Multiplets

In Physics terminology, the *multiplets* of some symmetry are often confused with the *representations* of the symmetry group. The two concepts are related but not synonymous, so let me explain the difference.

A *multiplet* is a set of objects — quantum states, operators, fields, whatever — which transform into each other by the symmetry action. For example, consider the $|n,\ell,m\rangle$ states of a hydrogen atom (for simplicity, let's ignore the spin). With respect to the $SO(3)$ rotational symmetry group, the states with the same n and ℓ but different m form a multiplet (n,ℓ): they transform into each other according to

$$R : |n,\ell,m\rangle \;\to\; \sum_{m'} \hat{\mathcal{D}}^{(\ell)}_{m,m'}(R)\,\big|n,\ell,m'\big\rangle\,, \tag{1}$$

but the transformation law does not involve any states with different n or ℓ. Instead, such states form different multiplets (n',ℓ'). In particular, the multiplets (n,ℓ) and (n',ℓ) — with the same ℓ but different $n'\neq n$ — are different multiplets comprising different states, even though they transform in a similar way under the $SO(3)$ symmetry.

By comparison, a *representation* is a mathematical object — a homomorphism from the symmetry group G into a space of matrices or linear operators. For example, the homomorphisms

$$R\in SO(3) \;\;\mapsto\;\; \text{matrix}\hat{\mathcal{D}}^{(\ell)}_{m,m'}(R) \tag{2}$$

for $\ell = 0,1,2,3,\ldots$ form finite representations of the $SO(3)$ group, but for any such homomorphism the representation does not care what the $\hat{\mathcal{D}}^{(\ell)}_{m,m'}(R)$ matrices act upon. Thus, for each ℓ there is only one representation of the $SO(3)$ by $(2\ell+1)\times(2\ell+1)$ matrices.

From the multiplet point of view, *a representation is the type of a multiplet*: it specifies how the members of a multiplet transform into each other, regardless of what these members happen to be. In the hydrogen atom example, the multiplets (n,ℓ) and (n',ℓ) (with the same ℓ but different $n'\neq n$) are of the same type — the (ℓ) representation of the $SO(3)$ — but they are different multiplets because they comprise different states.

In gauge theory, the matter fields form multiplets of the gauge theory group G, and there can be several distinct multiplets of the same type. For example, in QCD (quantum chromodynamics) the gauge group is $SU(3)$ while the matter fields are the quarks — Dirac spinor fields $\Psi_\alpha^{if}(x)$ which come in 3 colors $i = 1,2,3$ and 6 flavors $f = u,d,s,c,b,t$. The $SU(3)$ gauge symmetry acts only on the colors,

$$\Psi_\alpha^{\prime if}(x) \;=\; \sum_j U^i{}_j(x)\Psi_\alpha^{jf} \quad \langle\!\langle \text{ for the same } f \,\rangle\!\rangle, \tag{3}$$

so the quarks form 6 distinct multiplets according to their flavors: the 3 quark fields of different colors $i = 1,2,3$ but of the same flavor form a multiplet. But all 6 of these multiplet are of the same type — the fundamental **3** representation of the $SU(3)$ group.