PHY–396 K. Mid-term Exam. Due Wednesday, October 23 (2024)

Please do not waste your time — and also my time — by copying posted homework solutions or class notes. If you need to use any homework result, simply reference the appropriate question or equation and go ahead; likewise for the class notes or anything else explicitly derived in class. Similarly, you may quote the textbook or any other book you have happened to read, but in that case *make sure to spell out all the intermediate steps.*

The first three problems of the exam are about the topologically massive gauge fields in $2+1$ spacetime dimensions: Problem 1 deals with the classical abelian fields, problem 2 with the classical non-abelian fields, and problem 3 with the quantum abelian fields. Finally, problem 4 takes you back to $3+1$ dimensions and deals with the chiral symmetries of massless fermions.

1. In three spacetime dimensions (two space plus one time) an antisymmetric Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$ is equivalent to an axial Lorentz vector, $F^{\mu\nu} = \epsilon^{\mu\nu\lambda}F_\lambda$. Consequently, in 3D one can make the photons massive without breaking the gauge invariance of the electromagnetic field $A_\mu(x)$. Indeed, consider the following Lagrangian:

$$\mathcal{L} \;=\; -\frac{1}{2}\,F_\lambda F^\lambda \;+\; \frac{m}{2}\;F_\lambda A^\lambda \tag{1}$$

where

$$F_\lambda(x) \;=\; \tfrac{1}{2}\epsilon_{\lambda\mu\nu}F^{\mu\nu}(x) \;=\; \epsilon_{\lambda\mu\nu}\,\partial^\mu A^\nu(x), \tag{2}$$

or in components, $F^1 = -E^2$, $F^2 = +E^1$, and $F^0 = -B$. (In $2+1$ dimensions, the magnetic field is a pseudoscalar rather than an axial vector.) The parity-breaking mass term in the Lagrangian (1) is called the *topological mass term.*

   (a) Although the topological mass term is not gauge invariant, the action $S = \int d^3x\,\mathcal{L}$ is gauge invariant. (Assuming the $A^\mu(x)$ vanish fast enough when $x \to \infty$.)
   Prove this.

(b) The massive analogues of the 3D Maxwell equations are

$$\partial_\lambda F^\lambda \;=\; 0 \quad \text{and} \quad \epsilon^{\lambda\mu\nu}\partial_\mu F_\nu \;=\; mF^\lambda. \tag{3}$$

Derive these equations. Also, spell them out in 2D vector notations.

(c) Show that eqs. (3) imply the Klein–Gordon equations for the tension fields, $(\partial^2 + m^2)F_\lambda(x) = 0$.
Hint: in $2+1$ dimension $\epsilon^{\alpha\beta\gamma}\epsilon_\alpha^{\mu\nu} = g^{\beta\mu}g^{\gamma\nu} - g^{\beta\nu}g^{\gamma\mu}$.

(d) Write down the plane-wave solutions to eqs. (3) and show that for each on-shell momentum $k^\mu = (+\omega_{\mathbf{k}}, \mathbf{k})$ there is only one physical polarization. Or rather, one polarization for the positive-frequency wave and another polarization for the negative frequency wave.

(e) Show that the polarization of the $\mathbf{k} = 0$ wave is circular: $\mathbf{E}(t)$ rotates left for $m > 0$ but right for $m < 0$. Then argue that this means that the 2D spin state $|s\rangle$ of a topologically massive photon follows the sign of the topological mass m, namely $s = 1 \times \operatorname{sign}(m)$.

2. The non-abelian analogue of the topological mass term for the 3D gauge fields is the Chern–Simons term. Combining it with the usual Yang–Mills Lagrangian for the non-abelian gauge fields, we get

$$\begin{aligned}\mathcal{L} \;=\; \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{CS}} \;&=\; -\frac{1}{2g^2}\operatorname{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \frac{k}{4\pi}\,\epsilon^{\lambda\mu\nu}\operatorname{tr}\left(\mathcal{A}_\lambda\,\partial_\mu\mathcal{A}_\nu \;+\; \frac{2i}{3}\,\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\right)\\ &=\; -\frac{1}{2g^2}\operatorname{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \frac{k}{8\pi}\,\epsilon^{\lambda\mu\nu}\operatorname{tr}\left(\mathcal{A}_\lambda\,\mathcal{F}_{\mu\nu} \;-\; \frac{2i}{3}\,\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\right).\end{aligned} \tag{4}$$

For simplicity, let's assume an $SU(N)$ gauge symmetry group, and all the traces are in the fundamental representation $\mathbf{N}$ of the group.

(a) Expand the Lagrangian (4) in terms of the canonically normalized component fields $A^a_\mu(x)$. The Yang–Mills part was done in class, so focus on the Chern–Simons term: Expand it into a topological mass term $m = kg^2/4\pi$ for each gluon and a derivative-less 3–gluon interaction term.

