Problem **1**(a):

Similar to the massless vector field discussed in class, the derivatives $\partial_\mu A_\nu$ enter the Lagrangian density only via $F_{\mu\nu}$, so just like for the EM field, $\partial\mathcal{L}/\partial(\partial_\mu A_\nu) = -F^{\mu\nu}$. At the same time, for the massive field $\partial\mathcal{L}/\partial(A_\nu) = +m^2 A^\nu - J^\nu$. Thus, the Euler–Lagrange field equation for the massive vector field is

$$-\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu \mathcal{A}_\nu)} + \frac{\partial\mathcal{L}}{\partial(\mathcal{A}_\nu)} \equiv \partial_\mu F^{\mu\nu} + m^2 A^\nu - J^\nu = 0. \tag{S.1}$$

In terms of the A^ν and their explicit derivatives,

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \partial^2 A^\mu - \partial^\nu(\partial_\mu A^\mu), \tag{S.2}$$

so the Euler–Lagrange field equation becomes

$$\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) + m^2 A^\nu - J^\nu = 0. \tag{S.3}$$

Problem **1**(b):

Take the divergence ∂_ν of the field equation (S.3); the first two terms cancel out while the rest becomes

$$m^2\, \partial_\nu A^\nu - \partial_\nu J^\nu = 0. \tag{S.4}$$

In the massless case, this equation enforces the current conservation $\partial_\nu J^\nu = 0$ regardless of the 4–vector potential $A^\nu(x)$, but there is no such constraint in the massive case at hand. Instead, eq. (S.4) simply relates the current divergence to the 4–potential divergence,

$$\partial_\nu A^\nu = \frac{1}{m^2}\, \partial_\nu J^\nu. \tag{S.5}$$

Consequently, eq. (S.3) for the massive vector field becomes

$$(\partial^2 + m^2) A^\nu = J^\nu + \frac{1}{m^2}\, \partial^\nu(\partial_\mu J^\mu). \tag{S.6}$$

In particular, *when the current happens to be conserved,* $\partial_\nu J^\nu = 0$, *then — and only then —*

eqs. (S.5) *and* (S.6) *become*

$$\partial_\nu A^\nu \;=\; 0 \quad\text{and}\quad (\partial^2 + m^2)A^\nu \;=\; J^\nu. \tag{2}$$

Quod erat demonstrandum.

Problem **2**(a):

For the complex scalar field $\Phi(x)$ with Lagrangian density (3), we have

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} &= \partial^\mu\Phi^*, \\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)} &= \partial^\mu\Phi, \\ \frac{\partial\mathcal{L}}{\partial\Phi} &= -\left(m^2 + \frac{\lambda}{2}\Phi^*\Phi\right)\Phi^*, \\ \frac{\partial\mathcal{L}}{\partial\Phi^*} &= -\left(m^2 + \frac{\lambda}{2}\Phi^*\Phi\right)\Phi. \end{aligned} \tag{S.7}$$

Consequently, the Euler–Lagrange equations (4) and (5) become respectively

$$\partial_\mu\partial^\mu\Phi^* \;+\; \left(m^2 + \frac{\lambda}{2}\Phi^*\Phi\right)\Phi^* \;=\; 0 \tag{S.8}$$

and

$$\partial_\mu\partial^\mu\Phi \;+\; \left(m^2 + \frac{\lambda}{2}\Phi^*\Phi\right)\Phi \;=\; 0. \tag{S.9}$$

Note that the two equations are complex conjugates of each other.

Separating linear from non-linear parts of the field equations (S.8) and (S.9), we may rewrite them as

$$(\partial^2 + m^2)\Phi \;=\; -\frac{\lambda}{2}\Phi^*\Phi^2, \qquad (\partial^2 + m^2)\Phi^* \;=\; -\frac{\lambda}{2}\Phi^{*2}\Phi, \tag{S.10}$$

then from the linear left-hand sides of these equations we see that both Φ and Φ^* fields have mass$^2 = m^2$. Hence, once we quantize the fields, their quanta would be scalar particles of mass $m +$ quantum corrections.

Problem **2**(b):

By definition (6) of the current J^μ, its divergence is

$$\begin{aligned}\partial_\mu J^\mu \;&=\; -i\partial_\mu(\Phi^*\partial^\mu\Phi) \;+\; i\partial_\mu(\Phi\partial^\mu\Phi^*)\\ &=\; -i(\partial_\mu\Phi^*)\times(\partial^\mu\Phi) \;-\; i\Phi^*\times(\partial^2\Phi) \;+\; i(\partial_\mu\Phi)\times(\partial^\mu\Phi^*) \;+\; i\Phi\times(\partial^2\Phi^*)\\ &=\; -i\Phi^*\times(\partial^2\Phi) \;+\; i\Phi\times(\partial^2\Phi^*).\end{aligned} \tag{S.11}$$

Now let's make use of the field equations (S.8) and (S.9). For compactness sake, let's rewrite those equations as

$$\partial^2\Phi \;=\; -T\times\Phi \quad\text{and}\quad \partial^2\Phi^* \;=\; -T\times\Phi^* \tag{S.12}$$

for the same real

$$T \;=\; m^2 \;+\; \frac{\lambda}{2}\,\Phi^*\Phi. \tag{S.13}$$

Plugging eqs. (S.12) into the bottom line of eq. (S.11), we arrive at

$$\partial_\mu J^\mu \;=\; +i\Phi^*\times T\times\Phi \;-\; i\Phi\times T\times\Phi^* \;=\; 0. \tag{S.14}$$

Thus, *when the fields Φ and Φ^* obey their equations of motion, the current J^μ is conserved, quod erat demonstrandum.*

Problem **3**(a):

The definition (7) is manifestly symmetric with respect to cyclic permutations of indices λ,μ,ν, thus

$$H_{\lambda\mu\nu} \;=\; H_{\mu\nu\lambda} \;=\; H_{\nu\lambda\mu}\,. \tag{S.15}$$

