitudes of two pions to two pions, $\mathcal{M}_{\text{tree}}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m)$.
- (c) Show that due to specific relations between the quartic and the cubic couplings in the Lagrangian (3), in the low-energy limit $E_{\text{tot}} \ll M_\sigma$, all the amplitudes $\mathcal{M}_{\text{tree}}(\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m)$ become small as $O(E_{\text{tot}}^2/M_\sigma^2)$ or smaller.
- (d) Use Mandelstam's variables s, t, u to show that when any of the incoming or outgoing pions' energy becomes small (while the other pions' energies are $O(M_\sigma)$), the scattering amplitudes become small as $O(E_{\text{small}}/M_\sigma)$ or smaller.
- Later in class, we shall learn that this behavior stems from the *Goldstone–Nambu theorem*.
- (e) Write down specific tree-level amplitudes, partial cross-sections (in the CM frame), and total cross-sections for the processes
- (i) $\pi^1 + \pi^2 \rightarrow \pi^1 + \pi^2$
 - (ii) $\pi^1 + \pi^1 \rightarrow \pi^2 + \pi^2$
 - (iii) $\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1$

in the low-energy limit $E_{\text{cm}} \ll M_\sigma$.