INSTANTONS

Topological Sectors of Yang–Mills Theories

Consider any kind of a quantum field theory. Its Euclidean path integral

$$\iiint\mathcal{D}[\text{fields}(x)]\,\exp\bigl(-S_E[\text{fields}(x)]\bigr) \tag{1}$$

explores the entire space of fields' configurations, or rather the space $\mathcal{C}$ of fields' configurations that have finite Euclidean actions S_E. If that space $\mathcal{C}$ happens to be discontinuous, then the perturbation theory cannot possibly explore all of $\mathcal{C}$, not even as a formal sum of the perturbative series. Instead, the perturbation theory is limited to the subspace $\mathcal{C}_0 \subset \mathcal{C}$ of configurations continuously connected to the vacuum state. Formally, this subspace comprises configurations $\varphi^a(x)$ such that there exists a continuous family $\varphi^a_t(x)$ for $0 \le t \le 1$ where: (1) for any t, the action $S_E[\varphi^a_t(x)]$ is finite; (2) for $t = 1$, $\varphi^a_1(x) = \varphi^a(x)$ in question; (3) for $t = 0$, $\varphi^a_0(x) = \langle\text{vac}|\,\varphi^a\,|\text{vac}\rangle$.

So IF the (finite action) configuration space $\mathcal{C}$ has several disconnected sectors, THEN only the connected sector $\mathcal{C}_0$ is accessible to the perturbative expansion, BUT both connected and disconnected sectors contribute to the path integral (1) and hence to the partition function $Z[\text{sources}]$ of the theory. Thus, *the disconnected sectors of the configuration space give rise to the non-perturbative effects!*

In these notes, we shall see that the non-abelian gauge theories indeed have disconnected configuration spaces of gauge fields $A^a_\mu(x)$, and we shall explore some of the non-perturbative effects due to disconnected sectors of such configuration spaces.

Let's start with the Yang–Mills theory with some simple gauge group G and the classical Euclidean action

$$S_E[\mathcal{A}_\mu(x)] \;=\; \frac{1}{2g^2}\int\!d^4x_e\;\mathrm{tr}\bigl(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\bigr). \tag{2}$$

Theorem 1: if this Euclidean action for some field configuration $\mathcal{A}(x)$ is finite, then its *index*

$$I[\mathcal{A}_\mu(x)] \;\stackrel{\text{def}}{=}\; \frac{1}{16\pi^2}\int\!d^4x_e\;\mathrm{tr}\bigl(\widetilde{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu}\bigr) \;=\; \frac{1}{32\pi^2}\int\!d^4x_e\;\mathrm{tr}\bigl(\epsilon^{\alpha\beta\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu}\bigr) \tag{3}$$

must have an integer value.

I shall prove this theorem in a moment. But first, please note its immediate consequence: Since an integer-valued index of cannot change continuously, the space $\mathcal{C}$ of finite-action gauge field configurations is discontinuous, with a separate sector $\mathcal{C}_I$ for each integer value I of the index,

$$\mathcal{C} \;=\; \biguplus_{I=-\infty}^{+\infty} \mathcal{C}_I\,. \tag{4}$$

Consequently, the partition function of the quantum YM theory

$$Z_{\rm net} \;=\; \iiint\limits_{\text{whole }\mathcal{C}} \mathcal{D}[\mathcal{A}_\mu]\, D_{\rm FP}\, \exp(-S_E) \tag{5}$$

becomes

$$Z_{\rm net} \;=\; \sum_{I=-\infty}^{+\infty} Z(\text{sector } I) \tag{6}$$

where

$$Z(\text{sector } I) \;=\; \iiint\limits_{\text{sector}\,\mathcal{C}_I} \mathcal{D}[\mathcal{A}_\mu]\, D_{\rm FP}\, \exp(-S_E). \tag{7}$$

The perturbation theory — even if we somehow manage to sum up the entire perturbative series — would yield only $Z(\text{sector } I=0)$ rather than the whole $Z_{\rm net}$. So contributions $Z(\text{sectors } I\neq 0)$ from the other sectors correspond to the non-perturbative effects in the Yang–Mills theory.

Proof of the theorem 1: Let's parametrize the 4D Euclidean space by the radius $r=|\vec{x}|$ and the direction $\vec{n}=\vec{x}/r$, a unit 4-vector in the direction $\vec{x}$. A finite Euclidean action of some configuration $\mathcal{A}_\mu(\vec{x})$ imposes a restriction on the field behavior at large radii $r\to\infty$. Indeed, if

$$S_E \;=\; \frac{1}{4g^2}\int d^4\vec{x}\,\mathcal{F}^a_{\mu\nu}(\vec{x})\mathcal{F}^a_{\mu\nu}(\vec{x}) \;<\; \infty, \tag{8}$$

then the integrand should diminish for $r\to\infty$ faster than $r^{-4-\epsilon}$ (for some $\epsilon>0$), which means that each component $\mathcal{F}^a_{\mu\nu}(\vec{x})$ should decrease faster than $r^{-2-\epsilon}$. Naively, this calls for potential fields $\mathcal{A}^a_\mu(\vec{x})$ to decrease as

$$|\mathcal{A}^a_\mu(\vec{x})| \;<\; O\left(\frac{1}{r^{1+\epsilon}}\right) \quad \text{for } r\to\infty, \tag{9}$$

but this ignores the gauge symmetry of the YM action. Instead, the potentials $\mathcal{A}^a_\mu(\vec{x})$ must be *gauge equivalent* to some $\mathcal{A}'^a_\mu(\vec{x})$ which decrease with $r\to\infty$ as in eq. (9). Thus,

$$\mathcal{A}_\mu(\vec{x}) \;=\; U(\vec{x})\mathcal{A}'_\mu(\vec{x})U^{-1}(\vec{x}) \;+\; i\partial_\mu U(\vec{x})\,U^{-1}(\vec{x}), \tag{10}$$

where the first term decreases for $r\to\infty$ faster than $r^{-1-\epsilon}$, but the second term is under no such limitation. Now, suppose

$$U(r,\vec{n}) \;\xrightarrow[r\to\infty]{}\; U(\vec{n}), \tag{11}$$

with a non-trivial direction dependence of the $U(\vec{n})$ at the 4D space infinity. Then

$$i\partial_\mu U(\vec{x})\,U^{-1}(\vec{x}) \;=\; O\left(\frac{1}{r}\right) \quad \text{for } r\to\infty, \tag{12}$$

hence the second term on the RHS of eq. (10) is much larger than the first term,

$$\left|i\partial_\mu U(\vec{x})\,U^{-1}(\vec{x})\right| \;\gg\; O\left(\frac{1}{r^{1+\epsilon}}\right) \;>\; \left|U(\vec{x})\mathcal{A}'_\mu(\vec{x})U^{-1}(\vec{x})\right|, \tag{13}$$

so at very large radii we may approximate

$$\mathcal{A}_\mu(r,\vec{n}) \;\xrightarrow[r\to\infty]{}\; i\partial_\mu U(\vec{n})\,U^{-1}(\vec{n}). \tag{14}$$

Now let's relate this formula to the index

$$I \;=\; \frac{1}{32\pi^2}\int\! d^4\vec{x}\;\mathrm{tr}\big(\epsilon^{\alpha\beta\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu}\big) \tag{3}$$

of the gauge field configuration $\mathcal{A}_\mu(x)$. The integrand here happens to be a total derivative:

$$\frac{1}{32\pi^2}\;\mathrm{tr}\big(\epsilon_{\alpha\beta\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu}\big) \;=\; \partial_\alpha W_\alpha \tag{15}$$

where

$$W_\alpha \;=\; \frac{1}{16\pi^2}\,\epsilon_{\alpha\beta\mu\nu}\,\mathrm{tr}\big(\mathcal{A}_\lambda\mathcal{F}_{\mu\nu} \;-\; \tfrac{2i}{3}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\big). \tag{16}$$

Verifying these formulae is a part of your next homework set#24a. The 3-form $W_{\lambda\mu\nu} = \epsilon_{\alpha\lambda\mu\nu}W_\alpha$ dual to this 4-vector is called the *Chern–Simons form*.

