Problem **1**(a):

By the Leibniz rule for the commutators

$$\left[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma\right] \;=\; \hat{a}^\dagger_\alpha\left[\hat{a}_\beta, \hat{a}^\dagger_\gamma\right] \;+\; \left[\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\gamma\right]\hat{a}_\beta \;=\; \hat{a}^\dagger_\alpha \times \delta_{\beta\gamma} \;+\; 0\times\hat{a}_\beta \;=\; \delta_{\beta\gamma}\hat{a}^\dagger_\alpha \tag{S.1}$$

and likewise

$$\left[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta\right] \;=\; \hat{a}^\dagger_\alpha\left[\hat{a}_\beta, \hat{a}_\delta\right] \;+\; \left[\hat{a}^\dagger_\alpha, \hat{a}_\delta\right]\hat{a}_\beta \;=\; \hat{a}^\dagger_\alpha \times 0 \;+\; \left(-\delta_{\alpha\delta}\right)\times\hat{a}_\beta \;=\; -\delta_{\alpha\delta}\hat{a}_\beta\,. \tag{S.2}$$

Consequently, applying the Leibniz rule once again we get

$$\begin{aligned}
\left[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta\right] \;&=\; \hat{a}^\dagger_\gamma\left[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta\right] \;+\; \left[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma\right]\hat{a}_\delta \\
&=\; \hat{a}^\dagger_\gamma \times \left(-\delta_{\alpha\delta}\hat{a}_\beta\right) \;+\; \left(+\delta_{\beta\gamma}\hat{a}^\dagger_\alpha\right)\hat{a}_\delta \\
&=\; \delta_{\beta\gamma}\hat{a}^\dagger_\alpha\hat{a}_\delta \;-\; \delta_{\alpha\delta}\hat{a}^\dagger_\gamma\hat{a}_\beta\,.
\end{aligned} \tag{S.3}$$

Finally,

$$\begin{aligned}
\left[\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu\right] \;&=\; \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \times \left[\hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu\right] \;+\; \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \times \left[\hat{a}_\gamma, \hat{a}^\dagger_\mu \hat{a}_\nu\right] \times \hat{a}_\delta \\
&\qquad +\; \hat{a}^\dagger_\alpha \times \left[\hat{a}^\dagger_\beta, \hat{a}^\dagger_\mu \hat{a}_\nu\right] \times \hat{a}_\gamma \hat{a}_\delta \;+\; \left[\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\mu \hat{a}_\nu\right] \times \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \\
&=\; \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \times \left(+\delta_{\delta\mu}\hat{a}_\nu\right) \;+\; \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \times \left(+\delta_{\gamma\mu}\hat{a}_\nu\right) \times \hat{a}_\delta \\
&\qquad +\; \hat{a}^\dagger_\alpha \times \left(-\delta_{\beta\nu}\hat{a}^\dagger_\mu\right) \times \hat{a}_\gamma \hat{a}_\delta \;+\; \left(-\delta_{\alpha\nu}\hat{a}^\dagger_\mu\right) \times \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \\
&=\; +\delta_{\delta\mu}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu \;+\; \delta_{\gamma\mu}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta \;-\; \delta_{\beta\nu}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\gamma \hat{a}_\delta \;-\; \delta_{\alpha\nu}\hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta\,.
\end{aligned} \tag{S.4}$$

Problem **2**(a):

In 3D notations (but $\hbar = c = 1$ units), the Lagrangian density (3) for the massive vector field is

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) + \tfrac{1}{2}m^2\left(A_0^2 - \mathbf{A}^2\right) - J_0 A_0 + \mathbf{J}\cdot\mathbf{A} \\ &= \tfrac{1}{2}(-\dot{\mathbf{A}} - \nabla A_0)^2 + \tfrac{1}{2}(\nabla\times\mathbf{A})^2 + \tfrac{1}{2}m^2\left(A_0^2 - \mathbf{A}^2\right) - J_0 A_0 + \mathbf{J}\cdot\mathbf{A}\,. \end{aligned} \tag{S.5}$$

Note that only the first term on the last line contains any time derivatives at all, and it does not contain the $\dot{A}_0$ but only the $\dot{\mathbf{A}}$. Consequently, $\partial\mathcal{L}/\partial\dot{A}_0 = 0$ and the scalar potential $A_0(\mathbf{x})$ does not have a canonical conjugate field. On the other hand, the vector potential $\mathbf{A}(\mathbf{x})$ does have a canonical conjugate, namely

$$\frac{\delta L}{\delta\dot{\mathbf{A}}(x)} = \left.\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{A}}}\right|_{\mathbf{x}} = -(-\dot{\mathbf{A}}(\mathbf{x}) - \nabla A_0(\mathbf{x})) = -\mathbf{E}(\mathbf{x}). \tag{S.6}$$