Hence, to prove the total antisymmetry of the $H_{\lambda\mu\nu}$ tensor it is enough to show that $H_{\lambda\nu\mu} = -H_{\lambda\mu\nu}$ — antisymmetry with respect to other index pairs then follows by the cyclic symmetry (S.15). And indeed, antisymmetry of the B tensor leads to

$$\begin{aligned}H_{\lambda\nu\mu} \;&=\; \partial_\lambda B_{\nu\mu} \;+\; \partial_\nu B_{\mu\lambda} \;+\; \partial_\mu B_{\lambda\nu}\\ &=\; -\partial_\lambda B_{\mu\nu} \;-\; \partial_\nu B_{\lambda\mu} \;-\; \partial_\mu B_{\nu\lambda}\\ &=\; -H_{\lambda\mu\nu}\end{aligned} \tag{S.16}$$

and hence total antisymmetry of the H tensor. *Quod erat demonstrandum.*

Alternative proof:

Let us redefine the H tensor as

$$H_{\lambda\mu\nu} = \tfrac{1}{2}\partial_{[\lambda}B_{\mu\nu]} \tag{S.17}$$

where $[\lambda\mu\nu]$ imply total antisymmetrization with respect to the λ, μ, ν, *i.e.* summing over all possible permutations of those indices with appropriate signs. Obviously, this new definition makes $H_{\lambda\mu\nu}$ a totally antisymmetric tensor.

To see that the new definition (S.17) is equivalent to the old definition (7) we use the fact that $B_{\mu\nu}$ is itself antisymmetric. Consequently, there is no need to antisymmetrize $\partial_{[\lambda}B_{\mu\nu]}$ with respect to the two indices belonging to the B tensor, thus

$$\begin{aligned} H_{\lambda\mu\nu} &= \tfrac{1}{2}(\partial_\lambda B_{\mu\nu} - \partial_\lambda B_{\nu\mu}) + \tfrac{1}{2}(\partial_\mu B_{\nu\lambda} - \partial_\mu B_{\lambda\nu}) + \tfrac{1}{2}(\partial_\nu B_{\lambda\mu} - \partial_\nu B_{\mu\lambda}) \\ &= \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}\,, \end{aligned} \tag{S.18}$$

exactly as in eq. (7).

Problem **3**(b):

Thanks to eq. (S.17),

$$\partial_{[\kappa}H_{\lambda\mu\nu]} = \tfrac{1}{2}\partial_{[\kappa}\partial_\lambda B_{\mu\nu]}\,, \tag{S.19}$$

where all the indices are totally antisymmetrized. In particular, the indices of the two derivatives are antisymmetrized, which yields $\partial_{[\kappa}\partial_{\lambda]} = [\partial_\kappa, \partial_\lambda] = 0$ since the spacetime derivatives commute with each other. Consequently,

$$\partial_{[\kappa}H_{\lambda\mu\nu]} = 0. \tag{S.20}$$

Mathematically, this is the differential identity for the tensor $H_{\lambda\mu\nu}(x)$ defined according to eq. (S.17), or equivalently according to eq. (7).

Note that $\partial_{[\kappa}H_{\lambda\mu\nu]}$ stand for the signed sum of $4! = 24$ terms according to permutations of the indices $\kappa, \lambda, \mu, \nu$. Fortunately, total antisymmetry of the H tensor means that there is a 6-fold redundancy and only 4 of those 24 terms are different, thus

$$\tfrac{1}{6}\,\partial_{[\kappa}H_{\lambda\mu\nu]} = \partial_\kappa H_{\lambda\mu\nu} - \partial_\lambda H_{\mu\nu\kappa} + \partial_\mu H_{\nu\kappa\lambda} - \partial_\nu H_{\kappa\lambda\mu}\,. \tag{S.21}$$

Consequently, the differential identity (S.20) for the $H_{\lambda\mu\nu}(x)$ field may be written as (8).

Problem **3**(c):

Given the Lagrangian (9) as a function of $B_{\mu\nu}$ fields and their derivatives, we have

$$\frac{\partial\mathcal{L}(B,\partial B)}{\partial B_{\mu\nu}} \;=\; 0 \tag{S.22}$$

while

$$\begin{aligned} \frac{\partial\mathcal{L}(B,\partial B)}{\partial(\partial_\lambda B_{\mu\nu})} &= \tfrac{1}{6}\, H^{\alpha\beta\gamma} \times \frac{\partial H_{\alpha\beta\gamma}}{\partial(\partial_\lambda B_{\mu\nu})} \\ &= \tfrac{1}{6}\, H^{\alpha\beta\gamma} \times \tfrac{1}{2}\delta_\alpha^{[\lambda}\delta_\beta^{\mu}\delta_\gamma^{\nu]} \\ &= \tfrac{1}{2} H^{\lambda\mu\nu}. \end{aligned} \tag{S.23}$$

Consequently, the Euler–Lagrange field equations

$$\partial_\lambda\left(\frac{\partial\mathcal{L}(B,\partial B)}{\partial(\partial_\lambda B_{\mu\nu})}\right) \;-\; \frac{\partial\mathcal{L}(B,\partial B)}{\partial B_{\mu\nu}} \;=\; 0 \tag{S.24}$$

for the B fields become

$$\partial_\lambda H^{\lambda\mu\nu} \;=\; 0. \tag{S.25}$$

Problem **3**(d):

Rewriting eq. (10) as

$$B'_{\mu\nu}(x) \;=\; B_{\mu\nu}(x) \;+\; \partial_{[\mu}\Lambda_{\nu]}(x), \tag{S.26}$$

we have

$$\begin{aligned} H'_{\lambda\mu\nu}(x) &= \tfrac{1}{2}\partial_{[\lambda}B'_{\mu\nu]}(x) \\ &= \tfrac{1}{2}\partial_{[\lambda}B_{\mu\nu]}(x) \;+\; \tfrac{1}{2}\partial_{[\lambda}\partial_\mu\Lambda_{\nu]}(x) \\ &= H_{\lambda\mu\nu}(x) \;+\; 0, \end{aligned} \tag{S.27}$$

where the last equality follows from $\partial_{[\lambda}\partial_{\mu]} = 0$. Thus, the tension tensor $H_{\lambda\mu\nu}(x)$ — and hence the Lagrangian (9) — is invariant under the gauge transforms (10).

