1. In class we have focused on QCD and QCD-like theories of non-abelian gauge fields coupled to Dirac fermions in some multiplet(s) of the gauge group G, *cf.* my notes on QCD Feynman rules and Ward identities. This problem is about the scalar QCD, or more generally a non-abelian gauge theory with some gauge group G and complex scalar fields $\Phi^i(x)$ in some multiplet (r) of G.

   (a) Write down the physical Lagrangian of this theory, the complete bare Lagrangian of the quantum theory in the Feynman gauge, and the Feynman rules.

   Now consider the annihilation process $\Phi + \Phi^* \to 2$ gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

   (b) Draw the diagrams and write down the tree-level annihilation amplitude.

   As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: *On-shell Amplitudes involving* **a** *longitudinally polarized gauge bosons vanish, provided all the other gauge bosons are transversely polarized.* In other words,

$$\mathcal{M} \;\equiv\; e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n}\, \mathcal{M}_{\mu_1\mu_2\cdots\mu_n}(\text{momenta}) \;=\; 0$$
$$\text{when}\quad e_1^\mu \propto k_1^\mu \quad \text{but}\quad e_2^\nu k_{2\nu} \;=\; \cdots \;=\; e_n^\nu k_{n\nu} \;=\; 0.$$

   (c) Verify this identity for the scalar annihilation amplitude: Show that IF $e_2^\nu k_{2\nu} = 0$ THEN $k_{1\mu}\mathcal{M}^{\mu\nu} e_{2\nu} = 0$.

   Similar to what we had in class for the quark-antiquark annihilation, there are non-zero amplitudes for the scalar 'quark' and 'antiquark' annihilating into a pair of longitudinal gluons or a ghost-antighost pair, but the cross-sections for these two unphysical processes cancel each other.

   (d) Take both final-state gluons to be longitudinally polarized; specifically, in the center-of-mass frame let $e_1^\mu = (1, +\mathbf{n}_1)/\sqrt{2}$ for the first gluon and $e_2^\nu = (1, -\mathbf{n}_2)/\sqrt{2}$ for the second gluon.
   Calculate the tree-level annihilation amplitude $\Phi + \Phi^* \to g_L + g_L$ for these polarizations.

(e) Next, calculate the tree amplitude for the $\Phi + \Phi^* \to \text{gh} + \overline{\text{gh}}$. There is only one tree graph for this process, so evaluating it should not be hard.

(f) Compare the two un-physical amplitudes and show that the corresponding partial cross-sections cancel each other, thus

$$\frac{d\sigma_{\text{net}}}{d\Omega} = \frac{d\sigma_{\text{physical}}}{d\Omega}\,. \tag{1}$$

2. Now let's go back to the ordinary QCD and evaluate a few one-loop diagrams. In class, I have calculated the (infinite parts of the) δ_2 and δ_1 counterterms for the quarks, *cf.* my notes on QCD beta-function. Your task is to calculate the analogous $\delta_2^{(\text{gh})}$ and $\delta_1^{(\text{gh})}$ counterterms for the *ghosts fields.*

   (a) Draw one-loop diagrams whose divergences are canceled by the respective counterterms $\delta_2^{(\text{gh})}$ and $\delta_1^{(\text{gh})}$, and calculate the group factors for each diagrams.

   (b) Calculate the momentum integrals for the diagrams. Focus on the UV divergences and ignore the finite parts of the integrals.

   (c) Assemble your results and show that the *difference* $\delta_1^{(\text{gh})} - \delta_2^{(\text{gh})}$ for the ghosts is exactly the same as the $\delta_1 - \delta_2$ difference for the quarks.

3. Finally, consider the three gauge couplings of the $SU(3) \times SU(2) \times U(1)$ Standard Model and their one-loop beta-functions

$$\beta_1^{\text{1 loop}} = \frac{b_1 g_1^3}{16\pi^2}\,, \quad \beta_2^{\text{1 loop}} = \frac{b_2 g_2^3}{16\pi^2}\,, \quad \beta_3^{\text{1 loop}} = \frac{b_3 g_3^3}{16\pi^2}\,. \tag{2}$$

   In this exercise, you do not need to calculate these beta-function from scratch by evaluating the UV divergences of a bunch of loop diagrams. Instead, use eqs. (122) and (124–5) from my notes on QCD beta-function (pages 25–26).

   (a) Calculate the b_1, b_2, b_3 coefficients for the minimal version of the Standard Model: the $SU(3) \times SU(2) \times U(1)$ gauge fields, one Higgs doublet, three families of quarks and leptons, and nothing else.