Problem **2**(b):

In terms of the Hamiltonian and Lagrangian densities, eq. (4) means

$$\mathcal{H} = -\dot{\mathbf{A}}\cdot\mathbf{E} - \mathcal{L}\,. \tag{5$'$}$$

Expressing all fields in terms $\mathbf{A}$, $\mathbf{E}$, and A_0, we get

$$\begin{aligned} \dot{\mathbf{A}} &= -\mathbf{E} - \nabla A_0, \\ -\dot{\mathbf{A}}\cdot\mathbf{E} &= \mathbf{E}^2 + \mathbf{E}\cdot\nabla A_0, \\ \mathcal{L} &= \tfrac{1}{2}\left(\mathbf{E}^2 - (\nabla\times\mathbf{A})^2\right) + \tfrac{1}{2}m^2(A_0^2 - \mathbf{A}^2) - (A_0 J_0 - \mathbf{A}\cdot\mathbf{J}), \end{aligned} \tag{S.7}$$

and consequently,

$$\mathcal{H} = \tfrac{1}{2}\mathbf{E}^2 + \mathbf{E}\cdot\nabla A_0 - \tfrac{1}{2}m^2 A_0^2 + A_0 J_0 + \tfrac{1}{2}(\nabla\times\mathbf{A})^2 + \tfrac{1}{2}m^2\mathbf{A}^2 - \mathbf{A}\cdot\mathbf{J}. \tag{S.8}$$

Taking the $\int d^3\mathbf{x}$ integral of this density and integrating by parts the $\mathbf{E}\cdot\nabla A_0$ term, we arrive at the Hamiltonian (5). *Quod erat demonstrandum.*

Problem **2**(c):

Evaluating the derivatives of $\mathcal{H}$ in eq. (6) gives us

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \frac{\partial\mathcal{H}}{\partial(A_0)} - \nabla_i \frac{\partial\mathcal{H}}{\partial(\nabla_i A_0)} = -m^2 A_0 + J_0 - \nabla_i E^i. \tag{S.9}$$

Were there a canonical conjugate $\pi_0(\mathbf{x},t)$ of the $A_0(\mathbf{x},t)$, its time derivative $\partial\pi_0/\partial t$ would be given by the right hand side of eq. (S.9). But the $A_0(\mathbf{x},t)$ does not have a canonical conjugate, so instead of a Hamilton equation of motion we have a time-independent *constraint* (6), namely

$$m^2 A_0 = J_0 - \nabla\cdot\mathbf{E}. \tag{S.10}$$

In the massless EM case, a similar constraint gives rise to the Gauss Law $\nabla\cdot\mathbf{E} = J_0$. But the massive vector field does not obey the Gauss Law; instead, eq. (S.10) gives us a formula for the scalar potential A_0 in terms of $\mathbf{E}$ and J_0.

On the other hand, the Hamilton equations for the vector fields $\mathbf{A}$ and $\mathbf{E}$ are honest equations of motion. Specifically, evaluating the derivatives of $\mathcal{H}$ in eq. (7), we find

$$\frac{\delta H}{\delta E^i(\mathbf{x})} \equiv \frac{\partial\mathcal{H}}{\partial(E^i)} - \nabla_j \frac{\partial\mathcal{H}}{\partial(\nabla_j E^i)} = E^i + \nabla_i A_0, \tag{S.11}$$

which leads to Hamilton equation

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = -\mathbf{E}(\mathbf{x},t) - \nabla A_0(\mathbf{x},t). \tag{S.12}$$

Similarly, in eq. (8) we have

$$\frac{\delta H}{\delta A^i(\mathbf{x})} \equiv \frac{\partial\mathcal{H}}{\partial(A^i)} - \nabla_j \frac{\partial\mathcal{H}}{\partial(\nabla_j A^i)} = m^2 A^i - J^i - \nabla_j(\epsilon^{jik}(\nabla\times\mathbf{A})^k) \tag{S.13}$$

and hence Hamilton equation

$$\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) = m^2\mathbf{A} - \mathbf{J} + \nabla\times(\nabla\times\mathbf{A}). \tag{S.14}$$

Problem **2**(d):