Problem **3**(e):

Proceeding similarly to part (b), we have

$$\partial_{[\lambda}G_{\mu_1\mu_2\cdots\mu_{p+1}]}(x) \;=\; \frac{1}{p!}\,\partial_{[\lambda}\partial_{\mu_1}C_{\mu_2\cdots\mu_{p+1}]}(x) \;=\; 0 \tag{S.28}$$

because $\partial_{[\lambda}\partial_{\mu_1]} = 0$. Thus regardless of any equations obeyed or not obeyed by the $C(x)$ potentials, their very existence implies the differential identity

$$\partial_{[\lambda}G_{\mu_1\mu_2\cdots\mu_{p+1}]}(x) \;=\; 0 \tag{S.29}$$

for the tension fields $G(x)$.

As to the equations of motion, the Lagrangian (12) has derivatives

$$\begin{aligned}
\frac{\partial\mathcal{L}(C,\partial C)}{\partial C_{\mu_1\cdots\mu_p}} \;&=\; 0,\\
\frac{\partial\mathcal{L}(C,\partial C)}{\partial(\partial_\lambda C_{\mu_1\cdots\mu_p})} \;&=\; \frac{(-1)^p}{(p+1)!}\,G^{\alpha_1\cdots\alpha_{p+1}}\times\frac{\partial G_{\alpha_1\cdots\alpha_{p+1}}}{\partial(\partial_\lambda C_{\mu_1\cdots\mu_p})}\\
&=\; \frac{(-1)^p}{(p+1)!}\,G^{\alpha_1\cdots\alpha_{p+1}}\times\frac{1}{p!}\,\delta^{[\lambda}_{\alpha_1}\delta^{\mu_1}_{\alpha_2}\delta^{\mu_2}_{\alpha_3}\cdots\delta^{\mu_p]}_{\alpha_{p+1}}\\
&=\; \frac{(-1)^p}{p!}\,G^{\lambda\mu_1\cdots\mu_p}.
\end{aligned} \tag{S.30}$$

Hence, the Euler–Lagrange field equations

$$\partial_\lambda\left(\frac{\partial\mathcal{L}(C,\partial C)}{\partial(\partial_\lambda C_{\mu_1\cdots\mu_p})}\right) \;-\; \frac{\partial\mathcal{L}(C,\partial C)}{\partial C_{\mu_1\cdots\mu_p}} \;=\; 0 \tag{S.31}$$

for the $C_{\mu_1\cdots\mu_p}(x)$ fields become (up to an overall coefficient)

$$\partial_\lambda G^{\lambda\mu_1\cdots\mu_p}(x) \;=\; 0. \tag{S.32}$$

Problem **3**(f):

Under gauge transformations, the C tensor potential changes by

$$\Delta C_{\mu_1\cdots\mu_p}(x) \;=\; \frac{1}{(p-1)!}\,\partial_{[\mu_1}\Lambda_{\mu_2\cdots\mu_p]}(x) \tag{S.33}$$

for some arbitrary $(p-1)$–index antisymmetric tensor $\Lambda_{[\mu_2\cdot\mu_p]}(x)$. Hence the G tension tensor changes by

$$\Delta G_{\mu_1\mu_2\cdots\mu_{p+1}}(x) \;=\; \frac{1}{p!}\,\partial_{[\mu_1}\Delta C_{\mu_2\cdots\mu_{p+1}]}(x) \;=\; \frac{1}{(p-1)!}\,\partial_{[\mu_1}\partial_{\mu_2}\Lambda_{\mu_3\cdots\mu_{p+1}]}(x), \tag{S.34}$$

which vanishes because $\partial_{[\mu_1}\partial_{\mu_2]} = 0$. Thus, the tension tensor G is gauge invariant, and hence the Lagrangian (12) is also gauge invariant. *Quod erat demonstrandum.*

Problem **4**(a):

Let's start with Maxwell equations in 4D notations: the homogeneous equation(s)

$$\partial_\lambda F_{\mu\nu} \;+\; \partial_\mu F_{\nu\lambda} \;+\; \partial_\nu F_{\lambda\mu} \;=\; 0. \tag{S.35}$$

— or equivalently

$$\partial_{[\lambda}F_{\mu\nu]} \;=\; 0, \tag{S.36}$$

or even

$$\epsilon^{\kappa\lambda\mu\nu}\partial_\lambda F_{\mu\nu} \;=\; 0, \tag{S.37}$$

— and the inhomogeneous equation(s)

$$\partial_\mu F^{\mu\nu} \;=\; J^\nu \;\longrightarrow\; 0 \quad \text{in the absence of electric currents.} \tag{S.38}$$

In terms of the dual EM tensor (14), the homogeneous equation (S.37) for the original

EM tensor $F^{\mu\nu}(x)$ becomes

$$0 \;=\; \partial_\lambda\bigl(\epsilon^{\kappa\lambda\mu\nu}F_{\mu\nu}\bigr) \;=\; \partial_\lambda\bigl(2\tilde{F}^{\kappa\lambda}\bigr) \;=\; -2\partial_\lambda\tilde{F}^{\lambda\kappa}, \tag{S.39}$$

or equivalently

$$\partial_\lambda\tilde{F}^{\lambda\kappa} \;=\; 0, \tag{S.40}$$

which is the *in*-homogeneous Maxwell equation for the dual tensor $\tilde{F}^{\lambda\kappa}$ (in the absence of a dual current $\tilde{J}^\kappa$).