Anyway, by the 4D analogy of the Gauss Theorem, eq. (15) turns the 4D integral (3) to a 3D integral over a very large 3-sphere S^3 at space infinity $r \to \infty$,

$$I \;=\; \int\limits_{S^3 @ (r=\infty)} d^3\mathrm{vol}\, W_\alpha(\vec{n}) n_\alpha\,. \tag{17}$$

Next, let's take a closer look at eq. (16) for the W_α. At $r \to \infty$, we have

$$\mathcal{F}_{\mu\nu}(r,\vec{n}) \;<\; O(r^{-2-\epsilon}) \quad \text{while} \quad \mathcal{A}_\mu(r,\vec{n}) \;=\; O(r^{-1}), \tag{18}$$

hence in eq. (16),

$$\text{the first term} \;\sim\; \mathcal{F}\mathcal{A} \;<\; O(r^{-3-\epsilon}) \tag{19}$$

is much smaller than

$$\text{the second term} \;\sim\; \mathcal{A}\mathcal{A}\mathcal{A} \;=\; O(r^{-3}), \tag{20}$$

and therefore

$$\begin{aligned} W_\alpha(r,\vec{n}) \;\xrightarrow[r\to\infty]{}\; & \frac{-i}{24\pi^2}\, \epsilon_{\alpha\lambda\mu\nu}\, \mathrm{tr}(\mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\nu) \\ & \langle\!\langle \text{ in light of eq. (14) } \rangle\!\rangle \\ =\; & \frac{-1}{24\pi^2}\, \epsilon_{\alpha\lambda\mu\nu}\, \mathrm{tr}\Big((\partial_\lambda U) U^{-1} (\partial_\mu U) U^{-1} (\partial_\nu U) U^{-1} \Big). \end{aligned} \tag{21}$$

Thus, the index of a finite-action gauge field configuration obtains from the behavior of the $U(\vec{n})$ at 4-space infinity as

$$I \;=\; \frac{-1}{24\pi^2} \int\limits_{s^3 @ (r=\infty)} d^3\mathrm{vol}\, \epsilon_{\alpha\lambda\mu\nu}\, n_\alpha\, \mathrm{tr}\Big((\partial_\lambda U) U^{-1} (\partial_\mu U) U^{-1} (\partial_\nu U) U^{-1} \Big), \tag{22}$$

or in the differential form language,

$$I \;=\; \frac{-1}{24\pi^2} \int\limits_{s^3 @ (r=\infty)} \mathrm{tr}\Big((dU\cdot U^{-1}) \wedge (dU\cdot U^{-1}) \wedge (dU\cdot U^{-1}) \Big). \tag{23}$$

Note: this integral depends only on the *topology* of the $U(\vec{n})$ (at space infinity) and would be unaffected by any *continuous* variations of the $U(\vec{n})$.

To see how this works, suppose the gauge group G is $SU(2)$. The group manifold of $SU(2)$ is the unit 3-sphere S^3, so the group-element-valued function $U(\vec{n})$ maps the S^3 sphere at 4-space infinity to the S^3 group manifold. Moreover, the 3-volume form on the $SU(2)$ group manifold is

$$d^3V \;=\; -\tfrac{1}{12}\,\mathrm{tr}\Big((dU\cdot U^{-1})\wedge(dU\cdot U^{-1})\wedge(dU\cdot U^{-1})\Big) \tag{24}$$

while the volume of the whole 3-sphere is $2\pi^2$. Consequently, for $G = SU(2)$, the integral (23) measure the (oriented) volume of the U-image of the S^3 at 4-space infinity in units of the group manifold's volume. In other words, the index (23) counts how many times does the U-image of the S^3 at 4-space infinity wraps around the S^3 group manifold of the $SU(2)$. Obviously, this count depends only on the topology of the U-map from S^3 to S^3. Moreover, this count always has an integer value (positive, negative, or zero), and that's why the index I must have an integer value.

For other types of non-abelian gauge groups $G \neq SU(2)$, the group manifold of G has dimension > 3, so $U(\vec{n})$ maps the S^3 sphere at space infinity to a 3-dimensional sub-manifold of G rather than onto G itself. Nevertheless, for any simple compact non-abelian group G, its group manifold always has a topologically unique non-contractible 3-cycle, and one may count how many times the image of the S^3 at space infinity under the U-map wraps around this 3-cycle. Moreover, the integral (23) is precisely the number of times the U-map image of the S^3 at infinity maps onto that non-contractible 3-cycle, and that's why that integral — and hence the index of the gauge field configuration $\mathcal{A}_\mu(x)$ — must have integer value.

To actually prove the existence of that non-contractible 3-cycle, and that the integral (23) indeed counts how many times the U-map image of the S^3 at space infinity wraps around that cycle, I would need to work with homologies and cohomologies of group manifolds. Alas, this mathematical subject is quite beyond the scope of this introductory QFT class, so I am going to skip it. Instead, I ask you to either take my statements about this 3-cycle for granted, or to learn some differential topology and find the proof in some math textbook.

So here is the bottom line: For any finite-action configuration of the gauge fields, their behavior at $r \to \infty$ is related to a direction-dependent gauge transform $U(\vec{n})$, while the index I obtains from the same $U(\vec{n})$ according to eq. (23). The integral (23) depends only on the topology of the $U(\vec{n})$, and its values are always integer: I have proved that for $G = SU(2)$ and referred you to math generalizing the proof to larger gauge groups G. And the integer values of the topological integral (23) is precisely why the index I of any finite-action gauge field configurations is always integer, which proves theorem 1.

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Theorem 2: For any finite-action configuration of the gauge fields,

$$S_E \ \geq \ \frac{8\pi^2}{g^2}\,|I|\,. \tag{25}$$

Proof: In 4D Euclidean space,

$$\widetilde{\mathcal{F}}_{\alpha\beta} \ = \ \tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}\mathcal{F}_{\mu\nu}\,, \qquad \widetilde{\widetilde{\mathcal{F}}}_{\mu\nu} \ = \ \mathcal{F}_{\mu\nu}\,. \tag{26}$$

Let us define

$$\mathcal{F}^L_{\mu\nu} \ \stackrel{\text{def}}{=} \ \frac{1}{2}\Big(\mathcal{F}_{\mu\nu} + \widetilde{\mathcal{F}}_{\mu\nu}\Big), \qquad \mathcal{F}^R_{\mu\nu} \ \stackrel{\text{def}}{=} \ \frac{1}{2}\Big(\mathcal{F}_{\mu\nu} - \widetilde{\mathcal{F}}_{\mu\nu}\Big), \tag{27}$$

then the $\mathcal{F}^L_{\mu\nu}$ fields are self-dual while the $\mathcal{F}^R_{\mu\nu}$ fields are anti-self-dual,

$$\widetilde{\mathcal{F}}^L_{\mu\nu} \ = \ +\mathcal{F}^L_{\mu\nu}\,, \qquad \widetilde{\mathcal{F}}^R_{\mu\nu} \ = \ -\mathcal{F}^R_{\mu\nu}\,, \tag{28}$$

or in 3D terms

$$\begin{aligned} \mathbf{B}^L \ &= \ +\mathbf{E}^L, \qquad \mathbf{B}^R \ = \ -\mathbf{E}^R \qquad \text{(Euclidean)},\\ \mathbf{B}^L \ &= \ +i\mathbf{E}^L, \qquad \mathbf{B}^R \ = \ -i\mathbf{E}^R \qquad \text{(Minkowski)}. \end{aligned} \tag{29}$$

By construction (27) of the (anti) self-dual fields, and using

$$\operatorname{tr}\big(\widetilde{\mathcal{F}}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\big) \ = \ \operatorname{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\big), \tag{30}$$

we get

$$\begin{aligned} 2\operatorname{tr}\big(\mathcal{F}^L_{\mu\nu}\mathcal{F}^L_{\mu\nu}\big) \ &= \ \operatorname{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\big) \ + \ \operatorname{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\big),\\ 2\operatorname{tr}\big(\mathcal{F}^R_{\mu\nu}\mathcal{F}^R_{\mu\nu}\big) \ &= \ \operatorname{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\big) \ - \ \operatorname{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\big). \end{aligned} \tag{31}$$

The LH sides of both of these equations are non-negative, hence on the RH sides we must have

$$\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\big)\ \geq\ \pm\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\big) \tag{32}$$

for both signs $\pm$. Consequently, integrating both sides of this inequality over the 4D space, we get

$$\begin{aligned}\frac{g^2}{8\pi^2}\times S_E\ &=\ \frac{1}{16\pi^2}\int d^4x_e\ \mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\big)\\ &\geq\ \pm\frac{1}{16\pi^2}\int d^4x_e\ \mathrm{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\big)\\ &=\ \pm I\end{aligned} \tag{33}$$

for both $\pm$, and therefore

$$S_E\ \geq\ \frac{8\pi^2}{g^2}\times|I|. \tag{25}$$

Quod erat demonstrandum.

