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Complex scalar field $\phi(x)$

$U(1)$ gauge symmetry

$\phi(x) \to e^{i\theta(x)}\phi(x)$, indep $\theta(x)$ @ each x

Lattice, we sample $\phi(x)$ @ $x^\mu = a n^\mu$

Gauge: $\phi(\vec{n}) \to e^{i\theta(\vec{n})}\phi(\vec{n})$

indep. $\theta(\vec{n})$ @ each lattice site @ $\vec{n}$

Continuum: covariant $D_\mu \phi(x) = \partial_\mu \phi(x) + iA_\mu(x)\phi(x)$

$A'_\mu(x) = A_\mu(x) - \partial_\mu \theta(x)$

$(D_\mu \phi)'(x) = e^{i\theta(x)} D_\mu \phi(x)$.

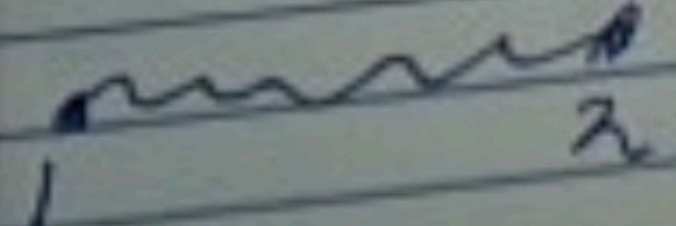
Lattice: $\partial_\mu \phi(x) \to \frac{1}{a}(\phi(\vec{n}+\hat{1}_\mu) - \phi(\vec{n}))$

diff phase transform

to make covariant derivative,

need a parallel transport from $\vec{n}$ to $\vec{n}+1_\mu$

Take any path (in the continuum)



let $W(\text{path}_1) = \exp\left(-i\int_1^2 A_\mu(x)\,dx^\mu\right)$

along the path.

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Gauge trans:

$$\int_1^2 A_\alpha dx^\alpha \to \text{itself} - \int_1^2 \partial_\alpha\theta\, dx^\alpha$$

$$= \text{itself} - \theta(2) + \theta(1).$$

$$W(\text{path}) \to \text{itself} \times e^{i\theta(2)} e^{-i\theta(1)}.$$

$$W(1\to 2)\,\Phi(1) \to e^{i\theta(2)}\, W(1\to 2)\,\Phi(1).$$

↳ transforms like $\Phi(2)$.

$\Phi(2) - W(1\to 2)\Phi(1)$ is covariant.

Lattice: $\forall$ lattice link from $\vec{n}$ to $\vec{n}+\hat{\mu}$

let $W_\mu(\vec{n})$ be W(that link)

$$= \exp\left(-i\int_{\vec{n}}^{\vec{n}+\hat{\mu}} A_\alpha dx^\alpha\right).$$

$$= 1 - ia A_\mu(\vec{n}) + O(a^2).$$

$$\text{Lattice } D_\mu\Phi(\vec{n}) = \frac{1}{a}\left[\Phi(\vec{n}+\hat{\mu}) - W_\mu(\vec{n})\Phi(\vec{n})\right]$$

Gauge-invariant lattice action

for Φ

$$S = a^4 \sum_{\vec{n}} \left\{ \frac{1}{a^2}\sum_\mu \left|\Phi(\vec{n}+\hat{\mu}) - W_\mu(\vec{n})\Phi(\vec{n})\right|^2 - m^2|\Phi(\vec{n})|^2 - \frac{\lambda}{4}|\Phi(\vec{n})|^4 \right\}$$

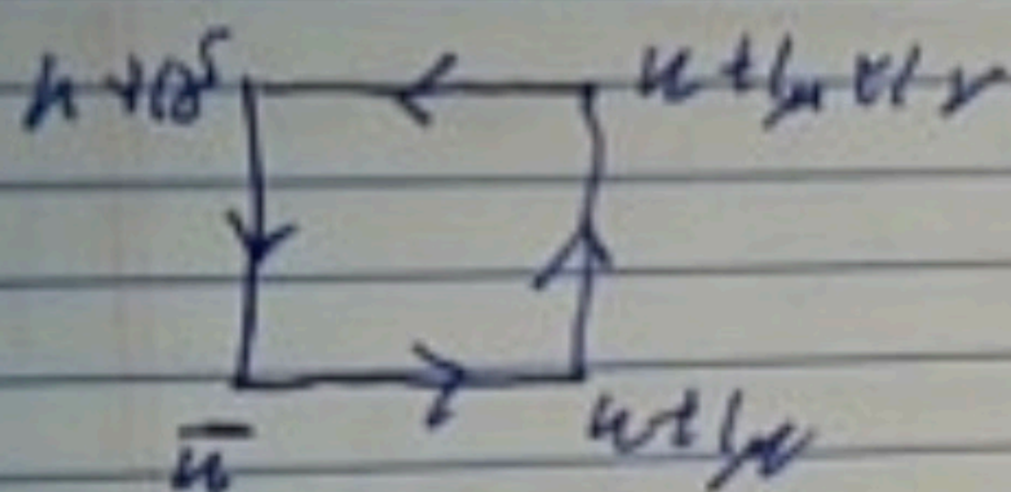
E5/3

Gauge field $A_\mu(x)$ is sampled via link variables $w_\mu(\vec{n})$.