On the other hand, reversing eq. (14) we get $F^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\tilde{F}_{\kappa\lambda}$. Plugging this relation into the in-homogeneous Maxwell equation (S.38) for the original EM tensor $F^{\mu\nu}(x)$, we get

$$0 \;=\; \partial_\mu F^{\mu\nu} \;=\; -\tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\partial_\mu\tilde{F}_{\kappa\lambda}, \tag{S.41}$$

or equivalently

$$\epsilon^{\nu\mu\kappa\lambda}\partial_\mu\tilde{F}_{\kappa\lambda} \;=\; 0, \tag{S.42}$$

which is the homogeneous Maxwell equation for the dual tensor $\tilde{F}^{\kappa\lambda}$.

Altogether, the complete set of Maxwell equations for the original EM tensor $F^{\mu\nu}(x)$ (in the absence of any electric currents) is equivalent to the complete set of Maxwell equations for the dual EM tensor $\tilde{F}^{\kappa\lambda}$ (also in the absence of dual currents $\tilde{J}^\kappa$). *Quod erat demonstrandum.*

PS: When the electric charges and/or currents do not vanish, $J^\mu(x) \not\equiv 0$, the electric-magnetic duality (14) turns them into the magnetic charges and currents $\tilde{J}^\kappa(x)$ — *i.e.*, the charge/current densities of magnetic monopoles. In presence of such magnetic monopoles, the homogeneous Maxwell equation (S.38) stops being homogeneous; instead, it becomes

$$\tfrac{1}{2}\epsilon^{\kappa\lambda\mu\nu}\partial_\lambda F_{\mu\nu} \;=\; -\tilde{J}^\kappa, \tag{S.43}$$

or equivalently

$$\partial_\lambda\tilde{F}^{\lambda\kappa} \;=\; \tilde{J}^\kappa, \tag{S.44}$$

while the in-homogeneous Maxwell eq. (S.38) remains the same,

$$\partial_\mu F^{\mu\nu} \;=\; J^\nu \tag{S.45}$$

or equivalently

$$\tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\partial_\nu \tilde{F}_{\kappa\lambda} \;=\; J^\mu . \tag{S.46}$$

Note that the no-longer-homogeneous eq. (S.43) disagrees with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ for any 4-vector potential $A^\mu(x)$. Consequently, the EM fields which couple to both electric and magnetic charges must be reformulated in a way which does not involve the potentials A^μ. Such reformulation is possible but it goes way beyond the scope of this class.

Problem **4**(b):

As we saw in problem **3**, the antisymmetric tension tensor $G^{\mu_1,\cdots\mu_{p+1}}(x)$ obeys the Euler–Lagrange equation

$$\partial_{\mu_1} G^{\mu_1,\mu_2\cdots,\mu_{p+1}} \;=\; 0 \tag{S.47}$$

and the differential identity

$$\partial_{[\mu} G_{\nu_1,\ldots,\nu_{p+1}]} \;=\; 0. \tag{S.48}$$

The LHS of this differential identity is a $(p+2)$ index totally antisymmetric tensor, so contracting it with the Levi-Civita tensor $\epsilon_{\nu_1,\ldots,\nu_D}$ turns it into a $(D-p-2=q)$ index antisymmetric tensor

$$\frac{1}{(p+1)!}\,\epsilon^{\lambda_1,\ldots,\lambda_q,\mu,\nu_1,\ldots,\nu_{p+1}}\partial_\mu G_{\nu_1,\ldots,\nu_{p+1}}\,,$$

so the differential identity (S.48) becomes

$$\begin{aligned} 0 \;&=\; \frac{1}{(p+1)!}\,\epsilon^{\lambda_1,\ldots,\lambda_q,\mu,\nu_1,\ldots,\nu_{p+1}}\partial_\mu E_{\nu_1,\ldots,\nu_{p+1}} \\ &=\; \partial_\mu\left(\frac{1}{(p+1)!}\,\epsilon^{\lambda_1,\ldots,\lambda_q,\mu,\nu_1,\ldots,\nu_{p+1}} E_{\nu_1,\ldots,\nu_{p+1}} \;=\; \tilde{G}^{\lambda_1,\ldots,\lambda_q,\mu}\right) \\ &=\; (-1)^q \partial_\mu \tilde{G}^{\mu,\lambda_1,\ldots,\lambda_q}. \end{aligned} \tag{S.49}$$

In other words, the differential identity for the original tension tensor G is equivalent to the Euler–Lagrange equation of motion for the dual tensor $\tilde{G}$.

In exactly the same way, the Euler–Lagrange equation of motion for the original tensor G is equivalent to the differential identity for the dual tensor $\tilde{G}$:

$$\begin{aligned} 0 &= \partial_\lambda G^{\lambda,\mu_1,\ldots,\mu_p} \\ &= \partial_\lambda \left(\frac{\pm 1}{(p+1)!} \, \epsilon^{\lambda,\mu_1,\ldots,\mu_p,\nu_1,\ldots,\nu_{q+1}} \tilde{G}_{\nu_1,\ldots,\nu_{q+1}} \right) \\ &= \frac{\pm' 1}{(p+1)!} \epsilon^{\mu_1,\ldots,\mu_p,\lambda,\nu_1,\ldots,\nu_{q+1}} \, \partial_\lambda \tilde{G}_{\nu_1,\ldots,\nu_{q+1}} \,, \end{aligned} \tag{S.50}$$

or equivalently

$$\partial_{[\lambda} \tilde{G}_{\nu_1,\ldots,\nu_{q+1}]} \;=\; 0. \tag{S.51}$$

Problem **4**(c):

