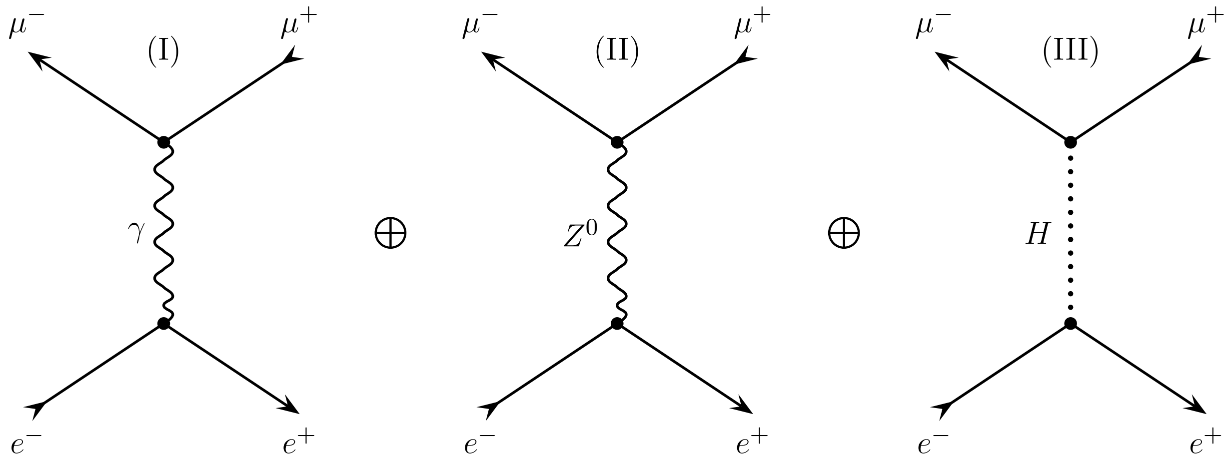
This is the second part of the over-large homework set 12. It has 2 problems about the electroweak interactions in the Standard Model. The first part (due 12/4) had 3 problems about Spontaneous Symmetry Breaking and the Higgs Mechanism.

1. Consider the $e^+e^- \to \mu^+\mu^-$ pair production in the Standard Model of electroweak interactions rather than in just QED. Unlike QED, the Standard Model has three tree-level diagrams contributing to this process: one with the virtual photon in the $s$ channel, one with the virtual $Z^0$ gauge boson, and one with the virtual Higgs scalar,

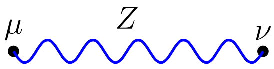



$$\mathcal{M}(e^-, e^+ \to \mu^-, \mu^+) \;=\; \mathcal{M}_\gamma \;+\; \mathcal{M}_Z \;+\; \mathcal{M}_H \,. \tag{1}$$

The first diagram was studied in class and also in the homework set#10. In this problem, we shall focus on the other two diagrams — especially the diagram (II) with a virtual $Z^0$ — and on their interference with the first diagram.

(a) Write down the amplitude $\mathcal{M}_H$ due to virtual Higgs scalar (diagram III). Also, relate the Yukawa couplings of the Higgs to the electrons and the muons to the fermion's masses, then argue that these couplings are so much smaller than the gauge couplings $e$ or $g'$ that the $\mathcal{M}_H$ is negligibly small compared to the $\mathcal{M}_Z$ or $\mathcal{M}_\gamma$ amplitudes.

Next, consider the amplitude due to a virtual $Z^0$ in the $s$ channel (diagram II). For simplicity, let's work in the unitary gauge where the 'eaten-up' scalars do not have any vertices or propagators, while the massive gauge bosons like $Z_0$ have propagators




$$= \frac{i}{q^2 - M_Z^2 + iM_Z\Gamma_Z} \times \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right). \tag{2}$$

(b) Derive the electron-$Z$ and muon-$Z$ vertices in diagram (II) from the neutral week current,

$$\mathcal{L} \supset -g' Z_\lambda \times \sum_{\substack{\text{quarks and}\\ \text{leptons}}} \overline{\Psi}\gamma^\lambda \left(T^3 \frac{1-\gamma^5}{2} - Q\sin^2\theta\right)\Psi, \tag{3}$$

*cf.* my notes on the electroweak interactions of quarks and leptons, and write down the amplitude $\mathcal{M}_Z$. For simplicity, approximate $\sin^2\theta \approx \frac{1}{4}$ (experimentally, $\sin^2\theta \approx 0.233$) so that for the charged leptons like the electron or the muon

$$T^3 \frac{1-\gamma^5}{2} - Q\sin^2\theta = \frac{-1+\gamma^5}{4} + \sin^2\theta \approx \frac{\gamma^5}{4}, \tag{4}$$

