1. In problem **3** from the previous homework set#20 you (should have) calculated the free energy of a free scalar field at a finite temperature using the functional integral methods. In this problem, you are going to apply similar methods to the free fermionic field and to the free EM field.

   As a warm-up exercise, consider a free fermion in $0 + 1$ dimensions, basically a two-level system in Quantum Mechanics. In the Hamiltonian formulation this means

$$\hat{H} \;=\; \omega\hat{\psi}^\dagger\hat{\psi} \quad \text{where} \quad \hat{\psi}^2 \;=\; (\hat{\psi}^\dagger)^2 \;=\; 0, \quad \{\hat{\psi},\hat{\psi}^\dagger\} \;=\; 1, \quad \text{and} \quad \omega = \text{constant} > 0, \tag{1}$$

   while in the Lagrangian formulation $\psi(t)$ and $\psi^*(t)$ are Grassmann-number-valued functions of time (real or imaginary) and

$$L_E \;=\; \psi^* \times \frac{d\psi}{dt_e} \;+\; \omega \times \psi^*\psi. \tag{2}$$

   At finite temperature, all measurable operators must be periodic in Euclidean time with period β, but for the fermionic fields this means that the bilinears must be periodic while the fermionic fields themselves can be either periodic or anti-periodic, $\psi(t_e + \beta) = \pm\psi(t_e)$.

   (a) To determine the right choice — periodic or anti-periodic, — use the functional integral to calculate the partition function for both types of boundary conditions for the fermionic variables in the Euclidean time, $\psi(t_E+\beta) = \pm\psi(t_E)$. Show that the periodic condition leads to an unphysical partition function, while the anti-periodic condition leads to the correct partition function of a two-level system.

   The lesson of part (a) applies to fermionic fields in all spacetime dimensions D: At a finite temperature $\mathcal{T} = 1/\beta$, all fermionic fields must be antiperiodic in the Euclidean time $x_D = t_e = ix_0$,

$$\Psi(\mathbf{x}, t_e + \beta) \;=\; -\Psi(\mathbf{x}, t_e). \tag{3}$$

   (b) Now let's apply this lesson to the free Dirac field in 3+1 dimensions. Use the functional integral over the fermionic fields to show that the free energy of this field is

$$F(\mathcal{T}) \;=\; -\mathcal{T}\log\mathrm{Det}_{\mathrm{functional}}[\partial\!\!\!/_E+m]. \tag{4}$$

Then evaluate the functional trace here in the momentum basis and show that the free energy *density* is

$$\mathcal{F}(\mathcal{T}) \stackrel{\mathrm{def}}{=} \frac{F(\mathcal{T})}{\mathrm{Volume}} \;=\; -\mathcal{T}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{p_4}\mathrm{tr}_{\mathrm{Dirac}}\big(\log(i\,p\!\!\!/_E+m)\big) \tag{5}$$

$$\;=\; -\mathcal{T}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{p_4}2\log(p_E^2+m^2). \tag{6}$$

where the sum over p_4 runs over

$$p_4 \;=\; 2\pi\mathcal{T}\times\big(\text{a half-integer}=\pm\tfrac{1}{2},\pm\tfrac{3}{2},\pm\tfrac{5}{2},\ldots\big). \tag{7}$$

(c) Use Poisson resummation — *cf.* the preamble of the previous homework#20 — to show that

$$\mathcal{F}(\mathcal{T})\;-\;\mathcal{F}(0) \;=\; 4\sum_{\ell=1}^{+\infty}(-1)^{\ell-1}\int\!\frac{d^4p_E}{(2\pi)^4}\,e^{i\beta\ell p_4}\times\log(p_E^2+m^2) \tag{8}$$

and hence

$$\mathcal{F}(\mathcal{T})\;-\;\mathcal{F}(0) \;=\; -4\mathcal{T}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\log\Big(1+e^{-\beta E(\mathbf{k})}\Big)\,, \tag{9}$$

in perfect agreement with the Fermi–Dirac statistics.

Finally, consider the free electromagnetic field $A_\mu(x)$. At finite temperature, the $A^\mu(x)$ — just like any other bosonic field — is periodic in the Euclidean time, $A^\mu(\mathbf{x},x_4+\beta)=+A^\mu(\mathbf{x},x_4)$. Calculate the partition function for the periodic EM field and mind the gauge-fixing terms in the Lagrangian and the Fadde'ev–Popov determinant in the functional integral.

(d) Show that formally, the EM free energy is

$$F(\mathcal{T}) \;=\; +\mathcal{T} \times \operatorname{Tr}\log\bigl(-\partial_E^2\bigr) \tag{10}$$

where the trace is over the Hilbert space of 4D wave functions periodic in the Euclidean time.