$|w_\mu(\vec{n})| = 1$.

Gauge $w_\mu(\vec{n}) \to e^{i\theta(n+1_\mu)} w_\mu(n) e^{-i\theta(\vec{n})}$

Action: Take a unit plaquette



Define $w_{\mu\nu}(\vec{n})$

$= w_\nu^{-1}(n)\, w_\mu^{-1}(n+1_\nu)\, w_\nu(n+1_\mu)\, w_\mu(n)$.

$w_{\mu\nu}$: gauge invariant.

$w_{\nu\mu} = w_{\mu\nu}^{-1}$

in terms of continuous $A_\mu(x)$.

$$w_{\mu\nu}(\vec{n}) = \prod_{4\text{ links}} \exp\left(-i\int_{\text{link}} A_\alpha dx^\alpha\right)$$

$$= \exp\left(-i\oint_\square A_\alpha dx^\alpha\right)$$

$$= \exp\left(-i\iint_\square F_{\alpha\beta}\, dx^\alpha dx^\beta\right)$$

$a \to 0$ limit $\iint = a^2 F_{\mu\nu}(\vec{n}) + O(a^3)$.

$$w_{\mu\nu}(\vec{n}) \simeq \exp\left(-ia^2 F_{\mu\nu}(\vec{n})\right).$$

E5(4)

$$2 - w_{\mu\nu} - w_{\mu\nu}^{-1} \approx 2 - 2\cos(a^2 F_{\mu\nu})$$
$$= +a^4 (F_{\mu\nu})^2 \quad (\text{no } \sum_{\mu\nu})$$

Lattice action

$$S = \frac{1}{4e^2} \sum_{\vec{n}} \sum_{\mu \neq \nu} \approx \frac{a^4}{4e^2} \sum_{n} \sum_{\mu\neq\nu} (F_{\mu\nu})^2$$

$$\longrightarrow \frac{1}{4e^2} \int d^4x \, F_{\mu\nu} F_{\mu\nu} = S_E(A_\mu).$$

$$\boxed{S_E^{\text{lattice}} = \frac{1}{4e^2} \sum_{n} \sum_{\mu\neq\nu} (2 - w_{\mu\nu} - w_{\mu\nu}^{-1}).}$$

↳ Eucl EM action in terms of the link variables $w_\mu(\vec{n})$

Fund. integral.

$$\int \mathcal{D}[\phi(x)] \int \mathcal{D}[\phi^*(x)] \, e^{-S_E}$$

$\rightarrow$ cont limit of $(\text{coeff}) \times \prod_{\vec{n}} (d\phi(\vec{n}) d\phi^*(\vec{n})) \; e^{-S_E^{\text{latt.}}}$

lattice variables: complex $\phi(\vec{n})$ at each site.

For A_μ, lattice variables are the $w_\mu(\vec{n})$ constrained to $|w_\mu(n)| = 1$

... transform: $w_\mu(\vec{n}) \rightarrow e^{i\theta(\ldots)} w_\mu(n) e^{-i\theta(\ldots)}$

E5|5 Lattice funct. integral

$\oint \mathcal{D}[A_\mu(x)] \to$ cont. limit of

$$\prod_{\substack{\text{links}\\ n,\mu}} \oint d\,\text{measure}(w_\mu(n))$$

each $w_\mu(n)$ spans a unit circle in $\mathbb{C}$

measure $d(w_\mu(n))$ must be gauge inv.

$$d\,\text{measure}(w_\mu(n)) = \frac{1}{2\pi}\, d\,\text{phase}(w_\mu(n)).$$

Gauge: $d\,\text{phase}(w_\mu(n))$

$$\to d\left(\text{phase}(w_\mu(n)) + \theta(n+\hat{\mu}) - \theta(n)\right).$$

$$= d\,\text{phase}(w_\mu(n)) : \text{gauge inv.}$$

$\frac{1}{2\pi}\oint$ factor makes gauge redundancy harmless.

$$\prod_{\substack{\text{all}\\ \text{links}}} \oint \frac{d\,\text{phase}(w_\mu(n))}{2\pi} = \oint \prod_{\substack{\text{loop}\\ \text{links}}} \oint \frac{1}{2\pi} d\,\text{phase}(w_\mu(n))$$

$$\times \underbrace{\prod_{\substack{\text{gauge}\\ \text{redundant}\\ \text{links}}} \oint \frac{1}{2\pi} d\,\text{phase}(w_\mu(n))}_{\text{is harmless } 1.}$$

$\int d\,\text{phase} = 1$

E5/6

On the lattice, we don't need to fix a gauge!