Moreover, to saturate the bound, we need to saturate the inequality (32) at all x, for the same sign $\pm$ as in eq. (33), namely $+$ for $I>0$ and $-$ for $I<0$. Consequently, we need

$$\begin{aligned}\text{self-dual}\quad \mathcal{F}_{\mu\nu}(x)\ &=\ +\widetilde{\mathcal{F}}_{\mu\nu}(x)\qquad \text{for } I>0,\\ \text{anti-self-dual}\quad \mathcal{F}_{\mu\nu}(x)\ &=\ -\widetilde{\mathcal{F}}_{\mu\nu}(x)\qquad \text{for } I<0.\end{aligned} \tag{34}$$

Either way, we have a first-order differential equation for the matrix-valued gauge potentials $\mathcal{A}_\mu(x)$, namely

$$\mathcal{F}_{\mu\nu}\ \stackrel{\text{def}}{=}\ \partial_\mu\mathcal{A}_\nu\ -\ \partial_\nu\mathcal{A}_\mu\ +\ i[\mathcal{A}_\mu,\mathcal{A}_\nu]\ =\ \pm\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\mathcal{F}_{\alpha\beta}. \tag{35}$$

Note that this (anti) self-duality equation automatically implies the Yang–Mills equations

$$D_\mu\widetilde{\mathcal{F}}_{\mu\nu}\ =\ 0\quad\text{and}\quad D_\mu\mathcal{F}_{\mu\nu}\ =\ 0. \tag{36}$$

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Now let's go back to the net Yang–Mills partition function

$$Z_{\text{net}} = \sum_{I=-\infty}^{+\infty} Z(\text{sector } I) \tag{37}$$

and consider the partition functions of various sectors. In particular, the $I = 0$ sector includes the vacuum configuration $\mathcal{A}_\mu(x) \equiv 0$. Moreover, all other configurations in this sector are continuously connected to the vacuum, so they obtain as finite-action fluctuations around the vacuum. Consequently, $Z(\text{sector } I = 0)$ obtains from the usual perturbation theory as

$$\log Z(\text{sector } I = 0)[\text{sources}] = \text{formal sum of all connected Feynman diagrams.} \tag{38}$$

The other sectors for $I \neq 0$ are more difficult as they do not contain the vacuum state. Instead, for any such sector we start with a base configuration $\mathcal{A}_\mu^{\text{base}}(x)$ which has the lowest action for the sector, $S_e^{\text{base}} = (8\pi^2/g^2)|I|$. Thus, the base tension fields $\mathcal{F}_{\mu\nu}^{\text{base}}(x)$ are self-dual (or anti-self-dual) and have the right index $= I$. Second, we look at the finite-action fluctuations around the base configuration,

$$\mathcal{A}_\mu(x) = \mathcal{A}_\mu^{\text{base}}(x) + \delta\mathcal{A}_\mu(x), \tag{39}$$

calculate their action as

$$S_E[\mathcal{A}_\mu(x)] = \frac{8\pi^2}{g^2} \times |I| + \Delta S_E[\delta\mathcal{A}_\mu(x)], \tag{40}$$

hence

$$Z(\text{sector } I) = \exp\left(-\frac{8\pi^2}{g^2} \times |I|\right) \times \iiiint\limits_{\text{shifted}\,\mathcal{C}_I} \mathcal{D}[\delta\mathcal{A}_\mu(x)] \exp(-\Delta S_E) \times D_{\text{FP}}\,. \tag{41}$$

Third, we calculate the post-exponential path integral factor here using the modified perturbation theory, much harder than the usual Feynman rules. To do that, we expand the ΔS_E functional in powers of the fluctuation $\delta\mathcal{A}_\mu(x)$, starting with the quadratic term since

$\mathcal{A}_\mu^{\rm base}$ is a minimum of the action. The quadratic part of the ΔS_E gives rise to the modified propagators for the sector $I\neq 0$, while the cubic, the quartic, *etc.*, parts give rise to the modified vertices. However, since the fluctuations $\delta\mathcal{A}_\mu(x)$ propagate in the background of the base fields $\mathcal{A}_\mu^{\rm base}(x)$ that are not translationally invariant, their propagators also lack translational invariance,

$$G_{\mu\nu}^{\rm modified}(x,y)\ \neq\ G^{\mu\nu}(x-y). \tag{42}$$

Likewise, the modified vertices also lack translational invariance — they depend on the $\mathcal{A}_\mu^{\rm base}(x)$ and hence on the x. Similarly, the ghost propagator and vertices — which obtain from the expansion of the Faddeev–Popov determinant — also lack translational invariance.

The bottom line is, the post-exponential factor in eq. (41) does obtain from a formal sum of an infinite series of Feynman diagrams, but the Feynman rules are significantly more complicated than in the vacuum sector $I=0$.

Yang–Mills Instantons

A Yang–Mills instanton is a self-dual gauge field configuration of index $I=+1$. As we shall see in a moment, this configuration is compact in all 4 Euclidean dimensions,

$$\mathcal{A}_\mu(r,\vec n)\ \xrightarrow[r\to\infty]{}\ 0\quad\text{for any direction }\vec n. \tag{43}$$

From the Euclidean time $t_e=x_4$ point of view, this makes the instanton configuration transient — it exists only for a brief instant of t_3 — hence the name *instanton.*

From the real (Minkowski) time point of view, a Yang–Mills instanton — like any other Euclidean configuration of finite S_E — is a tunneling event between distinct quantum states of the theory. To explain these states, I would need to canonically quantize the Yang–Mills theory, and then consider the topologically non-trivial gauge transforms in the Hilbert space of the quantum theory. I wish I had time to do it in this class, but I don't. Instead, let me refer you to Gerard 't˜Hooft 1999 lecture notes, chapter 4 about the instantons.

Note: a Yang–Mills instanton is transient event only in $4+0$ Euclidean dimensions. In higher spacetime dimensions, a Yang–Mills instanton is a topological defect of co-dimension 4, which can be a particle or an extended object. Specifically, in $4+1$ Minkowski dimensions, a

YM instanton is a particle; in 5+1 dimensions, it's a string, *etc.*, *etc.*; and in 9+1 dimensions it's a 5-brane. This is particularly relevant to the 9+1 dimensional superstring theory where the YM instantons and the D5 branes are continuously connected to each other.

Now let's take a closer look at the YM instanton fields, starting with the t' Hooft's instanton solution for the $SU(2)$ gauge group:

$$\mathcal{A}^a_\mu(x) \;=\; gA^a_\mu(x) \;=\; \frac{2\eta^a_{\mu\nu}x_\nu}{x_e^2+\rho^2}\,, \tag{44}$$

where ρ is the instanton's size — which may have any fixed positive value, — and $\eta^a_{\mu\nu}$ is a constant array. Specifically,

$$\text{for } i,j=1,2,3: \quad \eta^a_{ij} \;=\; \epsilon_{aij}\,, \quad \eta^a_{i4} \;=\; \delta_{ai}\,, \quad \eta^a_{4j} \;=\; -\delta_{aj}\,, \quad \eta^a_{44} \;=\; 0. \tag{45}$$

By this construction, the $\mu,\nu=1,2,3,4$ indices of the $\eta^a_{\mu\nu}$ form a self-dual antisymmetric tensor,

$$\eta^a_{\mu\nu} \;=\; -\eta^a_{\nu\mu} \;=\; +\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\eta^a_{\alpha\beta}\,. \tag{46}$$

Note that the antisymmetric tensors corresponds to the generators of the Euclidean rotation or spin group, which in 4 dimensions happens to be

$$\mathrm{Spin}(4) \;=\; SU(2)_L\times SU(2)_R\,, \tag{47}$$

where the self-dual antisymmetric tensors generate the $SU(2)_L$ while the anti-self-dual tensors generate the $SU(2)_R$. Thus, between their gauge and Euclidean indices, *the $\eta^a_{\mu\nu}$ map the $SU(2)_L$ generators onto the $SU(2)_{\rm gauge}$ generators and vice verse.* Consequently, the 't Hooft's instanton field configuration (44) has unbroken symmetry

$$SU(2)_{\rm L+gauge}\times SU_{\rm R}\,. \tag{48}$$

In the zero-size $\rho\to 0$ limit, the instanton gauge fields (44) become pure gauge, *i.e.*

gauge-equivalent to zero,

$$\mathcal{A}^a_\mu(x)\times\tfrac{1}{2}\tau^a \;=\; \frac{\tau^a\eta^a_{\mu\nu}x_\nu}{x_e^2} \;=\; i(\partial_\mu U(x))U^{-1}(x) \tag{49}$$

for

$$U(r,\vec{n}) \;=\; n_4 \;-\; i(n_1\tau_1+n_2\tau_2+n_3\tau_3) \;\in\; SU(2). \tag{50}$$

But for $\rho>0$, the instanton fields are not pure gauge anymore; instead, they have non-zero tensions

$$\mathcal{F}^a_{\mu\nu} \;=\; \frac{-4\rho^2}{(r^2+\rho^2)^2}\times\eta^a_{\mu\nu}\,. \tag{51}$$

Working out this formula for the tensions is left out as an optional exercise to the intersted students.

The self-duality of the tension fields (51) follows from that of the $\eta^a_{\mu\nu}$, while the index and the action obtain from

$$\mathcal{F}^a_{\mu\nu}\widetilde{\mathcal{F}}^a_{\mu\nu} \;=\; \frac{16\rho^4}{(r^2+\rho^2)^4}\times 12, \tag{52}$$

hence

$$I \;=\; \frac{1}{32\pi^2}\times\int\limits_0^\infty \frac{12\times 16\rho^4}{(r^2+\rho^2)^4}\times 2\pi^2 r^3\,dr \;=\; 1 \tag{53}$$

and

$$S_E \;=\; \frac{8\pi^2}{g^2}\times|I| \;=\; \frac{8\pi^2}{g^2}\,. \tag{54}$$

More generally, the self-dual $SU(2)$ gauge fields of index $I=+1$ form an 8-parameter family of instanton solutions:

$$\mathcal{A}^a_\mu(x) \;=\; \frac{2R^{ab}\eta^b_{\mu\nu}(x_\mu-y_\mu)}{(x-y)_e^2+\rho^2} \tag{55}$$

where y_μ are the 4 coordinates of the instanton's center, ρ is the instanton's size, and R^{ab} is an $SO(3)$ rotation matrix parametrizing the relative orientation between the $SU(2)_L\subset\mathrm{Spin}(4)$ and the $SU(2)_{\mathrm{gauge}}$ groups.