(c) Assume both the electrons and the muons to be ultra-relativistic ($E_{\text{c.m.}} = O(M_Z) \gg m_\mu, m_e$) and evaluate the amplitude $\mathcal{M}_Z$ for all possible particle helicities. (Use the center-of-mass frame.) I suggest you proceed exactly as in homework set#10 (problem 2) for the $\mathcal{M}_\gamma$ amplitude, but mind the $\gamma^5$ factors in the vertices and the massive vector propagator for the $Z_0$. Your answer should have form

$$\mathcal{M}_Z = \pm F(s) \times \mathcal{M}_\gamma \tag{5}$$

where the $\pm$ sign depends on the helicities while

$$F(s) = \left(\frac{e}{4g'}\right)^2 \frac{s}{s - M_Z^2 + i\Gamma_Z M_Z}. \tag{6}$$

(d) Combine the amplitudes due to virtual $Z$ and virtual photon and calculate the polarized partial cross-sections $d\sigma(e^-e^+ \to \mu^-\mu^+)/d\Omega$ as functions of CM energy$^2 = s$, scattering angle $\theta$, and helicities of all 4 fermions involved. Specifically, show that

$$\begin{aligned}
\frac{d\sigma(e_L^- + e_L^+ \to \mu_{\rm any}^- + \mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_R^- + e_R^+ \to \mu_{\rm any}^- + \mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} = 0, \\
\frac{d\sigma(e_{\rm any}^- + e_{\rm any}^+ \to \mu_L^- + \mu_L^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_{\rm any}^- + e_{\rm any}^+ \to \mu_R^- + \mu_R^+)}{d\Omega_{\rm c.m.}} = 0.
\end{aligned} \tag{7}$$

while

$$\begin{aligned}
\frac{d\sigma(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\rm c.m.}} \\
&= \frac{\alpha^2}{4s} \times |1+F(s)|^2 \times (1+\cos\theta)^2, \\
\frac{d\sigma(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\rm c.m.}} \\
&= \frac{\alpha^2}{4s} \times |1-F(s)|^2 \times (1-\cos\theta)^2,
\end{aligned} \tag{8}$$

where the $|1 \pm F(s)|^2$ factor stem from the interference between the virtual-photon and virtual-$Z$ diagrams.

(e) Finally, assume un-polarized electron and positron beams and a spin-blind muon detector. Calculate the total cross section $\sigma(e^+e^- \to \mu^+\mu^-)$ and the forward-backward asymmetry

$$A = \frac{\sigma(\theta < \pi/2) - \sigma(\theta > \pi/2)}{\sigma(\theta < \pi/2) + \sigma(\theta > \pi/2)} \tag{9}$$

as functions of the total energy $E_{\rm c.m.}$.

Note: In QED, the tree-level pair production is symmetric with respect to $\theta \to \pi - \theta$; the asymmetry in the Standard Model arises from the interference between the virtual-photon and virtual-$Z$ diagrams.

2. Here is another Standard Model problem. While most weak decays or quarks or leptons involve a virtual $W^+$ or $W^-$ gauge boson, the top quark is so heavy that it decays into a real (*i.e.*, on-shell) $W^+$ and the bottom quark. The interactions relevant to this process

are contained in

$$\mathcal{L} \supset -\frac{g}{2\sqrt{2}} W_\mu^- \overline{\Psi}_b \gamma^\mu (1-\gamma^5) \Psi_t - \frac{g}{2\sqrt{2}} W_\mu^+ \overline{\Psi}_t \gamma^\mu (1-\gamma^5) \Psi_b\,. \qquad (10)$$

For your information, $\alpha_w \equiv (g^2/4\pi) \approx 1/30$, $M_t \approx 173$ GeV, $M_W \approx 80.5$ GeV, and $M_b \approx 4.5$ GeV.

(a) Write down the Feynman vertices for the interactions (10), draw tree diagram(s) for the $t \to W^+ + b$ decay, and write down the tree-level decay amplitude.

(b) This amplitude does *not* satisfy the Ward identity. Write down a simple formula for $k_W^\mu \times \mathcal{M}_\mu$.

(c) The $W^+$ gauge boson is a massive particle, so it has 3 distinct spin/polarization states. Show that its polarization vectors $\mathcal{E}^\mu(k,\lambda)$ satisfy

$$\sum_\lambda \mathcal{E}^\mu(k,\lambda) \times \mathcal{E}^{*\nu}(k,\lambda) = -g^{\mu\nu} + \frac{k_W^\mu k_W^\nu}{M_W^2}\,, \qquad (11)$$

and consequently, for the $W$ emission amplitude of the form $\mathcal{M} = \mathcal{M}_\mu \times \mathcal{E}^\mu(k,\lambda)$, summing $|\mathcal{M}|^2$ over the $W$ polarizations yields

$$\sum_\lambda |\mathcal{M}|^2 = -\mathcal{M}^\mu \mathcal{M}_\mu^* + \frac{|\mathcal{M}_\mu k_W^\mu|^2}{M_W^2}\,. \qquad (12)$$

(d) Going back to the top quark decay $t \to b + W^+$, sum the $|\mathcal{M}|^2$ over both final particles' spins, average over the initial top quark's spin, and calculate the decay rate.

For simplicity, you may neglect the bottom quark's mass compared to masses of the top quark and of the $W$ boson.