Note that g^2 has dimensionality of mass in 3D while the k coefficient — called the *Chern–Simons level* — is dimensionless. In fact, k must be integer (positive, negative, or zero) to assure gauge invariance of the quantum theory.

(b) Verify invariance of the action $\int d^3x\,\mathcal{L}$ under the *infinitesimal* gauge transformations.

(b*) For extra credit, instead of part (b):
Show that under a finite gauge transformation $U(x)$, the action changes by a field-independent amount

$$\Delta S \;=\; \frac{-k}{12\pi}\int d^3x\,\epsilon^{\lambda\mu\nu}\,\operatorname{tr}\Bigl(U^{-1}\partial_\lambda U\cdot U^{-1}\partial_\mu U\cdot U^{-1}\partial_\nu U\Bigr). \tag{5}$$

FYI — but don't try to prove this during the exam — the integral here depends only on the topological properties of the $U(x)$, and its values are always integer $\times\, 24\pi^2$. Consequently, for integer k — and only for integer k — $\Delta S = 2\pi\times$ an integer, which makes e^{iS} gauge invariant and assures the gauge invariance of the path integral of the quantum theory.

(c) Derive the non-abelian analogues of eqs (3).

3. Next, consider the quantum theory of the massive EM fields in 3D; for simplicity, let's focus on the abelian fields of problem 1.

(a) Write down the canonical conjugate fields $\Pi^{1,2}(x)$ to the vector potentials $A^{1,2}(x)$, re-express the electric fields $E^{1,2}$ in terms of the $\Pi^{1,2}$ and the $A^{1,2}$ fields, then show that the canonical commutation relations between the quantum $\hat{\Pi}^{1,2}$ and $\hat{A}^{1,2}$ fields lead to

$$\Bigl[\hat{E}^i(\mathbf{x},t),\hat{E}^j(\mathbf{y},\text{same }t)\Bigr] \;=\; -im\epsilon^{ij}\,\delta^{(2)}(\mathbf{x}-\mathbf{y}). \tag{6}$$

(b) One of the classical equations of motion (3) amounts to $\nabla\cdot\mathbf{E} = mB$. Since this equation does not involve time derivatives, it becomes an operatorial identity in the Hilbert space of the quantum theory.
Use this identity to derive the equal-time commutations relations between the magnetic and the electric fields.

(c) Derive the Hamiltonian of the quantum theory from the classical Lagrangian (1) and show that *regardless of the mass m*

$$\hat{H} \;=\; \int d^2\mathbf{x} \Big(\tfrac{1}{2}\hat{\mathbf{E}}^2 \;+\; \tfrac{1}{2}\hat{B}^2 \Big). \tag{7}$$

Instead, the mass affects the equal-time commutation relations of the quantum fields. (And also the time-independent relation between B and $\nabla \cdot \mathbf{E}$.)

(d) Finally, use the Hamiltonian (7) and the commutation relations from parts (a) and (b) to show that in the Heisenberg picture, the quantum $\hat{\mathbf{E}}(x)$ and $\hat{B}(x)$ fields obey similar equations of motion to the classical fields.

4. Finally, let's go back to 3+1 spacetime dimensions and consider a theory of N massless Dirac spinor fields $\Psi^1(x), \ldots, \Psi^N(x)$ coupled to the electromagnetic field $A_\mu(x)$. All fermions have the same charge $q = -e$, thus

$$\mathcal{L} \;=\; -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \;+\; \overline{\Psi}_i (i \not{\partial} \,+\, e \not{A}) \Psi^i \qquad \langle\!\langle \,\text{implicit } \sum_{i=1}^{N} \rangle\!\rangle. \tag{8}$$

(a) List all *continuous* symmetries of the classical theory — global or local, space-time or internal — and specify how they act on the fields. For simplicity, skip the dilatation and special conformal symmetries of the classical massless theory.

The rest of this problem focuses on the $U(N)_L \otimes U(N)_R$ chiral symmetries which act on the Dirac fields according to

$$\Psi'^i(x) \;=\; \left(\frac{1-\gamma^5}{2}\, (U_L)^i{}_j \;+\; \frac{1+\gamma^5}{2}\, (U_R)^i{}_j \right) \Psi^j(x), \tag{9}$$

where U_L and U_R are two independent $N \times N$ unitary matrices. The infinitesimal chiral symmetries act as

$$\delta\Psi^i(x) \;=\; -i\epsilon \left(\frac{1-\gamma^5}{2}\, (T_L)^i{}_j \;+\; \frac{1+\gamma^5}{2}\, (T_R)^i{}_j \right) \Psi^j(x) \;=\; -i\epsilon \left((T_V)^i{}_j \;+\; \gamma^5 (T_A)^i{}_j \right) \Psi^j \tag{10}$$

for some hermitian matrices T_L and T_R, and hence $T_V = \frac{1}{2}(T_R + T_L)$ and $T_A = \frac{1}{2}(T_R - T_L)$.