In 3D notations, the Euler–Lagrange field equations (9) or $\partial_\mu F^{\mu\nu} + m^2 A^\nu = J^\nu$ become

$$\begin{aligned} \nabla\cdot\mathbf{E} \,+\, m^2 A^0 \;&=\; J^0, & \text{(S.15)}\\ -\dot{\mathbf{E}} \,+\, \nabla\times\mathbf{B} \,+\, m^2\mathbf{A} \;&=\; \mathbf{J}, & \text{(S.16)} \end{aligned}$$

where

$$\begin{aligned} \mathbf{E} \;&\stackrel{\text{def}}{=}\; -\dot{\mathbf{A}} \,-\, \nabla A^0, & \text{(S.17)}\\ \mathbf{B} \;&\stackrel{\text{def}}{=}\; \nabla\times\mathbf{A}. & \text{(S.18)} \end{aligned}$$

Clearly, eq. (S.15) is equivalent to eq. (S.10) while eq. (S.16) is equivalent to eq. (S.14) (provided **B** is defined as in eq. (S.18)). Finally, eq. (S.17) is equivalent to eq. (S.12), although their origins differ: In the Lagrangian formalism, eq. (S.17) is the definition of the **E** field in terms of A_0, **A** and their derivatives, while in the Hamiltonian formalism, **E** is an independent conjugate field and eq. (S.12) is the dynamical equation of motion for the $\dot{\mathbf{A}}$. *Quod erat demonstrandum.*

Problem **3**:

Let start with the $[\hat{\mathbf{A}}, \hat{H}]$ commutator. In light of eq. (13) for the Hamiltonian, we have

$$[\hat{A}^i(\mathbf{x}),\hat{H}] \;=\; \int\! d^3\mathbf{y} \left[\hat{A}^i(\mathbf{x}), \left(\tfrac{1}{2}\hat{\mathbf{E}}^2 \,+\, \frac{1}{2m^2}(\hat{J}_0 - \nabla\cdot\hat{\mathbf{E}})^2 \,+\, \tfrac{1}{2}(\nabla\times\hat{\mathbf{A}})^2 \,+\, \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 \,-\, \hat{\mathbf{J}}\cdot\hat{\mathbf{A}} \right)_{@\mathbf{y}} \right] \tag{S.19}$$

where all operators are at the same time t as the $\hat{A}^i(\mathbf{x},t)$. Since all the $\hat{A}^i(\mathbf{x})$ operators commute with each other at equal times, the last three terms in the Hamiltonian density do not contribute to the commutator (S.19). But for the first term we have

$$\begin{aligned} [\hat{A}^i(\mathbf{x}), \tfrac{1}{2}\hat{\mathbf{E}}^2(\mathbf{y})] \;&=\; \tfrac{1}{2}\{\hat{E}^j(\mathbf{y}), [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})]\}\\ &=\; \tfrac{1}{2}\{\hat{E}^j(\mathbf{y}), -i\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})\} & \text{(S.20)}\\ &=\; -i\delta^{(3)}(\mathbf{x}-\mathbf{y})\times\hat{E}^i(\mathbf{y}), \end{aligned}$$

while for the second term we have

$$\Big[\hat{A}^i(\mathbf{x}),\Big(\hat{J}_0(\mathbf{y})\;-\;\nabla\cdot\hat{\mathbf{E}}(\mathbf{y})\Big)\Big]\;=\;0\;-\;\frac{\partial}{\partial y^j}[\hat{A}^i(\mathbf{x}),\hat{E}^j(\mathbf{y})]\;=\;+i\delta^{ij}\,\frac{\partial}{\partial y^j}\delta^{(3)}(\mathbf{x}-\mathbf{y})\quad\text{(S.21)}$$

and hence

$$\begin{aligned}\left[\hat{A}^i(\mathbf{x}),\frac{1}{2m^2}\Big(\hat{J}_0(\mathbf{y})\;-\;\nabla\cdot\hat{\mathbf{E}}(\mathbf{y})\Big)^2\right]\;&=\;\frac{1}{m^2}\Big(\hat{J}_0(\mathbf{y})\;-\;\nabla\cdot\hat{\mathbf{E}}(\mathbf{y})\Big)\times+i\delta^{ij}\,\frac{\partial}{\partial y^j}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\ &=\;\hat{A}^0(\mathbf{y})\times i\frac{\partial}{\partial y^i}\delta^{(3)}(\mathbf{x}-\mathbf{y})\end{aligned}\quad\text{(S.22)}$$

where the second equality follows from eq. (12). Plugging these all these commutators into eq. (S.19) and integrating over $\mathbf{y}$, we obtain

$$\begin{aligned}[\hat{A}^i(\mathbf{x}),\hat{H}]\;&=\int d^3\mathbf{y}\left(-i\delta^{(3)}(\mathbf{x}-\mathbf{y})\times\hat{E}^i(\mathbf{y})\;+\;\hat{A}^0(\mathbf{y})\times i\frac{\partial}{\partial y^i}\delta^{(3)}(\mathbf{x}-\mathbf{y})\;+\;0\;+\;0\;+\;0\right)\\ &\qquad\langle\!\langle\,\text{integrating by parts}\,\rangle\!\rangle\\ &=\int d^3\mathbf{y}\,(-i)\delta^{(3)}(\mathbf{x}-\mathbf{y})\times\left(\hat{E}^i(\mathbf{y})\;+\;\frac{\partial}{\partial y^i}\hat{A}^0(\mathbf{y})\right)\\ &=\;-i\left(\hat{E}^i(\mathbf{x})\;+\;\frac{\partial}{\partial x^i}\hat{A}^0(\mathbf{x})\right).\end{aligned}\quad\text{(S.23)}$$