Non Abelian gauge Theories.

Let $G = SU(N)$, $\phi^i \in$ fundamental.

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{pmatrix}. \qquad V(\phi^\dagger\phi) = m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2$$

↳ $SU(N)$-invariant.

Under local $SU(N)$.

$$\phi'^i(x) = U^i{}_j(x)\phi^j(x)$$

$U^i{}_j(x)$ is an $SU(N)$ matrix

indep. matrix $\forall x$.

Lattice: @ each lattice site

$$\phi(\vec{n}) = \begin{pmatrix} \phi^1(\vec{n}) \\ \vdots \\ \phi^N(\vec{n}) \end{pmatrix}$$

Gauge tr. $\phi'(\vec{n}) = U(\vec{n})\,\phi(\vec{n})$

indep. $SU(N)$ matrix $U(\vec{n})$

@ each $\vec{n}$.

① Covariant derivative

$$D_\mu \phi^i(x) = \partial_\mu \phi^i(x) + i A_{\mu}{}^i{}_j(x)\,\phi^j(x).$$

A_μ: matrix-valued gauge field.

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$$A'_\mu(x) = U(x) A_\mu(x) U^{-1}(x) + i(\partial_\mu U(x)) U^{-1}(x).$$

Take a path $1 \rightsquigarrow 2$

define $W(\text{path}) = P\text{-}\exp\left(-i \int_{\text{path}} A_\mu dx^\mu\right)$.

$\downarrow$ path-ordered exp.

path $x^\mu(t)$ t: real

$$x(0) = x_1, \quad x(1) = x_2.$$

$$W = 1 + \sum_{k=1}^{\infty} (-i)^k \iiint_{0<t_1<t_2<\cdots<t_k<1} dt_1 \cdots dt_k \, A(t_k) A(t_{k-1}) \cdots A(t_1).$$

Theorem 1: For any hermitian, traceless $A_\mu(x)$
$W(\text{path})$ is an $SU(N)$ matrix

Theorem 2: under gauge transforms

$$W(\text{path } 1 \to 2) \to U(2)\, W(\text{path})\, U^{-1}(1).$$

$\hookrightarrow$ $W(1\to 2)\Phi(1)$ transforms like $\Phi(2)$.

E518

On the lattice, we sample $A_\mu(x)$ via $W_\mu(\vec n) = W$(path from $\vec n$ to $\vec n + 1_\mu$).
all $W_\mu(\vec n)$ are $SU(N)$ matrices.

covariant $\mathcal{D}_\mu \Phi(x) \to$

$$\frac{1}{a}\left[\Phi(\vec n + 1_\mu) - W_\mu(\vec n)\,\Phi(\vec n)\right]$$

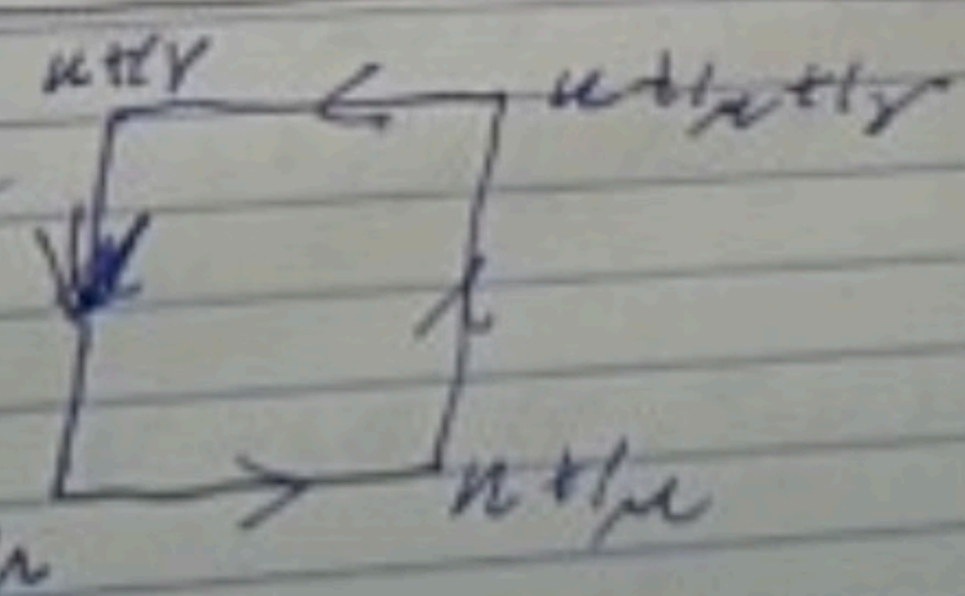
$$S^{\text{lattice}}[\Phi] = a^4 \sum_n \left\{ V(\Phi^\dagger(n)\Phi(n)) + \frac{1}{a^2}\sum_\mu \left(\Phi^\dagger_{n+1_\mu} - \Phi^\dagger_n W_\mu^\dagger(n)\right)\left(\Phi(n+1_\mu) - W_\mu(n)\Phi(n)\right)\right\}$$

