14/11

in a metal the Fermi sea state $|FS\rangle$

has $E_{FS} = E_{empty} + \sum_{p,s}^{filled} (E_{ps} - \mu)$

charge $Q_{FS} = Q_{empty} - e\, N_{filled}$.

E_{FS} : part of metal's binding energy

Q_{FS} is cancelled by Q_{ion}

include Q_{ion} into Q_{empty}

then $Q_{FS} = 0$.

$$Q_{net} = \sum_{p,s}^{unfilled} (-e) a_{ps}^\dagger a_{ps} + \sum_{p,s}^{filled} (+e) b_{ps}^\dagger b_{ps}$$

$+ Q$

Relativistic dims.

$|phys\ vac\rangle = |Dirac\ sea\rangle$

it has $E_{DS} = E_{empty} + \sum_{p,s}^{filled} (-E_p)$

$$Q_{DS} = \underset{\text{unknown}}{\underset{\downarrow}{Q_{empty}}} - e \underset{\text{infinite}}{\underset{\downarrow}{N_{filled}}}$$

Exp. fact : $Q(phys\ vac) = 0$.

(14/2) $\hat{Q} = \sum_{p,s} (-e \hat{a}^\dagger_{ps} a_{ps} + e b^\dagger_{ps} b_{ps}) + c.$

El. current $J^\mu_{el} = -e \bar{\psi} \gamma^\mu \psi$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \underbrace{(i\gamma^\mu D_\mu - m)}_{i\gamma^\mu \partial_\mu + e\hat{A}_\mu - m} \psi$$

$$= \mathcal{L}_{free} + e A_\mu \bar{\psi} \gamma^\mu \psi$$

$$\hat{Q}_{net} = Q_{empty} + \int d^3x \, (-e) \bar{\psi} \gamma^0 \psi$$

$$= Q_{empty} + \sum_{p,s} \underbrace{(-e\, a^\dagger_{ps} a_{ps} - e\, b_{ps} b^\dagger_{ps})}_{\text{box norm.} \quad -e a^\dagger_{ps} a_{ps} + e b^\dagger_{ps} b_{ps} - e}$$

$Q_{empty} + \sum_{p,s} (-e)$; $Q(\text{Dirac Sea}) = 0.$

14/3

Free Dirac fermion

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Global $U(1)_V$ symmetry $\psi'(x) = e^{-i\theta}\psi(x)$

$$\bar{\psi}'(x) = e^{i\theta}\bar{\psi}(x)$$

Noether current.

$$J_V^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\bigg|_{right} \times -i\psi$$

$$+ \cancel{\phi} i\bar{\psi} \times \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\bigg|_{\cancel{left}}$$

$$= i\bar{\psi}\gamma^\mu \times i\psi + 0 = \bar{\psi}\gamma^\mu\psi.$$

$$\boxed{J_V^\mu = V^\mu = \bar{\psi}\gamma^\mu\psi}$$

Continuous Lorentz tr.

$$V'^\mu = \bar{\psi}\underbrace{M^{-1}(L)\gamma^\mu \cancel{\psi} M(L)}_{L^\mu{}_\nu \gamma^\nu}\psi$$

$$V'^\mu = L^\mu{}_\nu V^\nu \rightarrow V^\mu \text{ is a vector.}$$

Next week: under parity (space reflection)

$$\vec{x} \rightarrow -\vec{x},\ t \rightarrow +t$$

V^μ is a true (polar) vector.

14/4

Dirac eq-n $i\gamma^\mu \partial_\mu \psi - m\psi = 0$

$$\rightarrow \gamma^\mu \partial_\mu \psi = -im\psi$$

$$\partial_\mu \bar{\psi} \gamma^\mu = +im\bar{\psi}$$

$$\partial_\mu V^\mu = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = \partial_\mu\bar{\psi}\gamma^\mu \psi + \bar{\psi}\gamma^\mu\partial_\mu\psi$$

$$= +im\bar{\psi}\psi - im\bar{\psi}\psi = 0.$$

For massless ψ, $\exists$ another

Global $U(1)_A$ symmetry

$$\psi'(x) = \exp(-i\theta\gamma^5)\psi(x)$$

$\gamma^5 \overset{def}{\equiv} i\gamma^0\gamma^1\gamma^2\gamma^3 \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$

