This is the first part of an oversize homework set#12. It has 3 problems about Spontaneous Symmetry Breaking and the Higgs Mechanism. The second part — due on the last class day (December 9) — will have 2 problems about the electroweak interactions in the Standard Model.

1. Let's start with the $SU(2)$ gauge symmetry coupled to a doublet of Higgs fields $\Phi^{1,2}(x)$,

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + D_\mu\Phi^\dagger D^\mu\Phi - \frac{\lambda}{2}\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2. \tag{1}$$

As discussed in class (*cf.* my notes on the Higgs mechanism, pages 8–10), the Higgs doublet develops a vacuum expectation value (VEV)

$$\langle\Phi\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \tag{2}$$

which completely breaks the $SU(2)$ gauge symmetry and gives the vector bosons equal masses $M_V = \frac{1}{2}gv$.

The reason all 3 vector fields get the same mass is a non-obvious global $SU(2)$ symmetry of the scalar fields. Indeed, the scalar potential is invariant under the $SO(4)$ symmetry which mixes the real and the imaginary parts of the scalar fields, and the $SO(4)$ or rather Spin(4) group happens to be isomorphic to $SU(2) \times SU(2)$. The gauge symmetry acts as one of these $SU(2)$ factors while the other $SU(2)$ factor of the $SO(4)$ remains a global symmetry.

To make both $SU(2)_{\rm local}$ and $SU(2)_{\rm global}$ symmetries manifest, consider a 2×2 matrix

$$W = \begin{pmatrix}+\Phi_2^* & +\Phi^1\\ -\Phi_1^* & +\Phi^2\end{pmatrix} \tag{3}$$

(a) Show that $\sigma_2 W^* \sigma_2 = W$, and that any matrix obeying this skewed reality condition has form (3) for come complex Φ^1 and Φ^2.

(b) Show that under a gauge transform of the $\Phi(x)$ doublet, the $W(x)$ matrix-valued field transforms in the similar manner,

$$\Phi'(x) \;=\; U(x)\Phi(x) \quad\Longrightarrow\quad W'(x) \;=\; U(x)W(x) \quad \forall U(x)\in SU(2). \tag{4}$$

(c) Show that in terms of the matrix-valued field $W(x)$ the Lagrangian (1) becomes

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; \tfrac{1}{2}\,\mathrm{tr}\bigl(D_\mu W^\dagger D^\mu W\bigr) \;-\; \frac{\lambda}{8}\bigl(\mathrm{tr}(W^\dagger W) \;-\; v^2\bigr)^2, \tag{5}$$

and that it is invariant under the $SU(2)_{\rm local}\times SU(2)_{\rm global}$ symmetry which acts as

$$\begin{aligned} W'(x) \;&=\; U_L(x)W(x)U_G^\dagger, \\ A'_\mu(x) \;&=\; U_L(x)A_\mu(x)U_L^{-1}(x) \;+\; \frac{i}{g}(\partial_\mu U_L(x))U_L^{-1}(x), \\ &\text{local } U_L(x)\in SU(2), \quad \text{global } U_G\in SU(2). \end{aligned} \tag{6}$$

The vacuum expectation value (2) — which translates to

$$\langle W\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{7}$$

— is not invariant under the $SU(2)_{\rm global}$ symmetry factor of (6). Nevertheless, there are no Goldstone bosons because $\langle W\rangle$ is invariant under a modified global $SU(2)$ symmetry

$$W'(x) \;=\; U_G W(x) U_G^\dagger, \quad \text{any } U_G\in SU(2). \tag{8}$$

Or terms of eq. (6), is a subgroup of the $SU(2)_{\rm local}\times SU(2)_{\rm global}$ parametrized by

- Any $U_G\in SU(2)$, accompanied by $U_L(x)=U_G\ \forall x$.

(d) Argue that the generators of this modified $SU(2)'_{\rm global}$ symmetry are

$$T^a[SU(2)'_{\rm global}] \;=\; T^a[SU(2)_{\rm global}] \;+\; T^a[SU(2)_{\rm local}] \tag{9}$$

and check that the VEV $\langle W\rangle$ is indeed invariant under the $SU(2)'_{\rm global}$.

Since the global $SU(2)$ symmetry is modified (by mixing with the gauge symmetry) rather than spontaneously broken, there are no Goldstone bosons for this symmetry. Moreover, the modification (9) make the symmetry act non-trivially on the massive vector fields.

(e) Show that the massive vector fields form a triplet of the $SU(2)'_{\text{global}}$, and that's why they all have the same mass.