gauge invariant.

Lattice action for gauge fields

$\rightarrow$ want $\frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$ in terms of $W_\mu(\vec n)$.

∇ plaquette



$$W_{\mu\nu}(\vec n) = W_\nu^{-1}(\vec n)\, W_\mu^{-1}(n+1_\nu)\, W_\nu(n+1_\mu)\, W_\mu(\vec n)$$

$\;\;\|$

$W(\square)$ $\quad$ <u>in this order.</u>

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Gauge transform

$$W_{\mu\nu}(\vec{n}) \to U(\vec{n})\, W_{\mu\nu}(\vec{n})\, U^{-1}(\vec{n}).$$

$\mathrm{tr}(W_{\mu\nu}(\vec{n}))$ is gauge invariant.

Cont. limit:

$$W_{\mu\nu}(\vec{n}) = P\text{-}\exp\left(-i\oint A_\alpha dx^\alpha\right).$$

$$= 1 - i a^2 F_{\mu\nu}(\vec{n}) + O(a^3).$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu].$$

$$W_\mu(\vec{n}) \simeq 1 - i a A_\mu(\vec{n})$$

$$W_{\mu\nu}(\vec{n}) = (1 + i a A_\mu(\vec{n}))(1 + i a A_\nu(\vec{n}+\hat{\mu}))$$
$$(1 - i a A_\mu(\vec{n}+\hat{\nu}))(1 - i a A_\nu(\vec{n})).$$
$$= 1 - i a^2 \partial_\mu A_\nu + i a^2 \partial_\nu A_\mu$$
$$- a^2 [A_\mu, A_\nu] + O(a^3).$$

$$\mathrm{tr}(W_{\mu\nu}) = N - i a^2 \underbrace{\mathrm{tr}(F_{\mu\nu})}_{0}$$
$$- \frac{a^4}{2}\mathrm{tr}((F_{\mu\nu})^2).$$

$$\mathrm{tr}(W_{\mu\nu}) - \mathrm{tr}(W_{\mu\nu}^{-1})$$
$$= a^4\, \mathrm{tr}((F_{\mu\nu})^2).$$

E5(10)

$$S^{\text{lattice}}(\{U_\mu\}) = \frac{1}{2g^2}\sum_n \sum_{\mu\neq\nu} \left(N - \mathrm{tr}(U_{\mu\nu} + U_{\mu\nu}^{-1})\right)$$

$$\xrightarrow{\text{cont. limit}} \frac{1}{2g^2}\int d^4x_E\, \mathrm{tr}(F_{\mu\nu}F_{\mu\nu}).$$

S^{lattice} : gauge invariant.

Functional integral

$\oint \mathcal{D}[A_\mu(x)]\, e^{-S_E}$ is cont. limit of

$$\prod_{\text{links}} \oint_G d\,\text{measure}(U_\mu(n))\; e^{-S_E^{\text{lattice}}}$$

∀ link: $\oint_G$ of $U_\mu(n)$ spans the group manifold of SU(N).

d measure $(U_\mu(n))$ must be invariant wrt

$$U \to U' = U(n+\hat\mu)\, U\, U^{-1}(n).$$

Fortunately, ∀ compact simple group ∃ unique invariant measure.

$d(U) = d(U_1 U U_2^{-1})$

for any fixed U_1, U_2.

E5/11 Ferex, $SU(2)$

group manifold is a 3-sphere S^3

$U = \alpha + i\vec{\beta}\cdot\vec{\sigma}$ ↳ Pauli

$$\alpha^2 + \vec{\beta}^2 = 1$$

$\rightarrow (\alpha, \vec{\beta})$ spans S^3

inv. measure is the 3volume measure on S^3.