As we saw in problem **3**, a 2-index tensor potential $B^{\mu\nu}$ has a 3-index antisymmetric tension tensor $H^{\lambda\mu\nu}$. In 4 Minkowski dimensions, such a 3-index antisymmetric tensor is dual to a 1-index tensor *i.e.* a vector

$$v^\kappa(x) \;=\; \tfrac{1}{6} \epsilon^{\kappa\lambda\mu\nu} H_{\lambda\mu\nu}(x) \quad \Leftrightarrow \quad H^{\lambda\mu\nu}(x) \;=\; \epsilon^{\lambda\mu\nu\kappa} v_\kappa(x). \tag{S.52}$$

The H tensor obeys the differential identity

$$\tfrac{1}{6} \epsilon_{\kappa\lambda\mu\nu} \partial^\kappa H^{\lambda\mu\nu} \;=\; 0 \tag{S.53}$$

and the Euler–Lagrange equation of motion

$$\partial_\lambda H^{\lambda\mu\nu} \;=\; 0. \tag{S.54}$$

In terms of the v^κ vector, eq. (S.53) becomes

$$\partial_\kappa v^\kappa \;=\; 0 \tag{S.55}$$

while eq. (S.54) becomes

$$\epsilon^{\lambda\mu\nu\kappa} \partial_\lambda v_\kappa \;=\; 0, \tag{S.56}$$

or equivalently

$$\partial_\lambda v_\kappa \;-\; \partial_\kappa v_\lambda \;=\; 0. \tag{S.57}$$

Mathematically, eq. (S.57) is the integrability condition for the vector field $v_\kappa(x)$ being a gradient of some scalar field $\phi(x)$. That is, eq. (S.57) is automatically true for $v_\kappa(x) = \partial_\kappa\phi(x)$ for any scalar field $\phi(x)$, and conversely, if eq. (S.57) holds true at all x, then $v_\kappa(x) = \partial_\kappa\phi(x)$ for some scalar field $\phi(x)$.

Now let's treat $\phi(x)$ as an independent scalar potential field whose tension is the vector field $v_\kappa(x)$. In terms of the $\phi(x)$, eq. (S.57) is automatic while eq. (S.55) becomes

$$\partial^2\phi \;=\; 0. \tag{S.58}$$

Physically, this is the equation of motion of a free massless scalar field with the Lagrangian density

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\kappa\phi)(\partial^\kappa\phi). \tag{S.59}$$

And that's why we say that in 4D, a free two-index antisymmetric tensor field $B^{\mu\nu}(x)$ is dual to a free massless scalar field $\phi(x)$.

Mathematical Supplement to Problems 3 and 4
Differential Forms.

Mathematics of various antisymmetric tensor fields becomes much simpler in the language of differential forms. Students interested in string theory should master this language and then go ahead and learn as much differential geometry and topology as they can; take a class on the subject or at least read a book. Wikipedia has a quick and dirty introduction to differential forms at http://en.wikipedia.org/wiki/Differential_form and related web pages.

Consider a space or spacetime of dimension D; it can be Euclidean or Minkowski, flat or curved; it might even be a smooth topological manifold without any metric at all. A differential form of rank $p \le D$ in such a space combines a tensor with p indices and a differential suitable for integration over a sub-manifold of dimension p (a line for $p = 1$, a surface for $p = 2$, *etc.*, *etc.*). For example,

$$A = A_\mu(x)\, dx^\mu, \qquad B = B_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu, \qquad C = C_{\lambda\mu\nu}(x)\, dx^\lambda \wedge dx^\mu \wedge dx^\nu, \ldots. \tag{S.60}$$

For $p = 2$, a 2–form should be integrated over an oriented surface, so the order of dx^μ and dx^ν matters; in fact they anticommute so $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. Likewise, the volume differential $dx^\lambda \wedge dx^\mu \wedge dx^\nu$ is totally antisymmetric with respect to permutation of indices $\lambda\mu\nu$. Consequently, in eq. (S.60), the $B_{\mu\nu}(x)$ tensor is antisymmetric, $B_{\nu\mu} = -B_{\mu\nu}$, while the $C_{\lambda\mu\nu}(x)$ tensor is totally antisymmetric in all of its 3 indices. And a general form of rank p

$$E = E_{\mu_1\mu_2\cdots\mu_p}(x)\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \tag{S.61}$$

involves a p-index *totally antisymmetric tensor* $E_{\mu_1\mu_2\cdots\mu_p}(x)$.

The *exterior derivative* of a rank-p form E is a form dE of rank $p+1$ defined as

$$dE = \big(dE_{\mu_1\mu_2\cdots\mu_p}(x)\big) \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = \partial_\lambda E_{\mu_1\mu_2\cdots\mu_p}(x)\, dx^\lambda \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{S.62}$$

but this compact formula hides the antisymmetrization due to anticommutativity of the dx^μ differentials. In the antisymmetric tensor form, $J = dE$ means

$$\begin{aligned} J_{\mu_1\cdots\mu_{p+1}}(x) &= \frac{1}{p!}\,\partial_{[\mu_1} E_{\mu_2\cdots\mu_{p+1}]}(x) = \sum_{j=1}^{p+1} (-1)^{j-1} \partial_{\mu_j} E_{\mu_1\cdots\not{\mu}_j\cdots\mu_{p+1}}(x) \\ &= \partial_{\mu_1} E_{\mu_2\cdots\mu_{p+1}} - \partial_{\mu_2} E_{\mu_1\mu_3\cdots\mu_{p+1}} \pm \cdots + (-1)^p \partial_{\mu_{p+1}} E_{\mu_1\cdots\mu_p}\,. \end{aligned} \tag{S.63}$$