2. Next, a more complicated problem on symmetry breaking. Consider an $N \times N$ matrix $\Phi(x)$ of complex scalar fields $\Phi^i{}_j(x)$, $i, j = 1, \ldots, N$. In matrix notations, the Lagrangian is

$$\mathcal{L} \;=\; \operatorname{tr}\left(\partial^\mu \Phi^\dagger\, \partial_\mu \Phi\right) \;-\; V(\Phi^\dagger \Phi) \tag{10}$$

where the potential is

$$V \;=\; \frac{\alpha}{2}\, \operatorname{tr}\left(\Phi^\dagger \Phi \Phi^\dagger \Phi\right) \;+\; \frac{\beta}{2}\left(\operatorname{tr}\left(\Phi^\dagger \Phi\right)\right)^2 \;+\; m^2\, \operatorname{tr}\left(\Phi^\dagger \Phi\right). \tag{11}$$

(a) Show that this theory has global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ acting as

$$\Phi(x) \;\to\; e^{i\theta} U_L \Phi(x) U_R^\dagger, \qquad U_L, U_R \in SU(N). \tag{12}$$

For $\alpha = 0$ the theory would have a much larger continuous symmetry group $SO(2N^2)$ which mixes up the real and the imaginary parts of all the $\Phi^i{}_j$ regardless of their complex structure or the $N \times N$ matrix structure of Φ. But for $\alpha \neq 0$ this huge symmetry group is reduced to $G = SU(N)_L \times SU(N)_R \times U(1)$. Note: it is easy to see that the α-term is invariant under the $SU(N)_L \times SU(N)_R \times U(1)$ subgroup of the $SO(2N^2)$, but showing that this term has no other continuous symmetries is a hard exercise in group theory. But since this is a homework on QFT rather than group theory, let's accept this statement without a proof.

From now on, we take $\alpha, \beta > 0$ but $m^2 < 0$. In this regime, V is minimized for non-zero vacuum expectation values $\langle \Phi \rangle \neq 0$ of the scalar fields.

(b) Let $(\kappa_1, \ldots, \kappa_N)$ be eigenvalues of the hermitian matrix $\Phi^\dagger\Phi$. Express the potential (11) in terms of these eigenvalues and show that the minimum lies at

$$\kappa_1 \;=\; \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; C^2 \;=\; \frac{-m^2}{\alpha + N\beta} > 0. \tag{13}$$

In terms of the matrix Φ, eq. (13) means $\Phi = C \times$ a unitary matrix. All such minima are related by symmetries (12) to $\Phi = C \times$ the unit matrix, so without loss of generality we may assume that the vacuum lies at

$$\langle\Phi\rangle \;=\; C \times \mathbf{1}_{N\times N} \quad i.e. \quad \left\langle \Phi^i{}_j \right\rangle \;=\; C \times \delta^i_j\,. \tag{14}$$

(c) Show that the symmetries (12) preserving these VEVs are limited to the $U_L = U_R \in SU(N)$ and $\theta = 0$. In other words, the $SU(N)_L \times SU(N)_R \times U(1)$ symmetry of the theory is spontaneously broken down to the $SU(N)_V$ subgroup.

Now that we know the vacuum state of the theory and its symmetry, let's find the particle spectrum of the theory.

(d) Expand the scalar potential in powers of the $\delta\Phi(x) = \Phi(x) - \langle\Phi\rangle$ and the $\delta\Phi^\dagger(x)$,

$$V(\delta\Phi^\dagger, \delta\Phi) \;=\; \text{const} \;+\; V_1 \;+\; V_2 \;+\; V_3 \;+\; V_4\,. \tag{15}$$

Show that $V_1 = 0$ while

$$V_2 \;=\; \frac{\alpha C^2}{2}\,\operatorname{tr}\big((\delta\Phi^\dagger + \delta\Phi)^2\big) \;+\; \frac{\beta C^2}{2}\,\operatorname{tr}^2(\delta\Phi^\dagger + \delta\Phi). \tag{16}$$

(e) Altogether, the N^2 complex scalar fields give rise to $2N^2$ particle species. Find the masses of all those particles from eq. (16).

Hint: Split the complex matrix Φ into its hermitian and antihermitian parts, and also into trace and traceless parts,

$$\delta\Phi(x) \;=\; \frac{\chi_1(x) + i\chi_2(x)}{\sqrt{2N}} \times \mathbf{1}_{N\times N} + \frac{\varphi_1(x) + i\varphi_2(x)}{\sqrt{2}} \tag{17}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are traceless hermitian matrices (or rather matrix-valued fields) while $\chi_1(x)$ and $\chi_2(x)$ are ordinary real fields.