$\gamma^5\gamma^5 = 1, \ \gamma^{5\dagger} = \gamma^5, \ \bar{\gamma^5} = -\gamma^5$

in Weyl basis $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$ in 2x2 blocks

$$\psi'(x) = e^{-i\theta\gamma^5}\psi(x)$$

$$\psi'^\dagger(x) = \psi^\dagger(x)e^{+i\theta\gamma^5}$$

$$\bar{\psi}'(x) = \bar{\psi}(x)e^{-i\theta\gamma^5}$$

bec $\gamma^0 e^{+i\theta\gamma^5}\gamma^0 = e^{-i\theta\gamma^5}$

14/5

Massless $\mathcal{L} = i\bar\psi\gamma^\mu\partial_\mu\psi$

$$\mathcal{L}' = i\bar\psi e^{-i\theta\gamma^5} \times \gamma^\mu \times e^{-i\theta\gamma^5}\partial_\mu\psi$$

$$e^{-i\theta\gamma^5}\gamma^\mu = \gamma^\mu \times e^{+i\theta\gamma^5}$$

$$e^{-i\theta\gamma^5}\gamma^\mu e^{-i\theta\gamma^5} = \gamma^\mu$$

$$\mathcal{L}' = i\bar\psi\gamma^\mu\partial_\mu\psi = \mathcal{L}.$$

→ Symmetry

$$\mathcal{L}_{mass} = -m\bar\psi\psi \to -m\bar\psi e^{-2i\theta\gamma^5}\psi$$

$$\neq \mathcal{L}_{mass}$$

Current $J^\mu_A = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Big|_\mu \times -i\cancel{\theta}\gamma^5\psi$

$$= i\bar\psi\gamma^\mu \times -i\gamma^5\psi.$$

$$\boxed{J^\mu_A = A^\mu = \bar\psi\gamma^\mu\gamma^5\psi}$$

HW: ~~γ^5 commutes with all $S^{\mu\nu}$~~

$$\to \gamma^5 M(\Lambda) = M(\Lambda)\gamma^5$$

under cont. Lorentz

$$A'^\mu = \bar\psi\,\underbrace{M^{-1}(\Lambda)\gamma^5\gamma^\mu M(\Lambda)}_{\gamma^5\times\Lambda^\mu{}_\nu\gamma^\nu}\psi \to \Lambda^\mu{}_\nu\bar\psi\gamma^\nu\gamma^5\psi.$$

14/6

under cont. Lorentz

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu \to A^\mu \text{ is a vector.}$$

Parity: A^μ is an <u>axial</u> vector

$U(1)_V$ $\psi' = e^{-i\alpha}\psi$ vector symmetry

$U(1)_A$ $\psi' = e^{-i\alpha\gamma^5}\psi$ axial symmetry.

conservation of A^μ

$$\partial_\mu (A^\mu = \bar\psi \gamma^\mu \gamma^5 \psi)$$

$$= \partial_\mu \bar\psi \gamma^\mu \times \gamma^5 \psi + \left(\bar\psi \gamma^\mu \gamma^5 \partial_\mu \psi \atop - \bar\psi \gamma^5 \times \gamma^\mu \partial_\mu \psi\right)$$

$$= 0 + im\bar\psi\gamma^5\psi + im\bar\psi\gamma^5\psi$$

$$= +2im\bar\psi\gamma^5\psi.$$

$$\boxed{\partial_\mu A^\mu = 0 \text{ \underline{iff} } m = 0}$$

14/7

1 massless Dirac ψ

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi$$

$\rightarrow$ global $U(1)_V \otimes U(1)_A$ symmetry

~~We~~ Weyl convention $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$

$$\psi_{Dirac} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$\psi_L(x)$: Left-handed Weyl spinor
$\underset{\sim}{2}$ of $Spin(3,1) = SL(2,\mathbb{C})$

$\psi_R(x)$ Right-handed Weyl spinor
$\bar{\underset{\sim}{2}}$ of $Spin(3,1)$.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \begin{aligned} \bar{\sigma}^\mu &= (1, +\vec{\sigma}) \\ \sigma^\mu &= (1, -\vec{\sigma}). \end{aligned}$$

$$\mathcal{L}_{Dirac} = i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R.$$

for $m = 0$.