In other words, $[\hat{\mathbf{A}}(\mathbf{x}),\hat{H}]=-i\hat{\mathbf{E}}(\mathbf{x})-i\nabla\hat{A}^0(\mathbf{x})$ and consequently in the Heisenberg picture,

$$\frac{\partial}{\partial t}\,\hat{\mathbf{A}}(\mathbf{x},t)\;=\;-i\,[\hat{\mathbf{A}}(\mathbf{x}),\hat{H}]\;=\;-\hat{\mathbf{E}}(\mathbf{x},t)\;-\;\nabla\hat{A}^0(\mathbf{x},t).\quad\text{(S.24)}$$

Clearly, this Heisenberg equation is the quantum equivalent of the classical Hamilton equation (S.12).

Now consider the $[\hat{\mathbf{E}},\hat{H}]$ commutator. Similarly to eq.(S.19), we have

$$[\hat{E}^i(\mathbf{x},t),\hat{H}]\;=\int d^3\mathbf{y}\left[\hat{E}^i(\mathbf{x},t),\left(\tfrac{1}{2}\hat{\mathbf{E}}^2\;+\;\tfrac{1}{2}m^2\hat{A}_0^2\;+\;\tfrac{1}{2}\hat{\mathbf{B}}^2\;+\;\tfrac{1}{2}m^2\hat{\mathbf{A}}^2\;-\;\hat{\mathbf{J}}\cdot\hat{\mathbf{A}}\right)(\mathbf{y},t)\right]\quad\text{(S.25)}$$

where $m^2\hat{A}^0=\hat{J}^0-\nabla\cdot\hat{\mathbf{E}}$ according to eq. (12) and $\hat{\mathbf{B}}\stackrel{\text{def}}{=}\nabla\times\hat{\mathbf{A}}$. At equal times, the $\hat{E}^i(\mathbf{x})$ operator commutes with all the $\hat{E}^j(\mathbf{y})$ and hence with the $\hat{\mathbf{E}}^2(\mathbf{y})$, and also with the $\hat{A}_0(\mathbf{y})$

and hence with the $\hat{A}_0^2(\mathbf{y})$; this eliminates the first two terms in the Hamiltonian density from the commutator (S.25). For the remaining three terms we have

$$\begin{aligned}
\left[\hat{E}^i(\mathbf{x}), (-\hat{\mathbf{J}}\cdot\hat{\mathbf{A}})(\mathbf{y})\right] &= -\hat{J}^j(\mathbf{y}) \times \left[\hat{E}^i(\mathbf{x}), \hat{A}^j(\mathbf{y})\right] \\
&= -\hat{J}^j(\mathbf{y}) \times +i\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= -i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \times \hat{J}^i(\mathbf{y}), \\
\left[\hat{E}^i(\mathbf{x}), \tfrac{1}{2}m^2\hat{\mathbf{A}}^2(\mathbf{y})\right] &= +im^2\delta^{(3)}(\mathbf{x}-\mathbf{y}) \times \hat{A}^i(\mathbf{y}), \\
\left[\hat{E}^i(\mathbf{x}), \hat{B}^j(\mathbf{y})\right] &= \epsilon^{jk\ell}\frac{\partial}{\partial y^k}\left[\hat{E}^i(\mathbf{x}), \hat{A}^\ell(\mathbf{y})\right] \\
&= \epsilon^{jk\ell}\frac{\partial}{\partial y^k}\left(+i\delta^{i\ell}\delta^{(3)}(\mathbf{x}-\mathbf{y})\right) \\
&= +i\epsilon^{jki}\frac{\partial}{\partial y^k}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \\
\left[\hat{E}^i(\mathbf{x}), \tfrac{1}{2}\hat{\mathbf{B}}^2(\mathbf{y})\right] &= \hat{B}^j(\mathbf{y}) \times +i\epsilon^{jki}\frac{\partial}{\partial y^k}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= -i\epsilon^{jki}\frac{\partial}{\partial y^k}\hat{B}^j(\mathbf{y}) \times \delta^{(3)}(\mathbf{x}-\mathbf{y}) \;+\; \text{a total derivative} \\
&= +i(\nabla\times\hat{\mathbf{B}})^i(\mathbf{y}) \times \delta^{(3)}(\mathbf{x}-\mathbf{y}) \;+\; \text{a total derivative.}
\end{aligned} \tag{S.26}$$

