1. Let's start with a few reading assignments:

   (a) §19.1 of *Peskin & Schroeder.* Read about the Hamiltonian picture of the axial anomaly in $1+1$ dimensions.

   (b) §19.2 of *Peskin & Schroeder,* first two subsections. Read how to evaluate the triangle diagrams using different UV regulators from what I had used in class, namely point-splitting (subsection 1) and dimensional regularization (subsection 2).

   (c) §19.2 of *Peskin & Schroeder,* third subsection, and §22.2–3 of *Weinberg.* Read about formal analysis of the axial anomaly stemming from the measure of the fermionic functional integral. Both *Peskin & Schroeder* and *Weinberg* explain regulating the Jacobian of the axial variable transform along the lines I used in class, but pay particular attention too the issue I did not explain, namely why the UV regulator should be a function of the $-\not{D}^2$ operator, $\hat{G} = G(-\not{D}^2/\Lambda^2)$.

2. Following up on *Weinberg*'s analysis of the axial anomaly of the fermionic functional integral's measure in $d = 4$ dimensions, let's generalize it to other *even* spacetime dimensions $d = 2n$. In any such dimension there a matrix Γ which acts as the γ^5 in 4D — $\Gamma^2 = +1$ while $\Gamma\gamma^\mu = -\gamma^\mu\Gamma$ for all $\mu = 1, 2, \ldots d$. Consequently, a massless Dirac fermion in $d = 2n$ dimensions has a classical axial symmetry

$$\Psi(x) \;\to\; \exp(i\theta\Gamma)\Psi(x), \qquad \overline{\Psi}(x) \;\to \Psi(x)\exp(i\theta\Gamma), \tag{1}$$

which leads to a classically conserved current

$$J^\mu_A \;=\; \overline{\Psi}\gamma^\mu\Gamma\Psi, \qquad \partial_\mu J^\mu_A \;= \text{classically} =\; 0. \tag{2}$$

But when the fermion Ψ is coupled to a gauge field — or a multiplet of such fermions is coupled to a non-abelian gauge field — the axial symmetry is broken by the anomaly, thus

$$\partial_\mu J^\mu_A \;=\; -\frac{2}{n!}\left(\frac{-1}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\,\operatorname{tr}\Big(\mathcal{F}_{\alpha_1\beta_1}\mathcal{F}_{\alpha_2\beta_2}\cdots\mathcal{F}_{\alpha_n\beta_n}\Big). \tag{3}$$

Your task is to derive this formula from the UV-regulated Jacobian of the fermionic path integral.

For your information, in any even Euclidean dimension $d = 2n$,

$$\{\gamma^\mu, \gamma^\nu\} = +2\delta^{\mu\nu}, \qquad [\gamma^\mu, \gamma^\nu] = +2i\sigma^{\mu\nu}, \tag{4}$$

$$\Gamma = i^n \gamma^1 \gamma^2 \cdots \gamma^{2n} \implies \Gamma^2 = +1, \quad \Gamma\gamma^\mu = -\gamma^\mu \Gamma \ \forall \mu = 1, \ldots, 2n, \tag{5}$$

$$\text{for any } d = 2n \text{ Dirac matrices } \gamma^\alpha, \ldots, \gamma^\omega : \quad \operatorname{tr}\bigl(\Gamma \gamma^\alpha \gamma^\beta \cdots \gamma^\omega\bigr) = (-2i)^n \epsilon^{\alpha\beta\cdots\omega}, \tag{6}$$

$$\text{for any } n \text{ spin matrices } \sigma^{\alpha\beta}, \ldots, \sigma^{\psi\omega} : \quad \operatorname{tr}\bigl(\Gamma \sigma^{\alpha\beta} \cdots \sigma^{\psi\omega}\bigr) = 2^n \, \epsilon^{\alpha\beta\cdots\psi\omega}. \tag{7}$$

3. This problem exists in two versions, 3A and 3B. If you are familiar with the differential form language, please solve problem 3A; otherwise, please solve problem 3B. The two versions are physically equivalent to each other, but the math is written in a different language.