For other non-abelian gauge groups G, the gauge fields of an instanton always belong to an $SU(2)$ subgroup of G. However, identifying a specific $SU(2)$ subgroup of G used by a particular instanton solution calls for additional parameters. Focusing on the level $= 1$ subgroups (where the generators of $SU(2) \subset G$ have the same normalization as the generators of G), we find that $G = SU(3)$ has a 4-parameter family of $SU(2)$ subgroups. Likewise, for larger $SU(N)$ groups, identifying an $SU(2)$ subgroups calls for $4(N-2)$ parameters, while for a different kind of a simple gauge group, the number of parameters is $4C(G) - 8$. And this number is on top of the 8 parameters of the instanton solutions for a specific $SU(2)$, thus *the instanton solutions of the gauge theory with a general simple gauge group G have* $4C(G)$ *continuous parameters*; for example, $4N$ parameters for a $SU(N)$ gauge theory.

Multi-Instanton Solutions

Besides the instantons — the self-dual gauge fields with index $= 1$ — there are also self-dual or anti-self-dual gauge fields with other values of the index $I \neq 0, 1$. For example, the anti-self-dual solutions with $I = -1$ — called the *anti-instantons* — obtain from the instanton solutions by 3-space reflections. Thus, for $G = SU(2)$, the general anti-instanton solution has form

$$\mathcal{A}_\mu^a(x) \;=\; \frac{2R^{ab}\bar{\eta}^b_{\mu\nu}(x_\mu - y_\mu)}{(x-y)_e^2 + \rho^2} \tag{56}$$

where

$$\bar{\eta}^a_{ij} \;=\; +\epsilon_{aij}\,, \quad \bar{\eta}^a_{i4} \;=\; -\delta_{ai}\,, \quad \bar{\eta}^a_{4j} \;=\; +\delta_{aj}\,, \quad \bar{\eta}^a_{44} \;=\; 0 \quad \text{for } i,j = 1,2,3, \tag{57}$$

hence

$$\bar{\eta}^a_{\mu\nu} \;=\; -\bar{\eta}^a_{\nu\mu} \;=\; -\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\eta}^a_{\alpha\beta}\,. \tag{58}$$

Similar to the instanton solutions, the anti-instanton solution form a continuous family of 8 parameters for $G = SU(2)$ or $4C(G)$ parameters for larger gauge groups.

For higher values $|I| > 1$ of the topological index, there also exist exactly self-dual multi-instanton solutions (for $I > +1$) or exactly anti-self-dual multi-anti-instanton solutions (for $I < -1$). However, the explicit form of such ADHM construction (named after its authors, M. F. Atiyah, V. G. Drinfeld, N. J. Hitchin, and Y. I. Manin) is much more complicated than the ’t Hooft’s singe-instanton solution, so I am not going to inflict it on the students. Instead, let me make just a couple of comments:

1. The exactly (anti) self-dual solutions for any given index $I \neq 0$ form a continuous family with $8 \times |I|$ real parameters for $G = SU(2)$ or $4C(G) \times |I|$ real parameters for larger gauge groups.

2. The multi-instanton parameter space has a corner where the solution looks like a superposition of I single instantons located far away from each other. In terms of the $(y_\mu, \rho, R)_i$ parameters of the individual instantons, we should have

$$|y_i - y_j|_e \gg \rho_i, \rho_j \quad \forall i, j = 1, \ldots, I;\ i \neq j, \tag{59}$$

then the multi-instanton solution can be approximated as

$$\mathcal{A}^a_\mu(x) \approx \sum_{i=1}^{I} \mathcal{A}^a_\mu(x)[\text{instanton}(y, \rho, R)_i] + O\left(\frac{\rho}{|y_i - y_j|_e^2}\right). \tag{60}$$

Likewise, the multi-anti-instanton solutions for $I < 0$ have a corner of the parameter space where they look line $|I|$ well-separated individual anti-instantons.

Cluster Expansion and the Instanton Angle

Let’s go back to the net partition function of the Yang–Mills theory,

$$Z_{\text{net}} = \sum_{I=-\infty}^{+\infty} Z(\text{sector } I), \tag{61}$$

where for each non-perturbative sector with $I \neq 0$, Z(that sector) obtains from fluctuation of the fields around an I-instanton or $-I$-anti-instanton solution. Since the sectors with different topological indices I are completely disjoint, we may try modifying the net partition function (61) to

$$Z_{\text{net}} = \sum_{I=-\infty}^{+\infty} C_I \times Z(\text{sector } I) \tag{62}$$

for some non-trivial coefficients C_I. For example, for $C_I = \delta_{I,0}$ we would get

$$Z_{\text{net}} = Z(\text{sector } I = 0) \text{ only}, \tag{63}$$

without any non-perturbative contributions.

So the question is: Are we allowed to modify the net partition function along the lines of eq. (62)? And the answer is: NO, because it would violate the cluster expansion principle, which says that *whatever happens in one region of space should not affect what happens in another distant region of space.* The quantum field theory is build on this principle, just like it's build on quantum mechanics and special relativity, so we should not allow any modifications which would violate the cluster expansion.

To see how the modifications (62) would ran afoul of the cluster expansion principle, consider the following configuration of the gauge fields: A small instanton centered at y_1, plus a small anti-instanton centered at a distant point y_2,

$$|y_1 - y_2|_e \;\gg\; \rho_1, \rho_2, \tag{64}$$

hence net gauge field

$$\mathcal{A}^a_\mu(x) \;\approx\; \mathcal{A}^\mu_a(x)[\text{instanton}(y_1,\rho_1,R_1)] + \mathcal{A}^\mu_a(x)[\text{antiinstanton}(y_2,\rho_2,R_2)] + O\left(\frac{\rho}{|y_1-y_2|^2}\right). \tag{65}$$

This gauge field configuration has net index $I = 0$ and net action

$$S_E \;=\; 2\times\frac{8\pi^2}{g^2} \;+\; \text{small positive correction} \;<\; \infty. \tag{66}$$

Consequently, this configuration belongs to the $I = 0$ sector and we cannot possibly exclude it or change the overall coefficient C_0 of its contribution to the net partition function.

But when the instanton and the anti-instanton are very far away from each other, the cluster expansion principle requires us to treat them independently of each other. That is, we should allow or disallow an instanton at y_1 regardless of whether we also have or do not have an antiinstanton at y_2, and vice verse. Hence, once we allow the instanton-antiinstanton pairs like (65), we should also allow the single-instanton configurations without an antiinstanton, or single antiinstanton configurations without the instanton. Since such configurations belong to sectors with $I = \pm 1$, this means we cannot throw away those sectors by setting C_{+1} or C_{-1} to zero. Moreover, to properly count the instanton's and the anti-instanton's contributions to the net partition function regardless of each other, we need

$$C_{+1}\times C_{-1}\ =\ C_0^2\ =\ 1. \tag{67}$$

Likewise, we can start with some configuration with index $I\neq 0$, and then add an instanton (hence $I'=I+1$), or an anti-instanton (hence $I'=I-1$), or both (hence $I'=I$). If the instanton and the anti-instanton are far away from each other, their existence or non-existence should not affect each other's contributions to the partition function, which calls for

$$C_{I'=I+1}\times C_{I'=I-1}\ =\ C_I^2\,. \tag{68}$$

Moreover, the coefficients C_I must obey these conditions for all I, which allows only one family of solutions:

$$C_I\ =\ \exp(i\Theta\times I) \tag{69}$$

for some real angle Θ called *the instanton angle* or *the vacuum angle*. Thus, the only allowed modification of the net partition function of the Yang–Mills theory is

$$Z_{\rm net}\ =\ \sum_{I=-\infty}^{+\infty}\exp(i\Theta I)\times Z(\text{sector }I). \tag{70}$$

Physically, we may reinterpret the $\exp(i\Theta I)$ factors in this modified partition function in terms of the modified Lagrangian of the Yang–Mills theory. Indeed, let's add a constant term $\Delta S_E(I)$ to the Euclidean action of some sector, than the partition function of that sector changes to

$$Z(\text{sector }I)\ =\ \iiint\limits_{\text{sector }\mathcal{C}_I}\mathcal{D}[\mathcal{A}_\mu(x)]\,e^{-S_E}\times D_{\rm FP}\ \longrightarrow\ Z(\text{sector }I)\times\exp(-\Delta S_E(I)). \tag{71}$$