Thus

$$\left[\hat{\mathbf{E}}(\mathbf{x}), \hat{\mathcal{H}}(\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\times\left(\nabla\times\hat{\mathbf{B}}(\mathbf{y}) + m^2\hat{\mathbf{A}}(\mathbf{y}) - \hat{\mathbf{J}}(\mathbf{y})\right) + \text{a total derivative,} \tag{S.27}$$

hence

$$\begin{aligned}
\left[\hat{\mathbf{E}}(\mathbf{x}), \hat{H}\right] &= \int d^3\mathbf{y}\begin{pmatrix} i\delta^{(3)}(\mathbf{x}-\mathbf{y})\times\left(\nabla\times\hat{\mathbf{B}}(\mathbf{y}) + m^2\hat{\mathbf{A}}(\mathbf{y}) - \hat{\mathbf{J}}(\mathbf{y})\right) \\ +\ \text{a total derivative}\end{pmatrix} \\
&= i\left(\nabla\times\hat{\mathbf{B}}(\mathbf{x}) + m^2\hat{\mathbf{A}}(\mathbf{x}) - \hat{\mathbf{J}}(\mathbf{x})\right),
\end{aligned} \tag{S.28}$$

and therefore in the Heisenberg picture

$$\frac{\partial}{\partial t}\hat{\mathbf{E}}(\mathbf{x},t) = -i\left[\hat{\mathbf{E}}(\mathbf{x},t), \hat{H}\right] = +\nabla\times\hat{\mathbf{B}}(\mathbf{x}) + m^2\hat{\mathbf{A}}(\mathbf{x}) - \hat{\mathbf{J}}(\mathbf{x}). \tag{S.29}$$

Again, this Heisenberg equation is the quantum equivalent of the classical Hamilton equation (S.14).

Problem **4**(a):

The relations $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$ follow from the hermiticity of the quantum fields $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$, and also from the convention

$$\mathbf{e}_\lambda(-\mathbf{k}) \;=\; -\mathbf{e}^*_\lambda(+\mathbf{k}) \tag{16.c}$$

for the polarization vectors. Indeed, taking the hermitian conjugate of eq. (18) for the $\hat{A}_{\mathbf{k},\lambda}$ we get

$$\begin{aligned}\hat{A}^\dagger_{\mathbf{k},\lambda} \;&=\; \left(\int\!d^3\mathbf{x}\,L^{-3/2}e^{+i\mathbf{kx}}\,\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{A}}(\mathbf{x})\right)^\dagger \;=\int\!d^3\mathbf{x}\,L^{-3/2}e^{-i\mathbf{kx}}\,\mathbf{e}_\lambda(\mathbf{k})\cdot\hat{\mathbf{A}}^\dagger(\mathbf{x})\\ &=\int\!d^3\mathbf{x}\,L^{-3/2}e^{+i(-\mathbf{k})\mathbf{x}}\,\big(-\mathbf{e}^*_\lambda(-\mathbf{k})\big)\cdot\hat{\mathbf{A}}(\mathbf{x}) \;=\; -\hat{A}_{-\mathbf{k},\lambda}\end{aligned} \tag{S.30}$$

and likewise for the $\hat{E}_{\mathbf{k},\lambda}$:

$$\begin{aligned}\hat{E}^\dagger_{\mathbf{k},\lambda} \;&=\; \left(\int\!d^3\mathbf{x}\,L^{-3/2}e^{+i\mathbf{kx}}\,\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{E}}(\mathbf{x})\right)^\dagger \;=\int\!d^3\mathbf{x}\,L^{-3/2}e^{-i\mathbf{kx}}\,\mathbf{e}_\lambda(\mathbf{k})\cdot\hat{\mathbf{E}}^\dagger(\mathbf{x})\\ &=\int\!d^3\mathbf{x}\,L^{-3/2}e^{+i(-\mathbf{k})\mathbf{x}}\,\big(-\mathbf{e}^*_\lambda(-\mathbf{k})\big)\cdot\hat{\mathbf{E}}(\mathbf{x}) \;=\; -\hat{E}_{-\mathbf{k},\lambda}\end{aligned} \tag{S.31}$$