(3A) In the differential form language, eqs. (3) become

$$d * J_A = -\frac{2}{n!} \left(\frac{-1}{4\pi}\right)^n \operatorname{tr}\bigl(\mathcal{F} \wedge \mathcal{F} \wedge \cdots \wedge \mathcal{F}\bigr)_{n\,\text{times}}. \tag{8}$$

The $2n$-forms on the RHS of these formulae are exact:

$$Q_{(2n)} \stackrel{\text{def}}{=} \operatorname{tr}\bigl(\mathcal{F} \wedge \mathcal{F} \wedge \cdots \wedge \mathcal{F}\bigr)_{n\,\text{times}} = d\Omega_{(2n-1)} \tag{9}$$

where $\Omega_{(2n-1)}$ — constructed as traces of appropriated products of the $\mathcal{A} = gA$ gauge fields (1 forms) and $\mathcal{F} = gF$ tensions fields (2 forms) — are the *Chern–Simons* forms. Specifically,

$$\Omega_{(1)} = \operatorname{tr}(\mathcal{A}) \qquad [\text{abelian } \mathcal{A} \text{ only}], \tag{10.a}$$

$$\Omega_{(3)} = \operatorname{tr}\Bigl(\mathcal{A} \wedge \mathcal{F} - \tfrac{i}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Bigr), \tag{10.b}$$

$$\Omega_{(5)} = \operatorname{tr}\Bigl(\mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F} - \tfrac{i}{2} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{F} - \tfrac{1}{10} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Bigr), \tag{10.c}$$

etc.

(a) Verify eq. (9) for $2n = 2, 4, 6$ and $\Omega_{(2n-1)}$ as in eqs. (10).

The Chern–Simons forms allow us define a conserved axial current as

$$*J_{AC} \;=\; *J_A \;+\; \frac{2}{n!}\left(\frac{-1}{4\pi}\right)^n \Omega_{(2n-1)}\,, \qquad d*J_{AC} \;=\; 0. \tag{11}$$

However, the price of this conservation is the loss of gauge invariance: Alas, the Chern–Simons forms are non guage invariant.

Nevertheless, the gauge variations of the Chern–Simons forms are closed, $d(\delta\Omega_{(2n-1)}) = 0$. Moverover, for the infinitesimal gauge transforms

$$\delta\mathcal{A} \;=\; -D\Lambda \;=\; -d\Lambda \;-\; i\mathcal{A}\Lambda \;+\; i\Lambda\mathcal{A}, \qquad \delta\mathcal{F} \;=\; -i\mathcal{F}\Lambda \;+\; i\Lambda\mathcal{F} \tag{12}$$

(for infinitesimal $\Lambda(x)$), the first variations of the Chern–Simons forms are not only closed but exact,

$$\delta\Omega_{(2n-1)} \;=\; -dH_{(2n-2)} \tag{13}$$

where $H_{(2n-2)}$ is a $(2n-2)$-form constructed as a trace of a product of Λ and a polynomial of the $\mathcal{A}$ and $\mathcal{F}$ forms. In particular,

$$H_{(0)} \;=\; \mathrm{tr}(\Lambda) \qquad [\text{abelian } \mathcal{A} \text{ only}], \tag{14}$$

$$H_{(2)} \;=\; \mathrm{tr}\big(\Lambda\, d\mathcal{A}\big) \;=\; \mathrm{tr}\big(\Lambda(\mathcal{F} \,-\, i\mathcal{A}\wedge\mathcal{A})\big), \tag{15}$$

$$\begin{aligned} H_{(4)} \;&=\; \mathrm{tr}\big(\Lambda\, d(\mathcal{A}\wedge d\mathcal{A} \,+\, \tfrac{i}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\big) \\ &=\; \mathrm{tr}\Big(\Lambda\,\big(\mathcal{F}\wedge\mathcal{F} \,-\, \tfrac{i}{2}(\mathcal{F}\wedge\mathcal{A}\wedge\mathcal{A} + \mathcal{A}\wedge\mathcal{F}\wedge\mathcal{A} + \mathcal{A}\wedge\mathcal{A}\wedge\mathcal{F}) \,-\, \tfrac{1}{2}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)\Big). \end{aligned} \tag{16}$$

Verify these formulae for $2n = 2, 4, 6$.
Note: eq. (14) is trivial, while eq. (15) should be similar to problem 2 of the Fall 2024 midterm exam. But eq. (16) needs to be verified from scratch.

PS: Besides the axial anomaly in $d = 2n$ dimensions, the $2n$-forms $Q_{(2n)}$, the Chern–Simons forms $\Omega_{(2n-1)}$, and the $H_{(2n-2)}$ forms are also useful in other spacetime dimensions. In particular, the Chern–Simons 3-form can be used in 3D to give the gauge bosons a topological mass term, as we saw during the Fall 2024 midterm exam.