- ⋆ FYI, each family comprises 8 left-handed Weyl fields in the $(\mathbf{3},\mathbf{2},y=+\frac{1}{6})$ and $(\mathbf{1},\mathbf{2},y=-\frac{1}{2})$ multiplets of the gauge symmetry and 7 right-handed Weyl fermions in the $(\mathbf{3},\mathbf{1},y=+\frac{2}{3})$, $(\mathbf{3},\mathbf{1},y=-\frac{1}{3})$, and $(\mathbf{1},\mathbf{1},y=-1)$ multiplets.

(b) Re-calculate the b_1, b_2, b_3 for the MSSM — the Minimal Supersymmetric Standard Model. FYI, here is complete list of the MSSM fields:

- ∘ The $SU(3)\times SU(2)\times U(1)$ gauge fields, same as the non-SUSY SM.
- • For each vector field there is a Majorana fermion (a gaugino) with similar $SU(3)\times SU(2)\times U(1)$ quantum numbers. Altogether, there is an adjoint multiplet of gauginos for each factor of the gauge symmetry.
- ∘ 3 families of quarks and leptons, same as the non-SUSY SM.
- • For each Weyl fermion — left-handed or right-handed — in these three families, the MSSM also have a complex scalar field (a squark or a slepton) with similar $SU(3)\times SU(2)\times U(1)$ quantum numbers. Altogether, this makes 45 complex scalar fields in similar multiplets to the quarks and leptons.
- • The Higgs sector of the MSSM comprises *two* $SU(2)$ doublets of complex scalars accompanied by one $SU(2)$ doublet of Dirac fermions (the higgsinos); all these doublets have $y=\frac{1}{2}$.
- — There are all kinds of Yukawa and ϕ^4 interactions between the MSSM fields, but you do not need them for the one-loop calculation of the gauge couplings' beta-functions.

In Grand Unified Theories

$$\alpha_3 \;=\; \alpha_2 \;=\; \tfrac{5}{3}\alpha_1 \;=\; \alpha_{\rm GUT} \quad \text{at the GUT scale.} \tag{3}$$

At lower energy scales $E\ll M_{\rm GUT}$ the SM couplings are given (lo the leading one-loop order) by the Georgi–Quinn–Weinberg equations

$$\begin{aligned} \frac{1}{\alpha_3(E)} &= \frac{1}{\alpha_{\rm GUT}} + b_3\times\frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,,\\ \frac{1}{\alpha_2(E)} &= \frac{1}{\alpha_{\rm GUT}} + b_2\times\frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,,\\ \frac{1}{\alpha_1(E)} &= \frac{5/3}{\alpha_{\rm GUT}} + b_1\times\frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,. \end{aligned} \tag{4}$$

(c) Derive these equations from eqs. (2).

The experimental data are usually interpreted in terms of the $\overline{\mathrm{MS}}$ gauge couplings at the Z^0 mass $M_Z \approx 91$ GeV; according to the latest particle data group publication

$$\frac{1}{\alpha_3(M_Z)} \approx 8.45 \pm 0.12, \quad \frac{1}{\alpha_2(M_Z)} \approx 29.585 \pm 0.005, \quad \frac{1}{\alpha_1(M_Z)} \approx 98.369 \pm 0.009. \quad (5)$$

Since the top quark and the Higgs boson are heavier than M_Z, let me translate these data to the $\overline{\mathrm{MS}}$ couplings at $E = M_{\mathrm{top}} \approx 173$ GeV:

$$\frac{1}{\alpha_3(M_t)} \approx 9.18 \pm 0.12, \quad \frac{1}{\alpha_2(M_t)} \approx 30.028 \pm 0.005, \quad \frac{1}{\alpha_1(M_t)} \approx 97.84 \pm 0.01. \quad (6)$$

(d) Check that these data are *not* consistent with eq. (4) for the minimal Standard Model.

(e) Now consider the Minimal Supersymmetric Standard Model. For simplicity, assume that all the super-partners — or rather all particles of the MSSM not present in the non-supersymmetric minimal SM — have masses $M \approx M_{\mathrm{top}} \approx 173$ GeV; this has been ruled out experimentally, but it's a useful toy model.

    Show that for this model — unlike for the minimal non-SUSY Standard Model, — the experimental gauge couplings (6) are consistent with the Georgi–Quinn–Weinberg eqs. (4). Also, calculate the GUT scale M_{GUT} for this model.

(f) Finally, consider a more realistic model, namely MSSM in which all the extra particles have the same mass $M_S = 2$ TeV, just out of LHS's reach.

    To check the consistency of this model, first extrapolate the experimental gauge couplings (6) from the M_{top} scale to the M_S scale using the beta-function coefficients $b_{1,2,3}$ of the non-SUSY Standard Model. And then check whether the resulting gauge couplings are consistent with eq. (4) for the MSSM.