As to the equal-time commutation relations for the $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ modes, they follow directly from eqs. (11): Since all $\hat{A}$'s commute with each other and all $\hat{E}$'s commute with each other, we obviously have

$$[\hat{A}_{\mathbf{k},\lambda},\hat{A}_{\mathbf{k}',\lambda'}] \;=\; 0 \quad\text{and}\quad [\hat{E}_{\mathbf{k},\lambda},\hat{E}_{\mathbf{k}',\lambda'}] \;=\; 0. \tag{S.32}$$

And the commutators between the $\hat{A}$ modes and the $\hat{E}$ modes obtains from eqs. (18):

$$\begin{aligned}[\hat{A}_{\mathbf{k},\lambda},\hat{E}^\dagger_{\mathbf{k}',\lambda'}] \;&=\; L^{-3}\int\!d^3\mathbf{x}\int\!d^3\mathbf{y}\,e^{-i\mathbf{kx}}\big(\mathbf{e}^*_\lambda(\mathbf{k})\big)^i\times e^{+i\mathbf{k}'\mathbf{y}}\big(\mathbf{e}_{\lambda'}(\mathbf{k}')\big)^j\times[\hat{A}^i(\mathbf{x}),\hat{E}^j(\mathbf{y})]\\ &=\; L^{-3}\int\!d^3\mathbf{x}\int\!d^3\mathbf{y}\,e^{-i\mathbf{kx}}\big(\mathbf{e}^*_\lambda(\mathbf{k})\big)^i\times e^{+i\mathbf{k}'\mathbf{y}}\big(\mathbf{e}_{\lambda'}(\mathbf{k}')\big)^j\times(-i)\delta^{(3)}(\mathbf{x}-\mathbf{y})\delta^{ij}\\ &=\; -iL^{-3}\int\limits_{\text{box}}\!d^3\mathbf{x}\,e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{x}}\times\big(\mathbf{e}^*_\lambda(\mathbf{k})\cdot\mathbf{e}_{\lambda'}(\mathbf{k}')\big)\\ &=\; -i\delta_{\mathbf{k},\mathbf{k}'}\times\big(\mathbf{e}^*_\lambda(\mathbf{k})\cdot\mathbf{e}_{\lambda'}(\mathbf{k})\big)\\ &=\; -i\delta_{\mathbf{k},\mathbf{k}'}\times\delta_{\lambda,\lambda'}\,,\end{aligned} \tag{S.33}$$

or equivalently,

$$[\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] \;=\; +i\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}} \times \delta_{\lambda,\lambda'}\,. \tag{S.34}$$

Problem **4**(b):

In the absence of the current $\hat{J}^\mu$, the Hamiltonian (13) reduces to 4 terms,

$$\hat{\mathcal{H}} \;=\; \int\!d^3\mathbf{x}\left(\frac{1}{2}\hat{\mathbf{E}}^2 \;+\; \frac{1}{2m^2}(\nabla\cdot\hat{\mathbf{E}})^2 \;+\; \frac{1}{2}(\nabla\times\hat{\mathbf{A}})^2 \;+\; \frac{m^2}{2}\hat{\mathbf{A}}^2\right). \tag{S.35}$$

Let's re-express each of the 4 terms here in terms of the field-mode operators $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$. For the first term, we have

$$\begin{aligned}\int\!d^3\mathbf{x}\,\hat{\mathbf{E}}^2(\mathbf{x}) \;&=\; \int\!d^3\mathbf{x}\,\hat{\mathbf{E}}^\dagger(\mathbf{x})\cdot\hat{\mathbf{E}}(\mathbf{x})\\ &=\; \int\!d^3\mathbf{x}\left(\sum_{\mathbf{k},\lambda}L^{-3/2}e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{e}^*_\lambda(\mathbf{k})\hat{E}^\dagger_{\mathbf{k},\lambda}\right)\cdot\left(\sum_{\mathbf{k}',\lambda'}L^{-3/2}e^{+i\mathbf{k}'\cdot\mathbf{x}}\,\mathbf{e}_{\lambda'}(\mathbf{k}')\hat{E}_{\mathbf{k}',\lambda'}\right)\\ &=\; \sum_{\mathbf{k},bk'}\sum_{\lambda,\lambda'}\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k}',\lambda'}\times L^{-3}\int\limits_{\text{box}}\!d^3\mathbf{x}\,e^{i\mathbf{x}\cdot(\mathbf{k}'-\mathbf{k})}\left(\mathbf{e}^*_\lambda(\mathbf{k})\cdot\mathbf{e}_{\lambda'}(\mathbf{k}')\right)\end{aligned} \tag{S.36}$$

where

$$L^{-3}\int\limits_{\text{box}}\!d^3\mathbf{x}\,e^{i\mathbf{x}\cdot(\mathbf{k}'-\mathbf{k})} \;=\; \delta_{\mathbf{k},\mathbf{k}'} \tag{S.37}$$

and then for $\mathbf{k}'=\mathbf{k}$

$$\mathbf{e}^*_\lambda(\mathbf{k})\cdot\mathbf{e}_{\lambda'}(\mathbf{k}'=\mathbf{k}) \;=\; \delta_{\lambda,\lambda'}\,. \tag{S.38}$$