Consequently, we may re-interpret

$$\exp(i\Theta I)\ =\ \exp(-\Delta S_E(I)), \tag{72}$$

hence

$$\Delta S_E(I) \;=\; -i\Theta \times I \;=\; \frac{-i\Theta}{16\pi^2}\int d^4x_e\, \mathrm{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\big) \tag{73}$$

and therefore

$$\Delta\mathcal{L}_E \;=\; \frac{-i\Theta}{16\pi^2}\, \mathrm{tr}\Big(\widetilde{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu}\Big). \tag{74}$$

In other words, the net Euclidean Lagrangian of the Yang–Mills theory becomes

$$\mathcal{L}_E \;=\; \frac{1}{2g^2}\, \mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}\Big) \;-\; \frac{i\Theta}{16\pi^2}\, \mathrm{tr}\Big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\Big). \tag{75}$$

Although the extra term here is imaginary, this is OK because it becomes real in the Minkowski space,

$$\mathcal{L}_M \;=\; -\frac{1}{2g^2}\, \mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) + \frac{\Theta}{32\pi^2}\, \mathrm{tr}\big(\epsilon^{\alpha\beta\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu}\big). \tag{76}$$

The extra Lagrangian term here happens to be a total derivative,

$$\frac{\Theta}{32\pi^2}\, \mathrm{tr}\big(\epsilon\mathcal{F}\mathcal{F}\big) \;=\; \Theta\,\partial_\alpha W^\alpha \tag{77}$$

for

$$W_\alpha \;=\; \frac{1}{16\pi^2}\,\epsilon^{\alpha\lambda\mu\nu}\, \mathrm{tr}\Big(\mathcal{A}_\lambda\mathcal{F}_{\mu\nu} \;-\; \tfrac{2i}{3}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\Big). \tag{16}$$

Consequently, the Θ term does not affect the Feynman rules of the YM theory, so the perturbation theory is completely Θ-independent. But the non-perturbative instanton effects do depend on Θ via the $e^{i\Theta I}$ factors in the net partition function.

In particular, the Θ term in the Lagrangian has odd parity and odd CP, so we expect the non-perturbative effects to break the P and CP symmetries of the Θ-less theory. If we add such Θ term to the Lagrangian of the QCD rather than the pure YM theory, it would produce non-perturbative CP-violating effects such as non-zero electric dipole moment of the neutron,

$$d_n \;=\; e\Theta \times \begin{cases} O(\alpha_{\mathrm{QCD}} \times \text{neutron's radius}), \\ \text{best estimate } 4.5\cdot 10^{-15}\text{ cm.}\end{cases} \tag{78}$$

But despite diligent experimental attempts to detect and measure this dipole moment, it turns out to be way too small; the current upper limit is $|d_n| < e \times 1.8\cdot 10^{-26}$ cm. In terms of the instanton angle, this limit gives $|\Theta| < 4 \cdot 10^{-12}$, so for all practical purposes $\Theta = 0$ and QCD happens to have perfect CP symmetry. There are many theories as to why $\Theta_{\rm QCD}$ happens to vanish, most likely being some kind of a Peccei–Quinn symmetry, see Roberto Peccei's lecture notes for the explanation.

But going beyond the Standard Model, you may have to deal with all kinds of gauge theories. And unlike QCD, some of these theories may non-zero instanton angles, with all kinds of interesting consequences thereof.

FERMIONS AND TOPOLOGY

In an earlier class I mentioned that the instantons of the electroweak $SU(2)_w$ gauge symmetry lead to the non-conservation of the baryon and lepton numbers, $\Delta L = \Delta B = 3$. In these notes, I explain how this happens. But for clarity's sake, I focus on a simpler theory — QCD with N_f flavors of exactly massless quarks — and the non-conservation of the axial quark number.

Thus, let J^μ_A be the net axial current of the massless quarks,

$$J^\mu_A \;=\; \sum_{\substack{\text{colors,}\\ \text{flavors}}} \overline{\Psi}\gamma^\mu\gamma^5\Psi, \tag{79}$$

with the corresponding global charge being

$$Q_A \;=\int\! d^3\mathbf{x}\, J^0_A(\mathbf{x}). \tag{80}$$

In terms of numbers of quarks and antiquarks of definite helicity, this axial charge counts

$$Q_A \;=\; \#\text{RHquarks} \;-\; \#\text{LHquarks} \;+\; \#\text{RHantiquarks} \;-\; \#\text{LHantiquarks}. \tag{81}$$

By the axial anomaly of QCD,

$$\partial_\mu J^\mu_A \;=\; -\frac{2N_f}{16\pi^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}\big), \tag{82}$$

so in a non-trivial background of gauge fields, the net axial charge of the quarks changes by

$$\Delta Q_A \;=\int\! dt\!\int\! d^3\mathbf{x}\,\partial_\mu J^\mu_A(\mathbf{x},t) \;=\; -\frac{2N_f}{16\pi^2}\int\! d^4x\;\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}\big) \;=\; -2N_f\times I[\text{gauge fields}]. \tag{83}$$

Thus, if the gauge field topology has a non-zero index $I \neq 0$, then somehow $N_f \times |I|$ quarks and antiquarks must change their helicities: from right-handed to left handed for $I > 0$, or from left-handed to right-handed for $I < 0$. But how the #$%& does this happen?

The Hamiltonian language explanation of this process is explained in detail (albeit for a 2D analogue of the axial anomaly) in the *Peskin and Schroeder* textbook, §19.1. Basically, in a slowly time-dependent background of gauge fields, the energy levels for the quarks also change with time. Some of these energy levels happen to cross zero, which changes the Dirac sea of the quarks. Consequently, if the energy of an occupied quark state drops below zero, then this quark becomes a part of the Dirac sea and disappears as a distinct particle. OOH, if the energy level of an occupied state that used to be negative becomes positive, then the quark which used to be a part of the Dirac sea becomes a distinct particle. In particular, in the instanton background, energies of N_f right-handed states drop below zero while energies of N_f left-handed state rise above zero. Consequently, N_f RH quarks dissolve into the Dirac sea — the physical vacuum of the quarks — while N_f LH quarks pop out from Dirac sea and become distinct particles. And that's how the net effect is N_f quarks changing their helicities from RH to LH.

But in these notes I focus on the functional quantization language rather than the Hamiltonian language. And in this language, changing quark's helicities and chiralities is related to the normalizable zero modes of the quark fields in the gauge field background. That is, normalizable in Euclidean space solutions of the Dirac equation,

$$\not{D}\psi(x) \;=\; 0, \qquad \int\! d^4x_e\, \psi^\dagger\psi \;<\; \infty. \tag{84}$$

As we shall see momentarily, these zero modes are chiral, $\gamma^5\psi = \pm\psi$, and the numbers of zero modes for each chirality are related to the topological index I of the gauge field configuration.

But before we start with the zero modes, consider the operator $-\not{D}^2$ in the Hilbert space of Dirac spinors in 4 Euclidean dimensions. This operator is Hermitian and non-negative, so we may diagonalize it. Thus, we get a basis of eigenstates $\psi_\lambda(x)$, $-\not{D}^2\psi_\lambda = C_\lambda^2\psi_\lambda(x)$, indexed by some parameter λ. For discrete eigenvalues C_λ^2 — and only for the discrete eigenvalues, — the eigenspinors $\psi_\lambda(x)$ are normalizable in 4D.

Furthermore, the operator $-\not{D}^2$ commutes with the γ^5, so we may diagonalize both of them at the same time, thus

$$-\not{D}^2\psi_\lambda(x) \;=\; C_\lambda^2\psi_\lambda(x), \qquad \gamma^5\psi_\lambda(x) \;=\; \pm_\lambda\psi_\lambda(x). \tag{85}$$

The operator $i\not{D}$ is also Hermitian; also, it squares to $-\not{D}^2$ and (b) anticommutes with the γ^5. Consequently, all the simultaneous eigenstates of $-\not{D}^2$ and γ^5 with $C_\lambda \neq 0$ come in pairs of the same C_λ^2 but opposite chiralities: Given $\psi_\lambda(x)$ obeying eqs. (85), we construct

$$\psi'(x) \;\stackrel{\text{def}}{=}\; \frac{1}{C_\lambda}\, i\not{D}\psi_\lambda(x) \tag{86}$$

which obeys

$$-\not{D}^2\psi'_\lambda(x) \;=\; C_\lambda^2\psi'_\lambda(x), \qquad \gamma^5\psi'_\lambda(x) \;=\; \mp_\lambda\psi'_\lambda(x), \tag{87}$$

thus if ψ_λ is LH then ψ'_λ is RH and vice verse. On the other hand, the eigenvalues with $C_\lambda = 0$ do not have to come in pairs; instead, they simply obey $i\not{D}\psi_\lambda(x) = 0$.