14/8

$U(1)_V$ & $U(1)_A$ symmetries

$$\psi_D(x) \to \underbrace{\exp(-i\theta_V - i\theta_A\gamma^5)}\psi_D(x).$$

$$\begin{pmatrix} e^{-i(\theta_V-\theta_A)} & 0 \\ 0 & e^{-i(\theta_V+\theta_A)} \end{pmatrix}$$

$$\to \begin{pmatrix} e^{-i\theta_L} & 0 \\ 0 & e^{-i\theta_R} \end{pmatrix} \qquad \begin{aligned} \theta_L &= \theta_V - \theta_A \\ \theta_R &= \theta_V + \theta_A. \end{aligned}$$

in terms of Weyl spinors

$$\left.\begin{aligned} \psi_L'(x) &= e^{-i\theta_L}\psi_L(x) \\ \psi_R'(x) &= e^{-i\theta_R}\psi_R(x) \end{aligned}\right\} \text{ independent } \theta_L, \theta_R.$$

$\to U(1)_L \otimes U(1)_R$ chiral symmetry

currents $J_L^\mu = \dfrac{V^\mu - A^\mu}{2} = \bar\psi\gamma^\mu\frac{1-\gamma^5}{2}\psi$

$$= \psi_L^\dagger\bar\sigma^\mu\psi_L$$

$$J_R^\mu = \frac{V^\mu + A^\mu}{2} = \bar\psi\gamma^\mu\frac{1+\gamma^5}{2}\psi = \psi_R^\dagger\sigma^\mu\psi_R.$$

Charged massless ψ

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar\psi\gamma^\mu D_\mu\psi$$

$\to$ same $U(1)_L \otimes U(1)_R$ (global)

14/9

N massless Dirac fermions

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_i \bar{\psi}_i (i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m)\psi^i$$

$i=1,\ldots,N$ same charge q, same mass m

→ global $U(N)_V$ symmetry

$$\psi'^i(x) = \sum_j U^i{}_j \psi^j(x)$$

unitary matrix $U^i{}_j$

$$\bar{\psi}'_i(x) = \sum_j \bar{\psi}_j (U^\dagger)^j{}_i$$

$G = U(N) = U(1) \times SU(N)$

$U(1)$ current $V^\mu = \sum_i \bar{\psi}_i \gamma^\mu \psi^i$

$SU(N)$ currents $V^{\mu a} = \sum_{ij} \bar{\psi}_i \left(\frac{\lambda^a}{2}\right)^i{}_j \gamma^\mu \psi^j$

→ combine into matrix-valued

currents $(V^\mu)^i{}_j = \bar{\psi}_j \gamma^\mu \psi^i$

$$= \sum_a 2V^{\mu a}(\lambda^a)^i{}_j + \frac{1}{N}V^\mu \delta^i_j$$

14/10

~~O~~ For N massless Dirac ψ^i

→ independent ~~O~~ Weyl ψ_L^i, $\psi_R^i(x)$

~~→ chiral $U(N)_L \otimes U(N)_R$ Global symmetries~~

$$\psi_L'^i(x) = (U_L)^i{}_j\, \psi_L^j(x) \quad (\text{implicit } \textstyle\sum_j)$$

$$\psi_R'^i(x) = (U_R)^i{}_j\, \psi_R^j(x)$$

independent $U(N)$ matrices U_L, U_R

in terms of ~~Ψ_R~~ Ψ_{Dirac}

$$\Psi'^i(x) = \left[(U_L)^i{}_j \times \frac{1-\gamma^5}{2} + (U_R)^i{}_j \times \frac{1+\gamma^5}{2}\right]\Psi^j(x)$$

As a group $U(N)_L \otimes U(N)_R \neq U(N)_V \otimes U(N)_A$

$U(N)_L \otimes U(N)_R = U(1)_V \otimes U(1)_A \otimes SU(N)_L \otimes SU(N)_R$

but generators of ~~infinitesimal $U(N)_L \times U(N)_R$ do contain both~~ vector & Axial generators.