The exterior derivative generalizes the 3D notions of *gradient*, *curl*, and *divergence*. Indeed, a scalar $\phi(x)$ is a 0–form and its gradient $\nabla\phi$ is a vector defining a 1-form $(\nabla\phi)_i\, dx^i = d\phi$. Likewise, for a vector $\vec{A}(x)$ and its curl $\vec{B}(x) = \nabla \times \vec{A}(x)$ we have a 1-form $A = A_i(x) dx^i$ and a 2-form $B = B_{ij}(x) dx^i \wedge dx^j = dA$ where $B_{ij} = \partial_i A_j - \partial_j A_i$; note that in 3D this antisymmetric tensor is equivalent to an axial vector, $B_{ij} = \epsilon_{ijk} B_k$. Finally, for a vector $\vec{E}(x)$ and its divergence $f(x) = \nabla \cdot \vec{E}$ we have an exterior derivative relation $f = dE$ for the 2-form $E = E_i(x)\epsilon_{ijk} dx^j \wedge dx^k$ equivalent to the vector $E_i(x)$ and a 3-form $f = f(x)\epsilon_{ijk} dx^i \wedge dx^j \wedge dx^k$ equivalent to the scalar $f(x)$.

The most important property of the exterior derivative is its nilpotency: for any differential form E, $ddE = 0$. This rule corresponds to the differential identities for all kinds of antisymmetric tensors, in particular the vector calculus identities $\nabla \times (\nabla\phi) = 0$ and $\nabla \cdot (\nabla \times \vec{A}) = 0$. The proof is very simple: If E is a form of rank p, $J = dE$ is a form of rank $p+1$, and $K = dJ$ is a form of rank $p+2$, then applying eq. (S.63) twice, we have

$$\begin{aligned} K_{\lambda\mu\nu_1\cdots\nu_p}(x) &= \frac{1}{(p+1)!}\,\partial_{[\lambda}J_{\mu\nu_1\cdots\nu_p]}(x) \\ &= \frac{1}{(p+1)!\,p!}\,\partial_{[\lambda}\partial_{[\mu}E_{\nu_1\cdots\nu_p]]}(x) \\ &= \frac{1}{p!}\,\partial_{[\lambda}\partial_{\mu}E_{\nu_1\cdots\nu_p]}(x) \\ &= 0 \end{aligned} \tag{S.64}$$

where the last equality follow from $\partial_{[\lambda}\partial_{\mu]} = 0$.

The application of the differential form language to electromagnetic fields and to the antisymmetric tensor fields in this homework is completely straightforward. In electromagnetism we work in Minkowski spacetime and identify the 4–vector potential $A_\mu(x)$ with a 1–form $A = A_\mu(x)dx^\mu$ and the tension tensor $F_{\mu\nu}(x)$ with a 2–form $F = F_{\mu\nu}(x)dx^\mu dx^\nu$. Clearly, F is the exterior derivative of A:

$$F \;=\; dA \quad\Longleftrightarrow\quad F_{\mu\nu} \;=\; \partial_{[\mu}A_{\nu]} \;=\; \partial_\mu A_\nu \,-\, \partial_\nu A_\mu\,. \tag{S.65}$$

The differential identity $\partial_{[\lambda}F_{\mu\nu]} = 0$ is simply $dF = 0$, which follows from $F = dA$ and $dF = ddA = 0$ by nilpotency of d. The gauge transform of the potentials is $A' = A + d\Lambda$ (where Λ is a 0–form, *i.e.* a scalar field), and the gauge invariance of the tension fields is simply

$$F' \;=\; dA' \;=\; dA \;+\; dd\Lambda \;=\; F \;+\; 0 \tag{S.66}$$

because $dd\Lambda = 0$.

Similarly, for the tensor potential $B_{\mu\nu}(x)$ and the tension tensor $H_{\lambda\mu\nu}$ in parts (a) through (d) of problem 3, we have a 2–form $B = B_{\mu\nu}(x)dx^\mu dx^\nu$ and a 3–form $H = H_{\lambda\mu\nu}(x)dx^\lambda dx^\mu dx^\nu$. Clearly, eq. (7) for the tension tensor translates to $H = dB$, which immediately gives rise to the differential identity $dH = 0$ because $ddB = 0$. Translating back to the tensor language, $dH = 0$ means eq. (8). And the gauge transform (10) is simply $B' = B + d\Lambda^{(1)}$ where $\Lambda^{(1)}$ is an arbitrary 1–form; the tension form H is gauge invariant because $dd\Lambda^{(1)} = 0$.

Finally, in parts (e) and (f) of problem 3, the totally-antisymmetric tensor potential $C_{\mu_1\cdots\mu_p}(x)$ with p indices corresponds to a form C of rank p and the tension tensor (11) corresponds to a form $G = dC$ of rank $p+1$. The differential identity is $dG = 0$, which follows from $ddC = 0$. And the gauge transform (13) is $C' = C + d\Lambda^{(p-1)}$ for an arbitrary rank $p-1$ form $\Lambda^{(p-1)}$; the tension form G is gauge invariant because $dd\Lambda^{(p-1)} = 0$.

The Lagrangians and the equations of motion for the EM and tensor fields may also be written in the differential form language, but they — as well as the electric-magnetic duality of in problem 4 — require an additional mathematical tool, the *Hodge duality*. There is a quick and dirty introduction to this subject at the Wikipedia page
https://en.wikipedia.org/wiki/Hodge_star_operator.