Next, consider the trace (in the Hilbert space of the 4D spinors)

$$N(t) \;=\; \mathrm{Tr}\Big(\gamma^5\exp(t\not{D}^2)\Big). \tag{88}$$

Since $-\not{D}^2$ is a positive operator, for any $t > 0$ the exponential factor in the trace rapidly approaches zero for high-momentum states, so the trace is UV-finite. As to a potential IR divergence, it may be regulated by putting the system in a large but finite box, but let's not worry about that issue. Instead, let's formally evaluate this trace in the common eigenbasis ψ_λ of the $-\not{D}^2$ and γ^5 operators. In this basis,

$$\gamma^5\exp(t\not{D}^2)\psi_\lambda(x) \;=\; \pm_\lambda\exp(-tC_\lambda^2)\psi_\lambda(x), \tag{89}$$

hence

$$N(t) \;=\; \sum_\lambda \pm_\lambda\exp(-tC_\lambda^2). \tag{90}$$

Moreover, for any $C_\lambda^2 \neq 0$, the eigenvalues come in pairs of opposite chiralities, so their contributions to the trace (90) cancel each other. Instead, the only un-cancelled contributions come from the zero modes: the RH zero modes contribute $+1$ while the LH zero modes contribute -1. Consequently, for any $t > 0$,

$$N(t) \;=\; \text{same } N \;=\; \#\binom{\text{right handed}}{\text{zero modes}} \;-\; \#\binom{\text{left handed}}{\text{zero modes}}. \tag{91}$$

Note: since only the zero modes do not pair-up by chiralities, they are distinct from all the $C_\lambda^2 > 0$ eigenstates, so the zero eigenvalue $C_\lambda^2 = 0$ is *discrete* and the the corresponding eigenstates are *normalizable*. Thus, the zero modes counted in eq. (91) — are the normalizable zero modes.

Atiyah–Singer theorem: For a QCD-like theory,

$$N \;=\; -N_f \times I[\text{gauge fields}] \;=\; -\frac{N_f}{16\pi^2}\int\! d^4x_e\, \operatorname{tr}\bigl(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\mu\nu}\bigr). \tag{92}$$

For a more general gauge theory with some gauge group G and massless Dirac fermions in some multiplets (m) of G, the theorem becomes

$$\#\binom{\text{LH zero modes}}{\text{of fermions in } (m)} \;-\; \#\binom{\text{RH zero modes}}{\text{of fermions in } (m)} \;=\; 2R(m) \times I[\text{gauge fields}]. \tag{93}$$

I am not going to prove the Atiyah–Singer theorem in class. Instead, let me simply relate it to my notes on the anomaly of the fermionic path integral, pages 21–28. Consider the trace (88) for a small $t = 1/\Lambda^2$ for some very large energy scale Λ acting as a UV regulator. In this context,

$$N \;=\; \operatorname{Tr}\bigl(\gamma^5 \times \exp(\not{D}^2/\Lambda^2)\bigr) \tag{94}$$

acts as a *UV-regulated*

$$N \;=\; \operatorname{Tr}_{\text{reg}}(\gamma^5) \;=\; \text{Anomaly}, \tag{95}$$

which evaluates to precisely $-N_f \times$ Index of the gauge field configuration.

Also, let me give a simple example of the Atiyah–Singer theorem in action. Let the gauge group be $SU(2)$ and the gauge field configuration is the 't Hooft's instanton

$$\mathcal{A}^a_\mu(x) \;=\; \frac{2\eta^a_{\mu\nu}x_\nu}{x_e^2 + \rho^2} \tag{96}$$

of index $I = +1$. Let's have a single $SU(2)$ doublet of massless Dirac fermions $\Psi^i(x)$, or equivalently a doublet of LH Weyl spinors $\psi^i_\alpha(x)$ plus a doublet of RH Weyl spinors $\chi^i_{\dot\alpha}(x)$.

(In my notations, $i = 1,2$ is the gauge index, $\alpha = 1,2$ is the LH Weyl spinor index, and $\dot{\alpha} = 1,2$ is the RH Weyl spinor index.) Then, in the instanton background (96), the LH Weyl spinors have a unique normalizable zero mode,

$$\psi^i_\alpha(x) \;=\; \delta^i_\alpha \times \frac{\text{const}}{(x_e^2+\rho^2)^{3/2}}, \qquad \not{D}\psi^i_\alpha \;=\; 0. \tag{97}$$

On the other hand, the RH Weyl spinors have no normalizable zero modes at all. Thus indeed,

$$\#\binom{\text{LH zero}}{\text{modes}} - \#\binom{\text{RH zero}}{\text{modes}} \;=\; 1 \;-\; 0 \;=\; +1 \;=\; (2\mathrm{R}(\mathbf{2})=1)\times(\mathrm{I}=+1) \;=\; +1. \tag{98}$$

Effects of Fermionic Zero Modes

Now that we know that in $I \neq 0$ backgrounds the fermionic fields have normalizable zero modes, let's consider the consequences of these zero modes. But as a training exercise, let's start with the Gaussian integral over finite numbers of fermionic variables,

$$\int\! d^N\bar\theta \int\! d^N\theta\; \exp\bigl(-\bar\theta^i A_{ij}\theta^j\bigr) \;=\; \det(A). \tag{99}$$

Clearly, if the matrix A has zero eigenvalues, then this integral vanishes. But if A has only one zero eigenvalue, then the Gaussian+ integral involving an extra $\theta\bar\theta$ pair does not vanish. Instead,

$$\begin{aligned}
&\int\! d^N\bar\theta \int\! d^N\theta\; \exp\bigl(-\bar\theta^i A_{ij}\theta^j\bigr) \times \theta^k\bar\theta^\ell \;= \\
&\qquad =\; [\mathrm{minor}(A)]^{k\ell} \\
&\qquad =\; (\text{zero eigenvector}^*)^k (\text{zero eigenvector})^\ell \times \prod(\text{nonzero eigenvalues}) \\
&\qquad \neq\; 0.
\end{aligned} \tag{100}$$

Likewise, if the matrix A has K zero eigenvalues, then the integrals

$$\int\! d^N\bar\theta \int\! d^N\theta\; \exp\bigl(-\bar\theta^i A_{ij}\theta^j\bigr) \times \theta^{i_1}\cdots\theta^{i_L}\,\bar\theta^{j_1}\cdots\bar\theta^{j_L} \tag{101}$$

vanish for $L < K$ but not for $L \geq K$ because K of the θ's and K of the $\bar\theta$'s would soak up all the zero modes of the matrix A. In particular, for $L = K$, the integral amounts to

$$\prod_{N-K} \begin{pmatrix} \text{nonzero} \\ \text{eigenvalues} \end{pmatrix} \times \prod_{K} \begin{pmatrix} \text{zero eigenvectors}^*, \\ \text{antisymmetrized} \end{pmatrix} \times \prod_{K} \begin{pmatrix} \text{zero eigenvectors}, \\ \text{antisymmetrized} \end{pmatrix}. \tag{102}$$

Now let's generalize these formulae to the fermionic functional integrals in the gauge-field background, namely the fermionic partition function

$$\hat{Z}[\mathcal{A}_\mu] \;=\; \iiint\!\mathcal{D}[\overline{\Psi}] \iiint\!\mathcal{D}[\Psi]\, \exp\left(-\int\! d^4x_e\, \overline{\Psi}\, \not{D} \Psi\right) \tag{103}$$

and $2L$-fermion amplitudes

$$\iiint\!\mathcal{D}[\overline{\Psi}] \iiint\!\mathcal{D}[\Psi]\, \exp\left(-\int\! d^4x_e\, \overline{\Psi}\, \not{D} \Psi\right) \times \Psi_1 \cdots \Psi_L \times \overline{\Psi}_1 \cdots \overline{\Psi}_L\,. \tag{104}$$

When the $\not{D}$ operator — which plays the role of matrix A — has $K > 0$ zero modes, the fermionic partition function (103) vanishes, but the amplitudes (104) involving $L \geq K$ fermionic pairs do not vanish! In particular, for $L = K$ the fermionic integral (104) yields

$$(\text{coefficient}) \times \prod_{K} (\text{zero modes of } \overline{\Psi}) \times \prod_{K} (\text{zero modes of } \Psi). \tag{105}$$

For example, consider QCD with N_f massless quark flavors. In the 1-instanton background, we have one zero mode of the left-handed $\Psi_{L,f}$ for each flavor f but no zero modes of the right-handed $\Psi_{R,f}$. On the $\overline{\Psi}$ side, the chirality is flipped, so we have one zero mode for each flavor of $\overline{\Psi}_{R,f}$ but no zero modes of $\overline{\Psi}_{L,f}$. Consequently, the leading fermionic correlation function in the one-instanton sector needs $2N_f$ fermionic fields to soak up all the zero modes, thus

$$\langle\Omega|\, \mathbf{T} \prod_{f=1}^{N_f} \Psi_{L,f} \overline{\Psi}_{R,f}\, |\Omega\rangle \;\neq\; 0. \tag{106}$$

In particle terms, this correlation function corresponds to a process which creates a LH quark for each Ψ_L and annihilates a RH quark for each $\overline{\Psi}_R$. The net effect of this process is to turn N_f RH quarks into LH quarks thus changing the axial charge by $\Delta Q_A = -2N_f$.