Let $U_L = 1 + i\epsilon_L$, $U_R = 1 + i\epsilon_R$

indep. hermitian ϵ_L, ϵ_R,

14/11

$$\delta\psi^i(x) = \left[i(\epsilon_L)^i{}_j \frac{1-\gamma^5}{2} + i(\epsilon_R)^i{}_j \frac{1+\gamma^5}{2} \right] \psi^j(x)$$

$$= \left[i(\epsilon_V)^i{}_j + i(\epsilon_A)^i{}_j \gamma^5 \right] \psi^j(x)$$

$$\left.\begin{aligned} \epsilon_V &= \frac{\epsilon_R + \epsilon_L}{2} \\ \epsilon_A &= \frac{\epsilon_R - \epsilon_L}{2} \end{aligned}\right\}$$ indep. infinitesimal hermitian matrices

→ currents

$$\boxed{\begin{aligned} (V^\mu)^i{}_j &= \bar{\psi}_j \gamma^\mu \psi^i \\ (A^\mu)^i{}_j &= \bar{\psi}_j \gamma^\mu \gamma^5 \psi^i \end{aligned}}$$

$$(J^\mu_L)^i{}_j = \tfrac{1}{2}(V^\mu - A^\mu)^i{}_j = \psi^\dagger_{Lj} \bar{\sigma}^\mu \psi_L^i$$

$$(J^\mu_R)^i{}_j = \tfrac{1}{2}(V^\mu + A^\mu)^i{}_j = \psi^\dagger_{Rj} \sigma^\mu \psi_R^i$$

QCD: quarks ψ^{if} have colors $i = 1,2,3$
& flavors f.

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_f \sum_i \bar{\psi}_{if} (i\gamma^\mu D_\mu - m_f) \psi^{if}$$

non-abelian

14/12

Global symmetries must commute with covariant derivatives

$$D_\mu \psi^{cf} = \partial_\mu \psi^{cf} + i g A_\mu^a \left(\tfrac{\lambda^a}{2}\right)^c{}_d \psi^{df}.$$

→ can act on flavor only

same action for all 3 colors.

if all m_f are same

$$\psi'^{cf} = \sum_{f'} U^f{}_{f'} \psi^{cf'}(x).$$

for $U \in U(N_f)$.

but if all m_f are different

then $U \in [U(1)]^{N_f}$

~~If all $m_f = 0$~~

Then global chiral symmetry

$U(N_f)_L \otimes U(N_f)_R$

$$\psi'^{cf}(x) = \left[(U_L)^f{}_{f'} \frac{1-\gamma^5}{2} + (U_R)^f{}_{f'} \frac{1+\gamma^5}{2} \right] \psi^{cf'}(x)$$

14/13

QCD in real life.

non-pert. strong couplings

at $E < \Lambda_S \sim 500\,\text{MeV}$.

2 flavors are very light

$m_u, m_d \sim$ few MeV. $\ll \Lambda_S$.

$\to$ approximate $m_u \simeq m_d \simeq 0$

$\to$ chiral $U(2)_L \otimes U(2)_R$

Another flavor s is heavier but

$m_s \sim 150\,\text{MeV}$

but lesser than Λ_S

$\to$ less good approximation $m_s \simeq 0$

$\to$ chiral $U(3)_L \otimes U(3)_R$

Remaining flavors c, b, t are heavy.

$U(2)_L \otimes U(2)_R = U(1)_V \times U(1)_A \otimes SU(2)_L \otimes SU(2)_R$

states in QCD.

$U(1)_V$: exact, Quark number $= 3 \times$ Baryon number

$U(1)A$: anomalous,

broken by quantum effects

14/14)

$SU(2)_L \otimes SU(2)_R$
is spontaneously broken down
to $SU(2)_V$ = isospin.

Ditto $U(3)_L \times U(3)_R$

$U(1)_V \times U(1)_A \times SU(3)_L \times SU(3)_R$

$U(1)_V$: exact
$U(1)_A$: anomalous

$SU(3)_L \times SU(3)_R$ is spontaneously
broken down to $SU(3)_V$

Chiral gauge Theories.

massless fermions → independent
ψ_L^i & ψ_R^i
→ they may transform differently
under a gauge symmetry
→ form different multiplets.
→ chiral gauge theory.

14/15

Feroex : Electroweak $SU(2)_W \times U(1)_Y$

under $SU(2)$ ψ_L form 12 doublets

ψ_R are 24 singlets.

Also, ψ_L & ψ_R have different

$U(1)_Y$ charges.