Plugging this into the bottom line of eq. (S.36), we get

$$\int\!d^3\mathbf{x}\,\hat{\mathbf{E}}^2(\mathbf{x}) \;=\; \sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k}',\lambda'}\times\delta_{\mathbf{k},bk'}\delta_{\lambda,\lambda'} \;=\; \sum_{\mathbf{k},\lambda}\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\,. \tag{S.39}$$

In exactly the same way, for the fourth term in the Hamiltonian (S.35) we have

$$\int\!d^3\mathbf{x}\,\hat{\mathbf{A}}^2(\mathbf{x}) \;=\; \sum_{\mathbf{k},\lambda}\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\,. \tag{S.40}$$

Next, consider the second term in the Hamiltonian (S.35). It involves

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}) \;=\; \nabla\cdot\left(\sum_{\mathbf{k},\lambda} L^{-3/2}e^{i\mathbf{x}\cdot\mathbf{k}}\mathbf{e}_\lambda(\mathbf{k})\hat{E}_{\mathbf{k},\lambda}\right) \;=\; \sum_{\mathbf{k},\lambda} L^{-3/2}\left(\nabla e^{i\mathbf{x}\cdot\mathbf{k}}\cdot\mathbf{e}_\lambda(\mathbf{k})\right)\hat{E}_{\mathbf{k},\lambda} \quad \text{(S.41)}$$

where

$$\nabla e^{i\mathbf{x}\cdot\mathbf{k}} \;=\; e^{i\mathbf{x}\cdot\mathbf{k}}\,i\mathbf{k} \quad \text{(S.42)}$$

and

$$\mathbf{k}\cdot\mathbf{e}_\lambda(\mathbf{k}) \;=\; |\mathbf{k}|\delta_{\lambda,0} \quad \text{(S.43)}$$

because in the helicity basis $\mathbf{e}_0(\mathbf{k})$ points in the direction of the $\mathbf{k}$ while $\mathbf{e}_{\pm1}\perp\mathbf{k}$. Consequently

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}) \;=\; \sum_{\mathbf{k}} i|\mathbf{k}|\,L^{-3/2}e^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{E}_{\mathbf{k},0}\,, \quad \text{(S.44)}$$

which involves only the longitudinal modes with $\lambda=0$. Therefore,

$$\begin{aligned}
\int\! d^3\mathbf{x}\,(\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}))^2 &= \int\! d^3\mathbf{x}\,(\nabla\cdot\hat{\mathbf{E}}^\dagger(\mathbf{x}))\cdot(\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}))\\
&= \int\! d^3\mathbf{x}\left(\sum_{\mathbf{k}}(-i)|\mathbf{k}|\,L^{-3/2}e^{-i\mathbf{k}\cdot\mathbf{x}}\,\hat{E}^\dagger_{\mathbf{k},0}\right)\cdot\left(\sum_{\mathbf{k}'}(+i)|\mathbf{k}'|\,L^{-3/2}e^{i\mathbf{k}'\cdot\mathbf{x}}\,\hat{E}_{\mathbf{k}',0}\right)\\
&= \sum_{\mathbf{k},\mathbf{k}'}|\mathbf{k}|\times|\mathbf{k}'|\times\hat{E}^\dagger_{\mathbf{k},0}\hat{E}_{\mathbf{k}',0}\times\left(L^{-3}\int_{\text{box}} d^3\mathbf{x}\,e^{i(\mathbf{k}'-\mathbf{k})\mathbf{x}} \;=\; \delta_{\mathbf{k},\mathbf{k}'}\right)\\
&= \sum_{\mathbf{k}}|\mathbf{k}|^2\times\hat{E}^\dagger_{\mathbf{k},0}\hat{E}_{\mathbf{k}',0}\,.
\end{aligned} \quad \text{(S.45)}$$

Finally, in the magnetic third term in the Hamiltonian (S.35), we have

$$\hat{\mathbf{B}}(\mathbf{x}) \;=\; \nabla\times\hat{\mathbf{A}}(\mathbf{x}) \;=\; \sum_{\mathbf{k},\lambda}\nabla\times\left(L^{-3/2}e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\right)\hat{A}_{\mathbf{k},\lambda}$$

where

$$\nabla\times\left(e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\right) \;=\; e^{i\mathbf{k}\cdot\mathbf{x}}\,(i\mathbf{k})\times\mathbf{e}_\lambda(\mathbf{k}) \;=\; e^{i\mathbf{k}\cdot\mathbf{x}}\,\lambda|\mathbf{k}|\,\mathbf{e}_\lambda(\mathbf{k}). \quad \text{(S.46)}$$