Likewise, in the electroweak theory, in the background of a single $SU(2)_W$ instanton, each $SU(2)$ doublet of left-handed leptons or quarks has one zero mode of the corresponding field. Consequently, the fermionic functional integral gives rise to a 12-fermion amplitude — 1 lepton field for each family, and 1 quark field for each family and each color. Each field here can create a LH quark or lepton or annihilate a RH quark or lepton, so the net lepton number changes by $\Delta L = +3$ and the net quark number changes by $3\Delta B = +9$.

Instanton Angle, Axial Symmetries, and Strong CP Violation

Earlier in these notes we have learned about the instanton angle Θ affecting the non-perturbative effects in a Yang–Mills theory. Obviously, we may include such angle into any gauge theory, so let's do it for QCD with a single massless quark flavor,

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \frac{\Theta}{16\pi^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}\big) \;+\; i\overline{\Psi}\,\not{D}\Psi. \tag{107}$$

Now let's try a local axial transform of the quark field,

$$\Psi(x) \;\to\; \exp\big(i\phi(x)\gamma^5\big)\,\Psi(x). \tag{108}$$

This is not a symmetry of the theory, not even classically. Indeed, such axial transform changes the Lagrangian (107) by

$$\begin{aligned}
\Delta\mathcal{L} \;&=\; -\partial_\mu\phi\times\overline{\Psi}\gamma^\mu\gamma^5\Psi \;=\; -\partial_\mu\phi\times\big(\text{axial current } J^{\mu 5}\big)\\
&\qquad \langle\!\langle\,\text{subtracting a total derivative}\,\rangle\!\rangle\\
&\cong\; \phi\times\partial_\mu J^{5\mu}\\
&\qquad \langle\!\langle\,\text{by axial anomaly}\,\rangle\!\rangle\\
&=\; \phi\times\frac{-1}{8\pi^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}\big).
\end{aligned} \tag{109}$$

In particular, for a global axial transform (where ϕ is the same for all x), we have

$$\Delta\mathcal{L} \;=\; -\frac{\phi}{8\pi^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}\big), \tag{110}$$

which amounts to changing the instanton angle

$$\Theta \;\to\; \Theta \;-\; 2\phi. \tag{111}$$

And since a global axial transform of a massless quark field is a valid field redefinition, the fact that this transform changes the instanton angle means that *in QCD with massless quarks the instanton angle is ill defined.*

Next, consider QCD with a single massive flavor. In the Weyl fermion language for the quarks,

$$\Psi_{L,R} = \frac{1 \mp \gamma^5}{2} \Psi_{\text{Dirac}}\,, \tag{112}$$

the fermionic part of the QCD Lagrangian becomes

$$\mathcal{L}_\psi \;=\; i\overline{\Psi}_L \not{D} \Psi_L \;+\; i\overline{\Psi}_R \not{D} \Psi_R \;-\; m\overline{\Psi}_R \Psi_L \;-\; m^* \overline{\Psi}_L \Psi_R \tag{113}$$

for a *complex mass parameter* m. The magnitude $|m|$ of this mass parameter is the physical mass of the quark, while the phase of m is a matter of convention, which may be changed by a suitable axial transform of the quark fields. Indeed, in the Weyl fermion language, the (global) axial transform

$$\Psi_{\text{Dirac}}(x) \;\to\; \exp(i\phi\gamma^5)\Psi_{\text{Dirac}}(x) \tag{114}$$

becomes

$$\Psi_L(x) \;\to\; e^{-i\phi}\Psi_L(x), \qquad \Psi_R(x) \;\to\; e^{+i\phi}\Psi_R(x), \tag{115}$$

so to keep the classical Lagrangian (113) invariant we should also change the phase of m according to

$$m \;\to\; e^{+2i\phi} m. \tag{116}$$

This way, we cal always make m real and non-negative.

However, in the quantum theory a global axial transform of the quark fields changes not only the phase of m but also the instanton angle Θ. Altogether, we have

$$m \;\to e^{2i\phi} m, \qquad \Theta \;\to\; \Theta \;-2\phi, \tag{117}$$

while

$$\overline{\Theta} \;\stackrel{\text{def}}{=}\; \Theta \;+\; \text{phase}(m) \tag{118}$$

remains invariant.

Now let's generalize our analysis to QCD with several massive quark flavors. In the $N_f \times N_f$ matrix form, the fermionic part of the Lagrangian is

$$\mathcal{L}_\psi = i\overline{\Psi}_L \not{D} \Psi_L + i\overline{\Psi}_R \not{D} \Psi_R - \overline{\Psi}_R M \Psi_L - \overline{\Psi}_L M^\dagger \Psi_R \tag{119}$$

where M is now a complex $N_f \times N_f$ mass matrix. However, this matrix can be made diagonal, real, and positive (or at least non-negative) by a (global) chiral $U(N_f)_L \times U(N_f)_R$ transform of the quark fields:

$$\Psi'_L(x) = U_L \Psi_L(x), \qquad \Psi'_R(x) = U_R \Psi_R(x), \tag{120}$$

for some independent unitary matrices U_L and U_R, hence

$$M' = U_R M U_L^\dagger \tag{121}$$

to keep the classical Lagrangian (119) invariant. For any complex matrix M, there always exist some unitary matrices U_L, U_R such that the resulting M' matrix is diagonal, real, and non-negative.

In the quantum theory, the chiral transform (120) is anomalous, so it generally changes the instanton angle Θ. Specifically, it's the $U(1)_A$ part of the $U(N_f)_L \times U(N_f)_R$ symmetry that is anomalous, so reorganizing the symmetry as $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$, we have

$$\phi_A = \tfrac{1}{2}\,\mathrm{phase}\det(U_R) - \tfrac{1}{2}\,\mathrm{phase}\det(U_L) \tag{122}$$

and therefore

$$\Theta' = \Theta - 2\phi_A = \Theta - \mathrm{phase}\det(U_R) + \mathrm{phase}\det(U_L). \tag{123}$$

At the same time,

$$\det(M') = \det(M) \times \frac{\det(U_R)}{\det(U_L)}\,, \tag{124}$$

hence

$$\mathrm{phase}\det(M') = \mathrm{phase}\det(M) + \mathrm{phase}\det(U_L) - \mathrm{phase}\det(U_L), \tag{125}$$

which makes the combined angle

$$\overline{\Theta} \stackrel{\text{def}}{=} \Theta + \text{phase}\det(M) \tag{126}$$

invariant under chiral transforms.

Now consider the possibility of strong CP violation in QCD. Naively, the QCD Lagrangian has two kinds of CP odd terms: the complex mass matrix for the quarks, and the instanton angle Θ. Indeed, the $\operatorname{tr}(\epsilon\mathcal{F}\mathcal{F})$ terms is C-even but P-odd, so it breaks the CP symmetry of the gluon sector. As to the quark sector, we saw back in November that in the Weyl fermion language

$$\begin{aligned} \mathbf{CP}: \quad & \Psi_L(\mathbf{x},t) \;\to\; \gamma^2\gamma^0\Psi_L^*(-\mathbf{x},t), \\ & \Psi_R(\mathbf{x},t) \;\to\; \gamma^2\gamma^0\Psi_R^*(-\mathbf{x},t), \\ \text{hence} \quad & \Big(i\overline{\Psi}_L \not{D}\Psi_L + i\overline{\Psi}_R \not{D}\Psi_R\Big) \;\to\; \Big(i\overline{\Psi}_L \not{D}\Psi_L + i\overline{\Psi}_R \not{D}\Psi_R\Big) \\ \text{but} \quad & \Big(\overline{\Psi}_R M\Psi_L + \overline{\Psi}_L M^\dagger\Psi_R\Big) \;\to\; \Big(\overline{\Psi}_R M^*\Psi_L + \overline{\Psi}_L M^\top\Psi_R\Big), \end{aligned} \tag{127}$$

so a complex mass matrix $M \neq M^*$ seems to break the CP symmetry. However, the CP action (127) can be modified to act as

$$\Psi'_L(\mathbf{x},t) \;\to\; \gamma^2\gamma^0\Psi'^*_L(-\mathbf{x},t), \qquad \Psi'_R(\mathbf{x},t) \;\to\; \gamma^2\gamma^0\Psi'^*_R(-\mathbf{x},t) \tag{128}$$

on the chirally transformed quark fields

$$\Psi'_L(x) \;=\; U_L\Psi_L(x), \qquad \Psi'_R(x) \;=\; U_R\Psi_R(x) \tag{129}$$

instead of the original Ψ_L and Ψ_R. And since the mass matrix can always be made real by such a chiral transform, $M' = M'^*$, the modified CP transform would be a good symmetry of the fermionic Lagrangian.

But any chiral transform that would make the mass matrix M' real would also change the instanton angle from Θ to $\Theta' = \overline{\Theta}$ as in eq. (126). And if that new instanton angle has non-zero value, then we would still have the strong CP violation in the gluon sector. But if the combined $\overline{\Theta}$ angle happens to vanish, then the (modified) CP would be an exact symmetry of QCD. So here is the bottom line: The strong CP violation in QCD depends only on the combined $\overline{\Theta}$ angle rather than the original Θ and/or the phases of the quark mass matrix .