Towards the end of the Spring semester, we shall see that the $H_{(4)}$ form — reduced from 6 spacetime dimensions down to 4 — governs the non-abelian gauge anomaly. In terms of the anomalous variation of the effective action of the gauge field,

$$\Delta_{\text{gauge}} S_E^{\text{eff}}[\mathcal{A}_\mu(x)] \;=\; -\frac{1}{16\pi^2} \int\limits_{\text{4d space}} H_{(4)}\,, \tag{17}$$

where the trace in eq. (16) is taken over the species of LH Weyl fermions minus a similar trace over the RH Weyl fermion species. Likewise, the non-abelian gauge anomalies in other even spacetime dimensions are also related to the $H_{(2n-2)}$ forms for $2n = d + 2$.

(3B) In any even dimension $d = 2n$, the right hand side of the anomaly equation (3) is always a total derivative,

$$\epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\,\operatorname{tr}\Big(\mathcal{F}_{\alpha_1\beta_1}\cdots\mathcal{F}_{\alpha_n\beta_n}\Big) \;=\; \partial_\mu \Omega^\mu_{(2n-1)} \tag{18}$$

where $\Omega^\mu_{(2n-1)}$ is some polynomial in gauge fields $\mathcal{A}^\nu = gA^\nu$ and $\mathcal{F}^{\rho\sigma} = gF^{\rho\sigma}$, for example

$$\begin{aligned}
\text{in } d=2,\ \Omega^\mu_{(1)} &= 2\epsilon^{\mu\nu}\operatorname{tr}(\mathcal{A}_\nu) \quad [\text{abelian } A_\nu \text{ only}],\\
\text{in } d=4,\ \Omega^\mu_{(3)} &= 2\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\left(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma} - \tfrac{2i}{3}\,\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\right),\\
\text{in } d=6,\ \Omega^\mu_{(5)} &= 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\operatorname{tr}\left(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta} - iA_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} - \tfrac{2}{5}\,\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha A_\beta\right),
\end{aligned} \tag{19}$$

etc., etc. The $\Omega^\mu_{(2n-1)}$ vectors are equivalent to $(2n-1)$–index totally antisymmetric tensors called the *Chern–Simons forms*, and those forms play many important roles in gauge theory and string theory. In particular, we may use the $\Omega^\mu_{(2n-1)}$ to define a conserved axial current

$$J^\mu_A \;\to\; J^\mu_{AC} \;=\; \overline{\Psi}\gamma^\mu\Gamma\Psi \;+\; \frac{1}{n!}\left(\frac{-1}{4\pi}\right)^n \times \Omega^\mu_{(2n-1)}\,. \tag{20}$$

(Its conservation follows from eqs. (3) and (18).) However, the price of this current conservation is the loss of gauge invariance: the original axial current J^μ_A is gauge invariant, but the J^μ_{AC} is not.

(a) You task is to verify eqs. (18) for $d = 2, 4, 6$.

The Chern–Simons vectors (19) are not gauge invariant, but their variations under the infinitesimal gauge transforms are total derivatives of antisymmetric tensors,

$$\delta\Omega^{\mu}_{(2n-1)} \;=\; -2\partial_\nu H^{\mu\nu}_{(2n-2)}, \qquad H^{\mu\nu}_{(2n-2)} \;=\; -H^{\nu\mu}_{(2n-2)}. \tag{21}$$

Specifically, for $d = 2n = 2, 4, 6$:

$$\begin{aligned}
\text{in } d=2,\ H^{\mu\nu}_{(0)} &= \epsilon^{\mu\nu}\,\mathrm{tr}(\Lambda) \quad [\text{abelian } \mathcal{A}_\nu \text{ only}],\\
\text{in } d=4,\ H^{\mu\nu}_{(2)} &= 2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\big(\Lambda\times\partial_\rho\mathcal{A}_\sigma\big),\\
\text{in } d=6,\ H^{\mu\nu}_{(4)} &= 4\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\left(\Lambda\times\partial_\rho\left(\mathcal{A}_\sigma\partial_\alpha\mathcal{A}_\beta + \tfrac{i}{2}A_\sigma\mathcal{A}_\alpha\mathcal{A}_\beta\right)\right).
\end{aligned} \tag{22}$$

(b) Verify eqs. (21) for these H tensors.

Note: for $d = 2$ eq. (21) is trivial, while for $d = 4$ it's very similar to problem 2(b) of the Fall˜2024 midterm exam. But for $d = 6$ you have to work it out from scratch.