Note: while the differential forms as such can be put on any smooth topological manifold even if it does not have a metric, the Hodge duality requires an orientable Riemannian manifold with some metric $g_{\mu\nu}(x)$ — which can be Euclidean or Minkowski — and the volume form

$$V = \sqrt{\pm g}\,\epsilon_{\mu_1,\ldots,\mu_D} dx^{\mu_1}\cdots dx^{\mu_D}. \tag{S.67}$$

In the antisymmetric tensor language, the Hodge duality works like this: First raise all the indices of a ($p' = p+1$)-index tensor $G_{\mu_1,\ldots,\mu_{p'}}$, and then contract it with the volume tensor (S.67); the remaining (un-contracted) $q' = D - p'$ indices of the volume tensor yield a ($q' = q+1$)–index antisymmetric tensor

$$\tilde{G}_{\nu_1,\ldots,\nu_{q'}} = \frac{1}{p'!}\sqrt{\pm g}\,\epsilon_{\nu_1,\ldots,\nu_D} \times g^{\nu_{q'+1}\mu_{q'+1}}\cdots g^{\nu_D\mu_D} \times G_{\mu_{p'+1},\ldots,\mu_D}\,. \tag{S.68}$$

In particular, in the flat Minkowski space, this formula becomes simply eq. (16).

In the differential form language, the Hodge duality (S.68) is denoted by the Hodge star $\star$:

$$\text{for } G \;=\; G_{\mu_1,\ldots,\mu_{p'}}\, dx^{\mu_1}\cdots dx^{\mu_{p'}}, \quad \text{we define } \star G \;=\; \tilde{G}_{\nu_1,\ldots,\nu_{q'}}\, dx^{\nu_1}\cdots dx^{\nu_{q'}}, \tag{S.69}$$

where the $G\mu_1,\ldots,\mu_{p'}$ and the $\tilde{G}_{\nu_1,\ldots,\nu_{q'}}$ tensors are defined exactly as in eq. (S.68). It is this Hodge star which allows us to write the the field equations for all kinds of antisymmetric tensor fields in a very compact form.

Indeed, take a p-index antisymmetric tensor field $C_{\mu_1,\ldots,\mu_p}(x)$; as we saw in problem (3), its tension tensor $G_{\mu_1,\ldots,\mu_{p+1}}(x)$ satisfies the differential identity

$$\partial_{[\mu}G_{\nu_1,\ldots,\nu_{p+1}]} \;=\; 0 \tag{S.48}$$

and the Euler–Lagrange equation

$$\partial_{\mu_1}G^{\mu_1,\mu_2\ldots,\mu_{p+1}} \;=\; 0. \tag{S.47}$$

In the differential form language the tension tensor is $G = dC$, the differential identity (S.48) becomes $dG = 0$, but the Euler–Lagrange equation (S.47) takes extra care. Fortunately, in problem 4(b) we saw that the Euler–Lagrange equation for the original tension tensor G is equivalent to the differential identity for the dual tensor $\tilde{G}$; in the differential form language, this means that the Hodge-dual tension form $\star G$ obeys $d\star G = 0$. Thus altogether, the field equations for the tension form $G = dC$ of a free p-form field C are

$$dG \;=\; 0 \quad \text{and} \quad d\star G \;=\; 0. \tag{S.70}$$

In particular, in 4 Minkowski dimensions the Maxwell eqs. for the free EM tension field $F_{\mu\nu}(x)$ in the differential form language become

$$dF \;=\; 0 \quad \text{and} \quad d\star F \;=\; 0. \tag{S.71}$$

In presence of an electric current $J^\mu(x) \neq 0$, these equations become

$$dF \;=\; 0 \quad \text{and} \quad d\star F \;=\; \star J, \tag{S.72}$$

where $\star J$ is a 3-form dual to the 1-form $J = J_\mu(x)\, dx^\mu$. Note that it this 3-form we should integrate over the 3D space to obtain the net electric charge

$$Q = \int_{\text{3D space}} \star J. \tag{S.73}$$

Finally, when both electric and magnetic charges and currents are present, the Maxwell equations become

$$dF = \star J_{\text{mag}} \quad \text{and} \quad d\star F = J_{\text{el}}, \tag{S.74}$$

and since $dF \neq 0$ we may no longer write $F = dA$ for a 1-form potential $A_\mu(x)\, dx^\mu$.

Likewise in problem 4(c) we have a duality between a free 2-form field B and a free massless scalar field $\phi(x)$ which we may treat as a 0-form ϕ. The corresponding tension forms $H = dB$ and $v = d\phi$ are Hodge-dual to each other (in 4D), $H = \star v$ and $v = \star H$, and obey field equations

$$dH = d\star v = 0 \quad \text{and} \quad d\star H = dv = 0. \tag{S.75}$$

As the last subject of this mathematical supplement, consider the Lagrangians for the various p-form fields. Let's start with the EM field in 4 Minkowski dimensions and consider the wedge product

$$F\wedge(\star F) = F_{\kappa\lambda}\, dx^\alpha \wedge dx^\lambda \wedge \tilde{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu. \tag{S.76}$$

In 4D, a wedge product of four dx^α's amounts to

$$dx^\alpha \wedge dx^\lambda \wedge dx^\mu \wedge dx^\nu = -\epsilon^{\kappa\lambda\mu\nu} \times dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \frac{-\epsilon^{\kappa\lambda\mu\nu}}{\sqrt{-g}} \times V \tag{S.77}$$

where $V = \sqrt{-g} d^4x$ is the 4D volume form. (The factor $\sqrt{-g}$ allows for a curved spacetime, if you ever care for the EM fields in such a background.) Consequently,

$$F \wedge (\star F) \;=\; V \times \frac{-\epsilon^{\kappa\lambda\mu\nu}}{\sqrt{-g}}\, F_{\kappa\lambda}\, \tilde{F}_{\mu\nu} \tag{S.78}$$

where

$$\begin{aligned} \frac{-\epsilon^{\kappa\lambda\mu\nu}}{\sqrt{-g}}\, F_{\kappa\lambda}\, \tilde{F}_{\mu\nu} \;&=\; \frac{-\epsilon^{\kappa\lambda\mu\nu}}{\sqrt{-g}}\, F_{\kappa\lambda} \times \tfrac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho} \\ &=\; F_{\kappa\lambda}\, F^{\sigma\rho} \times \Big[(-\tfrac{1}{2})\epsilon^{\kappa\lambda\mu\nu}\epsilon_{\mu\nu\sigma\rho} \;=\; +\delta[\kappa_\sigma \delta^{\lambda]}_\rho\Big] \\ &=\; +2F_{\kappa\lambda}F^{\kappa\lambda}. \end{aligned} \tag{S.79}$$