Furthermore, the CP-violating $\overline{\Theta}$ angle of QCD has no perturbative effects, as it affects only the topologically non-trivial configurations of gauge fields with $I \neq 0$. But it does affect the non-perturbative QCD effects such as quark confinement and hence the properties of the bound states — the mesons, the baryons, and the glueballs. In particular, $\overline{\Theta} \neq 0$ would give the neutron a CP-violating electric dipole moment[⋆]

$$d_n \;=\; e\overline{\Theta} \times \begin{cases} O(\alpha_{\rm QCD} \times \text{neutron's radius}), \\ \text{best estimate } 4.5 \cdot 10^{-15} \text{ cm.} \end{cases} \tag{131}$$

However, no strong CP-violating effects were ever discovered experimentally despite diligent searches. Instead, all we have are rather stringent upper limits on such effects. In particular, according to the 2024 Particle Data Group listing, the neutron's electric dipole moment is limited to $|d_n| < e \times 1.8 \cdot 10^{-26}$ cm, which translates to

$$|\overline{\Theta}| \;<\; 4 \cdot 10^{-12}. \tag{132}$$

Thus, for all practical purposes, there is no strong CP violation.

But from the theoretical point of view, having $\overline{\Theta} \approx 0$ with such a high accuracy is rather puzzling, especially in light of the Higgs origin of quark masses. As we have learned back in December (here are my notes on the subject), the mass matrices for the (u, c, t) and the (d, s, b) quark flavors cannot be diagonalized in the same basis respecting the $SU(2)_W$ doublet structure of the LH quarks. Instead, there is a non-trivial unitary Cabibbo–Kobayashi–Maskawa matrix relating the two bases, and this CKM matrix has a rather large complex phase. Thus, prior to diagonalization, the M_{uct} and the M_{dsb} quark mass matrices should

⋆ By Wigner's projection theorem, the matrix elements of a vector operator $\mathbf{d}$ between neutron state must be parallel to those of the spin operator $\mathbf{S}$,

$$\langle\cdots|\,\hat{\mathbf{d}}\,|\cdots\rangle \;=\; \tfrac{4}{3}\,\langle\cdots|\,\hat{\mathbf{d}}\,|\cdots\rangle\,\langle n|\,\hat{\mathbf{S}}\cdot\hat{\mathbf{d}}\,|n\rangle \;=\; (\text{coeff})\tfrac{4}{3}\,\langle\cdots|\,\hat{\mathbf{d}}\,|\cdots\rangle\,\langle n|\,\hat{\vec{\mu}}\cdot\hat{\mathbf{d}}\,|n\rangle \tag{130}$$

where $\hat{\vec{\mu}}$ is the neutron's magnetic dipole moment. But the dipole moment product $\hat{\vec{\mu}}\cdot\hat{\mathbf{d}}$ is C-even and parity-odd, so its non-zero value for the neutron would violate the CP symmetry.

be complex, and there is absolutely no reason to expect the determinant of the net 6×6 quark mass matrix

$$M = \begin{pmatrix} M_{uct} & 0 \\ 0 & M_{dsb} \end{pmatrix} \tag{133}$$

to have a real determinant. Consequently, to have the phase of this complex determinant being precisely canceled by the bare instanton angle Θ to obtain $\overline{\Theta} = 0$ would take either a miracle of fine tuning, or some new physics beyond the Standard Model.

The most likely explanation of $\overline{\Theta}_{\rm QCD} = 0$ seems to be some kind of a Peccei–Quinn symmetry, named after its first proposal by Roberto Peccei and Helen Quinn back in 1977. Let me give you a rather brief explanation in these notes; for details, please look up Roberto Peccei's 2006 lecture notes. Basically, let's extend the Standard Model to include 2 or more Higgs doublets, and let the scalar potential have a global $U(1)$ symmetry changing their relative phases. This symmetry is classically exact but spontaneously broken by the Higgses' VEVs, which gives us a continuous family of classically degenerate vacua. Moreover, let the Yukawa couplings of the several Higgs doublets to the quarks be such that the action of the Peccei–Quinn $U(1)$ symmetry on the Higgs VEVs changes the phases of the quark masses such that

$$U(1)_{\rm PQ}: \quad \det(M) \to \det(M) \times e^{i\alpha}. \tag{134}$$

At the same time, the bare instanton angle Θ of QCD remain unchanged, so that the CP-violating combined angle changes by ϕ,

$$\overline{\Theta} = \Theta + \operatorname{phase}\det(M) \to \overline{\Theta} + \alpha. \tag{135}$$

From the Goldstone–Nambu theorem point of view, we may promote the phase α to a massless Goldsone field $\alpha(x)$. Classically, the effective low-energy Lagrangian for this field is simply

$$\mathcal{L}_{\rm eff} = \frac{F^2}{2}\,(\partial_\mu\alpha)^2 \tag{136}$$

where $F \sim$ energy scale where the Peccei–Quinn symmetry is spontaneously broken. In terms of a canonically normalized Goldsone field

$$\Phi(x) \;=\; F\times\alpha(x) \;+\; \text{const} \tag{137}$$

this gives us

$$\mathcal{L}_{\text{eff}} \;=\; \tfrac{1}{2}(\partial_\mu\Phi)^2 \;+\; \text{no potential}, \tag{138}$$

while the constant in eq, (137) can be set so that

$$\overline{\Theta}(\Phi) \;=\; \frac{\Phi}{F}\,. \tag{139}$$

To be precise, the Lagrangian (138) does not include the QCD effects. Non-perturbatively, the value of $\overline{\Theta}(\Phi)$ affects the vacuum state of QCD and hence its vacuum energy density $V(\overline{\Theta})$. We do not know how exactly does this potential depend on the $\overline{\Theta}$, but we know some of its general features:

- Its general magnitude is $O(\Lambda_{\text{QCD}}^4)$, since all non-perturbative effects in QCD happen at the Λ_{QCD} scale.
- As a function of $\overline{\Theta}$, the potential $V(\overline{\Theta})$ is periodic WRT $\Theta\to\Theta+2\pi$ and even WRT $\Theta\to-\Theta$ (which is reuired by the CP symmetry of the Θ-less QCD). Consequently, $V(\overline{\Theta})$ should have a local extremum at $\overline{\Theta}=0$.

⋆ We *presume* that the extremum at $\overline{\Theta}=0$ is the global minimum of the potential $V(\overline{\Theta})$.

In terms of the Goldstone field Φ, this QCD vacuum energy $V(\overline{\Theta})$ becomes an effective potential

$$V_{\text{eff}}(\Phi) \;=\; V(\overline{\Theta}(\Phi)), \tag{140}$$

whose minimum corresponds to $\overline{\Theta}(\Phi_{\min}) = 0$. Thus, QCD effects lift the degeneracies between the vacua connected by the Peccei–Quinn symmetry, and the preferred vacuum state is precisely the state which leads to $\overline{\Theta} = 0$. And that's how the Peccei–Quinn symmetry solves the strong CP violation problem, or rather explains why such strong CP violations happen to be absent from the real world.

The flip side of the effective potential (140) is that it gives a small mass to the Goldstone field Φ of the Peccei–Quinn symmetry; this field is usually called the *axion*. Indeed, expanding $V(\overline{\Theta})$ in powers of $\overline{\Theta}$ we generally have

$$V(\overline{\Theta}) \;=\; \frac{A}{2}\times\overline{\Theta}^2 \;+\; \frac{B}{24}\times\overline{\Theta}^4 \;+\;\cdots \tag{141}$$

with coefficients $A, B, \ldots = O(\Lambda_{\rm QCD}^4)$. In terms of the canonically normalized axion field Φ,

$$V_{\rm eff}(\Phi) \;=\; \frac{A}{2F^2}\times\Phi^2 \;+\; \frac{B}{24F^4}\times\Phi^4 \;+\;\cdots \tag{142}$$

hence mass

$$M \;=\; \frac{\sqrt{A}}{F} \;=\; \frac{O(\Lambda_{\rm QCD}^2)}{F}\,. \tag{143}$$

BTW, the axion field is called that way (or rather its quanta are called *the axions*) because of its axial coupling to the gluon fields,

$$\mathcal{L}_{\rm axion} \;=\; \tfrac{1}{2}(\partial_\mu\Phi)^2 \;-\; \frac{M^2}{2}\,\Phi^2 \;+\; \frac{g^2}{16\pi^2 F}\times\Phi\times{\rm tr}\bigl(F_{\mu\nu}\widetilde{F}^{\mu\nu}\bigr) \;+\;\cdots$$

where $\cdots$ stand for a similar axial coupling to the electromagnetic fields.

The axions — if they exist — are very interesting particles, and there are zillions of papers written about them. For a good introduction to the subject, I recommend Roberto Peccei's 2006 lecture notes.