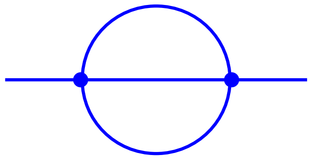
1. As discussed in class, in the $\lambda\Phi^4$ theory the field strength renormalization begins at the two-loop level. Specifically, the leading contribution to the $d\Sigma(p^2)/dp^2$ — and hence to the $Z-1$ — comes from the two-loop 1PI diagram



(1)

Your task is to evaluate this contribution.

(a) First, write the $\Sigma(p^2)$ from the diagram (1) as an integral over two independent loop momenta, say q_1^μ and q_2^μ, then use the Feynman's parameter trick — *cf.* eq. (F.d) of the homework set#13 — to write the product of three propagators as

$$\iiint d\xi\, d\eta\, d\zeta\, \delta(\xi+\eta+\zeta-1)\,\frac{2}{(\mathcal{D})^3} \tag{2}$$

where $\mathcal{D}$ is a quadratic polynomial of the momenta q_1, q_2, p, and mass m with Feynman-parameter dependent coefficients.

Warning: Do not set $p^2=m^2$ but keep p an independent variable.

(b) Next, change the independent loop momentum variables from q_1 and q_2 to $k_1 = q_1 + \text{something}\times q_2 + \text{something}\times p$ and $k_2 = q_2 + \text{something}\times p$ to give $\mathcal{D}$ a simpler form

$$\mathcal{D} \;=\; \alpha\times k_1^2 \;+\; \beta\times k_2^2 \;+\; \gamma\times p^2 \;-\; m^2 \;+\; i0 \tag{3}$$

for some (ξ,η,ζ)–dependent coefficients α,β,γ, for example

$$\alpha \;=\; (\xi+\zeta), \qquad \beta \;=\; \frac{\xi\eta+\xi\zeta+\eta\zeta}{\xi+\zeta}\,, \qquad \gamma \;=\; \frac{\xi\eta\zeta}{\xi\eta+\xi\zeta+\eta\zeta}\,. \tag{4}$$

Make sure the momentum shift has unit Jacobian $\partial(q_1,q_2)/\partial(k_1,k_2)=1$.

(c) Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1\, d^4k_2\, \frac{1}{\mathcal{D}^4}. \tag{5}$$

Note that although this momentum integral diverges as $k_{1,2} \to \infty$, the divergence is logarithmic rather than quadratic.

(d) To evaluate the momentum integral (5), Wick-rotate the momenta k_1 and k_2 to the Euclidean space, and then use the dimensional regularization. Here are some useful formulæ for this calculation:

$$\frac{6}{A^4} = \int_0^\infty dt\, t^3\, e^{-At}, \tag{6}$$

$$\int \frac{d^Dk}{(2\pi)^D}\, e^{-ctk^2} = \bigl(4\pi ct\bigr)^{-D/2}, \tag{7}$$

$$\Gamma(2\epsilon)X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \tfrac{1}{2}\log X + O(\epsilon). \tag{8}$$

(e) Assemble your results as

$$\frac{d\Sigma(p^2)}{dp^2} = -\frac{\lambda^2}{12(4\pi)^4} \iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \times \left(\frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} + \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2}\right). \tag{9}$$

(f) Before you evaluate the Feynman parameter integral (9) — which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually calculating the integrals

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3}, \tag{10}$$

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2},$$

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where $\xi,\eta \to 0$ while $\zeta \to 1$ (or $\xi,\zeta \to 0$, or $\eta,\zeta \to 0$).

- This calculation shows that

$$\frac{d\Sigma}{dp^2} = \frac{\text{constant}}{\epsilon} + \textit{a\_finite\_function}(p^2) \tag{11}$$

and hence

$$\begin{aligned}\Sigma(p^2) = \text{(a divergent constant)} &+ \text{(another divergent constant)} \times p^2 \\ &+ \textit{a\_finite\_function}(p^2)\end{aligned} \tag{12}$$

up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of $\Sigma(p^2)$ may be canceled with just two counterterms, δ^m and $\delta^Z \times p^2$.

For the purposes of calculating the field strength renormalization factor

$$Z = \left[1 - \frac{d\Sigma}{dp^2}\right]^{-1} \tag{13}$$

we need to evaluate the derivative $d\Sigma/dp^2$ at $p^2 = M_{\rm ph}^2$ — the physical mass^2 of the scalar particle. However, to the leading non-trivial order in λ we may approximate $M_{\rm ph}^2 \approx m_{\rm bare}^2$ and set $p^2 = m^2$ in the Feynman-parameter integral (9). Consequently, the second integral (10) becomes a little simpler, although it is still a frightful mess.

$\star$ Optional exercise: Evaluate the integrals (10) for $p^2 = m^2$ and show that

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} = \frac{1}{2}, \tag{14}$$

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta)^2} = -\frac{3}{4}.$$

Do not try to do this calculation by hand — it would take way too much time. Instead, use *Mathematica* or equivalent software. To help it along, replace the (ξ, η, ζ) variables with (x, w) according to

$$\begin{gathered}\xi = w \times x, \quad \eta = w \times (1-x), \quad \zeta = 1-w, \\ \iiint d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) = \int_0^1 dx \int_0^1 dw\, w,\end{gathered} \tag{15}$$

then integrate over w first and over x second.

Alternatively, you may evaluate the integrals like this numerically. In this case, don't bother changing variables, just use a simple 2D grid spanning a triangle defined by $\xi + \eta + \zeta = 1$, $\xi, \eta, \zeta \geq 0$; modern computers can sum up a billion grid points in less than a minute. But watch out for singularities at the corners of the triangle.

(g) Finally, assemble your results and calculate the field strength renormalization factor Z to the two-loop order.

2. [Based on *Peskin and Schroeder* problem 10.2(a).] Consider the Yukawa theory of a Dirac field $\Psi(x)$ and a real pseudoscalar field $\Phi(x)$, with the *physical Lagrangian*

$$\mathcal{L}_{\text{ph}} = \tfrac{1}{2}(\partial_\mu \Phi)^2 - \tfrac{1}{2}m^2\Phi^2 + \overline{\Psi}(i\not\partial - M)\Psi - ig\Phi \times \overline{\Psi}\gamma^5\Psi - \tfrac{1}{24}\lambda\Phi^4. \tag{16}$$

Note parity symmetry of this theory.

(a) Find all the superficially divergent amplitudes of this theory and the degrees of divergence of these amplitudes.

(b) Write down all the counterterms needed to cancel all the divergent amplitudes, and spell out the Feynman rules for the counterterm perturbation theory. Also, make sure that all the counterterms obtain from the *bare Lagrangian* of the Yukawa theory.

(c) Without actually evaluating any loop diagrams, argue that the δ^λ counterterm does not vanish when $\lambda = 0$ but $g \neq 0$. Consequently, the theory must have non-zero *bare* $\lambda_b \Phi_b^4$ coupling even if the physical λ_{ph} coupling happens to vanish. In other words, having $\lambda_{\text{ph}} = 0$ would be an accident of fine tuning but not a natural value of the physical coupling.

- The actual calculation of the counterterms (or at least their infinite parts) at the one-loop level is postponed to the next homework set.