(e) Recycle arguments from the scalar field case in the previous homework set to show that eq. (10) leads to

$$\mathcal{F}(T) \;-\; \mathcal{F}(0) \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\, 2T \times \left(1 - e^{-\beta|p|}\right). \tag{11}$$

2. Next, an exercise in group theory you would need for QCD and QCD-like gauge theories. Consider a generic simple non-abelian compact Lie group G and its generators T^a. For a suitable normalization of the generators,

$$\operatorname{tr}_{(r)}(T^aT^b) \;\equiv\; \operatorname{tr}\left(T^a_{(r)}T^b_{(r)}\right) \;=\; R(r)\delta^{ab} \tag{12}$$

where the trace is taken over any complete multiplet (r) — irreducible or reducible, it does not matter — and $T^a_{(r)}$ is the matrix representing the generator T^a in that multiplet. The coefficient $R(r)$ in eq. (12) depends on the multiplet (r) but it's the same for all generators T^a and T^b. This coefficient $R(r)$ is called the *index* of the multiplet (r).

The quadratic Casimir operator $C_2 = \sum_a T^aT^a$ of the Lie algebra commutes with all the algebra's generators, $\forall b : [C_2, T^b] = 0$. Consequently, when we restrict this operator to any *irreducible* multiplet (r) of the group G, it becomes a unit matrix times some number $C(r)$. In other words,

$$\text{for an irreducible } (r), \quad \sum_a T^a_{(r)}T^a_{(r)} \;=\; C(r) \times \mathbf{1}_{(r)}\,. \tag{13}$$

For example, for the isospin group $SU(2)$, the Casimir operator is $C_2 = \vec{I}^2$, the irreducible multiplets have definite isospins $I = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and the corresponding Casimir eigenvalues are $C(I) = I(I+1)$.

(a) Show that for any irreducible multiplet (r), its index $R(r)$ and its Casimir eigenvalue $C(r)$ are related to each other as

$$\frac{R(r)}{C(r)} = \frac{\dim(r)}{\dim(G)}\,. \tag{14}$$

In particular, for the $SU(2)$ group, this formula gives

$$R(I) = \tfrac{1}{3}I(I+1)(2I+1). \tag{15}$$

(b) Suppose the first three generators T^1, T^2, and T^3 of G generate an $SU(2)$ subgroup, thus

$$[T^1,T^2] = iT^3, \quad [T^2,T^3] = iT^1, \quad [T^3,T^1] = iT^2. \tag{16}$$

Show that if a multiplet (r) of G decomposes into several $SU(2)$ multiplets of isospins $I_1, I_2, \ldots, I_n$, then

$$R(r) = \sum_{i=1}^{n} \tfrac{1}{3}I_i(I_i+1)(2I_i+1). \tag{17}$$

(c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting only on the first two components of a complex N-vector. This complex N-vector is called the fundamental multiplet (of the $SU(N)$) and denoted (N) or $\mathbf{N}$. As far as the $SU(2)$ subgroup is concerned, (N) comprises one doublet and $N-2$ singlets, hence

$$R(N) = \frac{1}{2} \quad \text{and} \quad C(N) = \frac{N^2-1}{2N}\,. \tag{18}$$

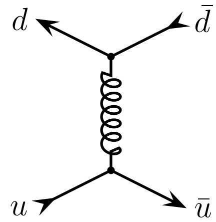
Show that the adjoint multiplet of the $SU(N)$ decomposes into one $SU(2)$ triplet, $2(N-2)$ doublets, and $(N-2)^2$ singlets, therefore

$$R(\text{adj}) = C(\text{adj}) \equiv C(G) = N. \tag{19}$$

Hint: $(N) \times (\overline{N}) = (\text{adj}) + (1)$.

(d) The symmetric and the anti-symmetric 2–index tensors form irreducible multiplets of the $SU(N)$ group. Find out the decomposition of these multiplets under the $SU(2) \subset SU(N)$ and calculate their respective indices R and Casimirs C.

3. Finally, let's apply this group theory to physics. Consider quark-antiquark pair production in QCD, specifically $u\bar{u} \to d\bar{d}$. There is only one tree diagram contributing to this process,



(20)

Evaluate this diagram, then sum/average the $|\mathcal{M}|^2$ over both spins and *colors* of the final/initial particles to calculate the total cross section. For simplicity, you may neglect the quark masses.

Note that the diagram (20) looks exactly like the QED pair production process $e^-e^+ \to$ virtual $\gamma \to \mu^-\mu^+$, so you can re-use the QED formula for summing/averaging over the spins, *cf.* my notes on Dirac traceology from the Fall semester, pages 10–13. But in QCD, you should also sum/average over the colors of all the quarks, and that's the whole point of this exercise.