(b) Show that the Noether currents of these symmetries are linear combinations of the vector and axial currents

$$\left(J_V^\mu\right)^j_{\ i} \;=\; \overline{\Psi}_i\gamma^\mu\Psi^j \qquad \text{and} \qquad \left(J_A^\mu\right)^j_{\ i} \;=\; \overline{\Psi}_i\gamma^\mu\gamma^5\Psi^j\,. \tag{11}$$

Also, verify that all these currents are indeed conserved (in the classical theory).

Now consider the net charge operators in the quantum theory,

$$\left(\hat{Q}_V\right)^i_{\ j} \;=\int d^3\mathbf{x}\left(\hat{J}_V^0(\mathbf{x},t)\right)^i_{\ j} \qquad \text{and} \qquad \left(\hat{Q}_A\right)^i_{\ j} \;=\int d^3\mathbf{x}\left(\hat{J}_A^0(\mathbf{x},t)\right)^i_{\ j}\,. \tag{12}$$

(c) Calculate the equal-time commutators of these charges with the fermionic fields $\hat{\Psi}^k(\mathbf{x},t)$ and $\hat{\overline{\Psi}}_k(\mathbf{x},t)$.

(d) Verify that all the charges (12) commute with the Hamiltonian operator for the fermionic fields. For simplicity, ignore the EM fields and their interactions with the fermions.

(e) Verify that the charges (12) obey the commutation relations of the $U(N)\times U(N)$ generators, namely

$$\begin{aligned}
\left[\left(\hat{Q}_V\right)^i_{\ j},\left(\hat{Q}_V\right)^k_{\ \ell}\right] &= \delta^i_\ell\left(\hat{Q}_V\right)^k_{\ j} \;-\; \delta^k_j\left(\hat{Q}_V\right)^i_{\ \ell}\,,\\
\left[\left(\hat{Q}_V\right)^i_{\ j},\left(\hat{Q}_A\right)^k_{\ \ell}\right] &= \delta^i_\ell\left(\hat{Q}_A\right)^k_{\ j} \;-\; \delta^k_j\left(\hat{Q}_A\right)^i_{\ \ell}\,,\\
\left[\left(\hat{Q}_A\right)^i_{\ j},\left(\hat{Q}_A\right)^k_{\ \ell}\right] &= \delta^i_\ell\left(\hat{Q}_V\right)^k_{\ j} \;-\; \delta^k_j\left(\hat{Q}_V\right)^i_{\ \ell}\,.
\end{aligned} \tag{13}$$

Finally, let's expand the charges (12) in terms of fermionic creation and annihilation operators. For simplicity, work in the Schrödinger picture and treat the fermionic fields as free.

(f) As a first step, use the results of homework set#7 to show that

$$u^\dagger(+\mathbf{p},s')v(-\mathbf{p},s) \;=\; 0 \;=\; v^\dagger(-\mathbf{p},s')u(+\mathbf{p},s) \quad \text{for any spin states } s,s'. \tag{14}$$

Also show that in the helicity basis for spin states of *massless* particles

$$\gamma^5 u(\mathbf{p},\lambda) \;=\; +2\lambda u(\mathbf{p},\lambda) \quad \text{but} \quad \gamma^5 v(\mathbf{p},\lambda) \;=\; -2\lambda v(\mathbf{p},\lambda). \tag{15}$$

Please do not redo the homework in your answer, simply quote the relevant formulae.

(g) And now expand the fermionic fields into creation and annihilation operators and show that in the helicity basis for the spin states Show that in the helicity basis of massless particles' spin states

$$(Q_V)^i{}_j \;=\int\frac{d^3\mathbf{p}}{(2\pi)^2\,2E_{\mathbf{p}}}\sum_\lambda\Bigl(\hat{a}^\dagger(\mathbf{p},\lambda,j)\hat{a}(\mathbf{p},\lambda,i)\;-\;\hat{b}^\dagger(\mathbf{p},\lambda,i)\hat{b}(\mathbf{p},\lambda,j)\Bigr), \tag{16}$$

$$(Q_A)^i{}_j \;=\int\frac{d^3\mathbf{p}}{(2\pi)^2\,2E_{\mathbf{p}}}\sum_\lambda 2\lambda\times\Bigl(\hat{a}^\dagger(\mathbf{p},\lambda,j)\hat{a}(\mathbf{p},\lambda,i)\;+\;\hat{b}^\dagger(\mathbf{p},\lambda,i)\hat{b}(\mathbf{p},\lambda,j)\Bigr). \tag{17}$$

Note: For $i = j$ you may get an extra infinite but c-number constant due to anticommuting $\hat{b}$ and $\hat{b}^\dagger$ operators to bring them to normal order ($\hat{b}^\dagger$ to the left of $\hat{b}$). Subtract this constant away to make sure the vacuum state is annihilated by all the charges.