(f) Finally, organize the $2N^2$ particles into multiplets of the unbroken $SU(N)_V$ symmetry and make sure that all members of each multiplet have the same mass.

Also, check the Nambu–Goldstone theorem for this model — verify that for each *spontaneously broken* generator of the symmetry (12) there is a massless particle with similar quantum numbers WRT the unbroken $SU(N)_V$ subgroup.

3. In the previous problem we had the continuous global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ spontaneously broken down to its $H = SU(N)_V$ subgroup. Now let's gauge the entire $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry and work out the Higgs mechanism.

Thus, consider a theory of N^2 complex scalar fields $\Phi^i{}_j(x)$ organized into an $N \times N$ matrix Φ, and $2N^2 - 1$ real vector fields $B_\mu(x)$, $L^a_\mu(x)$, and $R^a_\mu(x)$, the latter organized into traceless hermitian matrices $L_\mu(x) = \sum_a L^a_\mu(x) \times \frac{1}{2}\lambda^a$ and $R_\mu(x) = \sum_a R^a_\mu(x) \times \frac{1}{2}\lambda^a$, where $a = 1, \ldots, (N^2-1)$ and λ^a are the Gell-Mann matrices of $SU(N)$. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} - \tfrac{1}{2}\operatorname{tr}(L_{\mu\nu}L^{\mu\nu}) - \tfrac{1}{2}\operatorname{tr}(R_{\mu\nu}R^{\mu\nu}) + \operatorname{tr}\Big(D^\mu\Phi^\dagger D_\mu\Phi\Big) - V(\Phi^\dagger\Phi), \tag{18}$$

where

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\,, \\ L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu + ig[L_\mu, L_\nu], \\ R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu + ig[R_\mu, R_\nu], \\ D_\mu\Phi &= \partial_\mu\Phi + ig'B_\mu\Phi + igL_\mu\Phi - ig\Phi R_\mu\,, \\ D_\mu\Phi^\dagger &= (D_\mu\Phi)^\dagger = \partial_\mu\Phi^\dagger - ig'B_\mu\Phi^\dagger + igR_\mu\Phi^\dagger - ig\Phi^\dagger L_\mu\,. \end{aligned} \tag{19}$$

For simplicity, I assume equal gauge couplings $g_L = g_R = g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

Finally, the scalar potential V is precisely as in the previous problem,

$$V = \frac{\alpha}{2}\operatorname{tr}\big(\Phi^\dagger\Phi\Phi^\dagger\Phi\big) + \frac{\beta}{2}\operatorname{tr}^2\big(\Phi^\dagger\Phi\big) + m^2\operatorname{tr}\big(\Phi^\dagger\Phi\big), \qquad \alpha, \beta > 0, \quad m^2 < 0, \tag{20}$$

hence similar VEVs of the scalar fields: up to a gauge symmetry,

$$\langle\Phi\rangle = C \times \mathbf{1}_{N\times N} \quad \text{where} \quad C = \sqrt{\frac{-m^2}{\alpha + N\beta}}\,, \tag{21}$$

which breaks the $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry down to the $H = SU(N)_V$

subgroup.

(a) The Higgs mechanism makes N^2 out of $2N^2 - 1$ vector fields massive. Calculate their masses by plugging $\langle\Phi\rangle$ for the $\Phi(x)$ into the $\mathrm{tr}(D_\mu\Phi^\dagger\, D^\mu\Phi)$ term of the Lagrangian. In particular, show that the abelian gauge field B_μ and the $X_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a - R_\mu^a)$ combinations of the $SU(N)$ gauge fields become massive, while the $V_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a + R_\mu^a)$ combinations remain massless.

(b) Find the effective Lagrangian for the massless vector fields $V_\mu^a(x)$ by freezing all the other fields, *i.e.* setting $B_\mu(x) \equiv 0$, $X_\mu^a(x) \equiv 0$, and $\Phi(x) \equiv \langle\Phi\rangle$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.

⋆ For extra challenge, allow for un-equal gauge couplings $g_L \neq g_R$. Find which combinations of the $L_\mu^a(x)$ and $R_\mu^a(x)$ fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an $SU(N)$ Yang–Mills theory, but this time the gauge coupling is

$$g_v = \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}}\,. \tag{22}$$