1. As a warm-up exercise, consider the electric charge renormalization in QED.

   In general, a coupling g of an operator involving n fields $\hat{\phi}_a(x),\ldots,\hat{\phi}_n(x)$ has beta-function

$$\beta_g \;=\; (\gamma_1+\cdots+\gamma_n)\times\bigl(g+\delta^g\bigr)\;-\;\frac{d\delta^g}{d\log E}\,, \tag{1}$$

   cf. my notes on the renormalization group, eq. (131) on page 25. When applied to QED, this formula yields

$$\beta_e \;=\; (2\gamma_e+\gamma_\gamma)\times\bigl(e+e\delta_1\bigr)-\frac{d(e\delta_1)}{d\log E}\,. \tag{2}$$

   Use the Ward identity $\delta_1(E)=\delta_2(E)$ to reduce this formula to

$$\beta_e \;=\; \gamma_\gamma\times e. \tag{3}$$

2. Now consider the electron mass renormalization in QED. At high energies $E\gg m_e$, we may treat the electron mass m_e as a small coupling between the left-handed and the right-handed Weyl spinor components of the electron field $\hat{\Psi}(x)$. Consequently, we may write the renormalization group equation for the running electron mass $m_e(E)$ just as we would write it for any other kind of a small coupling,

$$\frac{dm_e(E)}{d\log E} \;=\; \beta_m(m_e,\alpha). \tag{4}$$

   Your task is to calculate the β-function here to the one-loop order, and then to solve the RGE (4).

   (a) Before you calculate anything, use the axial symmetry of a massless electron to argue that for a small but non-zero electron's mass, the counterterm δ_m should be proportional to m_e itself and therefore be logarithmically rather than quadratically divergent,

$$\delta_m\;\sim\;\alpha m_e\times\log\Lambda\quad\text{rather than}\quad\delta_m\;\sim\;\alpha\times\Lambda. \tag{5}$$

   (b) Now, a bit of hard work: Calculate to the one-loop order the UV-infinite parts of the δ_2 and δm counterterms as functions of the gauge-fixing parameter ξ. Assume

$E \gg m_e$ and use the off-shell renormalization condition for these counterterms:

$$\begin{aligned} \text{expand} \quad \Sigma_{\text{net}}(\not{p}) \;&=\; A(p^2) \times \not{p} \;+\; B(p^2) \times m \\ \text{and demand} \quad A \;=\; B \;&=\; 0 \quad \text{for} \quad p^2 \;=\; -E^2. \end{aligned} \tag{6}$$

The counterterms you obtain in part (a) should have form

$$\delta_2(E) \;=\; \frac{C_2(\xi)\alpha}{2\pi} \times \left(\frac{1}{\epsilon} \;+\; \log\frac{\mu^2}{E^2} \;+\; \text{const}\right), \tag{7}$$

$$\delta_m(E) \;=\; \frac{C_m(\xi)\alpha m}{2\pi} \times \left(\frac{1}{\epsilon} \;+\; \log\frac{\mu^2}{E^2} \;+\; \text{const}\right), \tag{8}$$

for some ξ–dependent coefficients $C_2(\xi)$ and $C_m(\xi)$.

(c) Check that the difference $C_m(\xi) - C_2(\xi)$ does not depend on the gauge parameter ξ. If it does, go back to part (b) and check for mistakes.

Applying the general formula (1) for the β-function of an n-field coupling to the running electron mass $m_e(E)$, we get

$$\beta_m \;=\; 2\gamma_e \times \big(m(E) + \delta_m(E)\big) \;-\; \frac{d\delta_m(E)}{d\log(E)}\,. \tag{9}$$

(d) Use eqs. (7) and (8) to show that to the one-loop order this formula yields

$$\frac{dm_e(E)}{d\log E} \;=\; \beta_m \;=\; 2(C_m - C_2) \times \frac{\alpha(E)m(E)}{2\pi} \;+\; O(\alpha^2 m). \tag{10}$$

Note: the running mass should be gauge invariant, that's why I asked you to check the ξ–independence of $C_m - C_2$ in part (c).

Finally, let's focus on the basic QED (EM and electron fields, and nothing else) where the running gauge coupling depends on the energy according to

$$\frac{d\alpha(E)}{d\log E} \;=\; \frac{2\alpha^2}{3\pi} \;+\; O(\alpha^3). \tag{11}$$

(e) Solve the differential equation (10) for the $\alpha(E)$ as in eq. (11). Show that the solution has form

$$m(E) \; = \; m_0 \times \left(\frac{\alpha(E)}{\alpha_0}\right)^r \tag{12}$$

for some power r, and calculate that power.

Hint: let $m(E) = F(\alpha(E))$, derive the differential equation for the $F(\alpha)$, and solve that equation.

3. Finally, a couple of optional reading assignment to the students who need them. There is no specific due date for these assignements, but I hope you can finish them by the end of the semester. Since they are rather long, I suggest you start reading them this week, and then continue after the modterm exam whenever you have free time.

   (a) In class I shall give you a quick and dirty introduction to the path integrals. But it would be very helpful — both for the QFT class and for your general education — to get more familiar with the subject. So please read *Quantum Mechanics and Path Integrals* by Richard Feynman and Albert Hibbs about care and use of the Path Integrals. The PMA library has a few paper colies of this book or you can read it online at scribd.com at this link.

   (b) Basic group theory is rather important to the non-abelian gauge theories like QCD. I shall try to explain the relevant issues in class, but due to time constraints I would have to be brief, and it would help if you already know the basics. So, if you are unfamiliar with the group theory — especially the continuous group theory, — read *Lie Algebras in Particle Physics: from Isospin to Unified Theories* by Howard Georgi (1999, Westview press, ISBN 9780813346113). This book is available in electronic format at the UT library at (this link). Since you cannot finish the whole book by the time you would need the knowledge — which should be about a week after the break, — start by *carefully* reading the first 3 chapters, and then browse through chapters on the $SU(2)$, the $SU(3)$, and the color.