Altogether, we have

$$F \wedge (\star F) \;=\; 2F_{\kappa\lambda}F^{\kappa\lambda} \times V \;=\; -8\mathcal{L}_{\mathrm{EM}} \times V, \tag{S.80}$$

which allows us to write the EM Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\kappa\lambda}F^{\kappa\lambda}$ — or rather the action integral

$$S \;=\; \int\limits_{\text{spacetime}} \mathcal{L} \times \big(V = \sqrt{g}\, d^4x\big) \tag{S.81}$$

as

$$S \;=\; \int\limits_{\text{spacetime}} \frac{-1}{8}\, F \wedge (\star F). \tag{S.82}$$

Likewise, for a p-form C in D Minkowski dimensions, let's take a wedge product of the $(p+1)$ form $G = dC$ and its Hodge-dual $(q+1 = D-p-1)$ form $\star G$:

$$G \wedge (\star G) \;=\; G_{\mu_1,\ldots,\mu_{p+1}}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \wedge \tilde{G}_{\nu_1,\ldots,\nu_{q+1}}\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{q+1}}, \tag{S.83}$$

and since $(p+1)+(1+1) = D$, we have

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}} \wedge dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{q+1}} \;=\; \frac{(-1)^{D-1}}{\sqrt{-g}}\, \epsilon^{\mu_1,\cdots\,\mu_{p+1},\nu_1,\ldots,\nu_{q+1}} \times V \tag{S.84}$$

where $V=\sqrt{-g}\,d^Dx$ is the D-dimensional spacetime volume form. Consequently,

$$\begin{aligned}
G\wedge(\star G) &= V\times\frac{(-1)^{D-1}}{\sqrt{-g}}\,\epsilon^{\mu_1,\dots\mu_{p+1},\nu_1,\dots,\nu_{q+1}}\,G_{\mu_1,\dots,\mu_{p+1}}\,\tilde{G}_{\nu_1,\dots,\nu_{q+1}}\\
&= V\times\frac{(-1)^{D-1}}{\sqrt{-g}}\,\epsilon^{\mu_1,\dots\mu_{p+1},\nu_1,\dots,\nu_{q+1}}\,G_{\mu_1,\dots,\mu_{p+1}}\times\\
&\quad\times\frac{\sqrt{-g}}{(p+1)!}\,\epsilon_{\nu_1,\dots,\nu_{q+1},\lambda_1,\dots,\lambda_{p+1}}\,G^{\lambda_1,\dots\lambda_{p+1}}\\
&= V\times\frac{(-1)^{D-1}}{(p+1)!}\times G_{\mu_1,\dots,\mu_{p+1}}\,G^{\lambda_1,\dots,\lambda_{p+1}}\times\\
&\quad\times(-1)^{D-1}(-1)^{(p+1)(q+1)}(q+1)!\,\delta^{[\lambda_1}_{\mu_1}\cdots\delta^{\lambda_{p+1}]}_{\mu_{p+1}}\\
&= V\times(-1)^{(p+1)(q+1)}\,\frac{(q+1)!}{(p+1)!}\,\delta^{[\lambda_1}_{\mu_1}\cdots\delta^{\lambda_{p+1}]}_{\mu_{p+1}}\times G_{\mu_1,\dots,\mu_{p+1}}\,G^{\lambda_1,\dots,\lambda_{p+1}}\\
&= V\times(-1)^{(p+1)(q+1)}\,(q+1)!\times G_{\mu_1,\dots,\mu_{p+1}}\,G^{\mu_1,\dots,\mu_{p+1}}.
\end{aligned}\tag{S.85}$$

Comparing this formula to the Lagrangian density (12) for the p-form field C and its tension G,

$$\mathcal{L}(C,G)\;=\;\frac{(-1)^p}{2(p+1)!}\,G_{\mu_1,\dots,\mu_{p+1}}\,G^{\mu_1,\dots,\mu_{p+1}},\tag{12}$$

we see that

$$G\wedge(\star G)\;=\;V\times\mathcal{L}\times 2(p+1)!(q+1)!(-1)^{p+(p+1)(q+1)}.\tag{S.86}$$

Therefore, the action integral becomes

$$S\;=\;\int\limits_{\text{spacetime}}\mathcal{L}\times V\;=\;\frac{\pm1}{2(p+1)!(q+1)!}\int\limits_{\text{spacetime}}G\wedge(\star G)\tag{S.87}$$

where the overall sign is

$$\pm1\;=\;(-1)^{p+(p+1)(q+1)}\;=\;\begin{cases}-1 & \text{for even } D,\\ (-1)^p & \text{for odd } D.\end{cases}\tag{S.88}$$

In particular, for the free EM field in D dimensions

$$S\;=\;\frac{-1}{4(D-2)!}\int\limits_{\text{spacetime}}F\wedge(\star F),\tag{S.89}$$

while for the 2-form B and its tension $H = dB$,

$$S = \frac{(-1)^{D-1}}{12(D-3)!} \int_{\text{spacetime}} H \wedge (\star H). \tag{S.90}$$

Even a free massless scalar field $\phi(x)$ fits the pattern since we may treat it as a zero-form ϕ with a 1-form tension $v = d\phi$, thus

$$S = \frac{(-1)^{D-1}}{2(D-1)!} \int_{\text{spacetime}} d\phi \wedge (\star d\phi). \tag{S.91}$$