Note that due the factor of λ, only the transverse modes $\hat{A}_{\mathbf{k},\lambda}$ with $\lambda=\pm1$ contribute to the magnetic field

$$\hat{\mathbf{B}}(\mathbf{x}) \;=\; \sum_{\mathbf{k},\lambda} L^{-3/2}\,\lambda|\mathbf{k}|\,e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{A}_{\mathbf{k},\lambda} \tag{S.47}$$

but the longitudinal modes with $\lambda = 0$ do not contribute. Consequently, the magnetic term in the Hamiltonian (S.35) becomes

$$\begin{aligned}
\int\! d^3\mathbf{x}\,\hat{\mathbf{B}}^2(\mathbf{x}) \;&=\int\! d^3\mathbf{x}\,\hat{\mathbf{B}}^\dagger(\mathbf{x})\cdot\hat{\mathbf{B}}(\mathbf{x})\\
&=\int\! d^3\mathbf{x}\left(\sum_{\mathbf{k},\lambda} L^{-3/2}\,\lambda|\mathbf{k}|\,e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\hat{A}^\dagger_{\mathbf{k},\lambda}\right)\cdot\\
&\qquad\cdot\left(\sum_{\mathbf{k}',\lambda'} L^{-3/2}\,\lambda'|\mathbf{k}'|\,e^{+i\mathbf{k}'\cdot\mathbf{x}}\mathbf{e}_{\lambda'}(\mathbf{k}')\hat{A}_{\mathbf{k}',\lambda'}\right)\\
&=\sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\lambda|\mathbf{k}|\lambda'|\mathbf{k}'|\,\hat{A}^\dagger_{\mathbf{k}',\lambda'}\hat{A}_{\mathbf{k},\lambda}\times L^{-3}\int_{\text{box}} d^3\mathbf{x}\,e^{i\mathbf{x}\cdot(\mathbf{k}'-\mathbf{k})}\times\big(\mathbf{e}^*_\lambda(\mathbf{k})\cdot\mathbf{e}_{\lambda'}(\mathbf{k}')\big)\\
&\quad\langle\!\langle\,\text{where } L^{-3}\textstyle\int d^3\mathbf{x}\,e^{i\mathbf{x}\cdot(\mathbf{k}'-\mathbf{k})} \;=\; \delta_{\mathbf{k},\mathbf{k}'}\,\rangle\!\rangle\\
&\quad\langle\!\langle\,\text{and } \mathbf{e}^*_\lambda(\mathbf{k})\cdot\mathbf{e}_{\lambda'}(\mathbf{k}') = \delta_{\lambda,\lambda'} \text{ for } \mathbf{k}=\mathbf{k}'\,\rangle\!\rangle\\
&=\sum_{\mathbf{k},\lambda}\lambda^2|\mathbf{k}|^2\,\hat{A}^\dagger_{\mathbf{k}',\lambda'}\hat{A}_{\mathbf{k},\lambda}
\end{aligned} \tag{S.48}$$

Altogether, the 4 terms in the Hamiltonian (S.35) add up to

$$\hat{H} \;=\; \sum_{\mathbf{k},\lambda}\left(\frac{1}{2}\left(1\,+\,\frac{\mathbf{k}^2}{m^2}\,\delta_{\lambda,0}\right)\times\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\;+\;\frac{m^2+\lambda^2\mathbf{k}^2}{2}\times\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\right). \tag{S.49}$$

Taking a closer look at the coefficients here and comparing them to the $\omega_{\mathbf{k}}$ and $C_{\mathbf{k},\lambda}$ in eq. (19.b–c), we see that

$$1\,+\,\frac{\mathbf{k}^2}{m^2}\times\delta_{\lambda,0} \;=\; \left\{\begin{matrix} 1 & \text{for } \lambda=\pm1\\ \omega^2_{\mathbf{k}}/m^2 & \text{for } \lambda=0\end{matrix}\right\} \;=\; C_{\mathbf{k},\lambda}$$

while

$$m^2\,+\,\lambda^2\mathbf{k}^2 \;=\; \left\{\begin{matrix} \omega^2_{\mathbf{k}} & \text{for } \lambda=\pm1\\ m^2 & \text{for } \lambda=0\end{matrix}\right\} \;=\; \frac{\omega^2_{\mathbf{k}}}{C_{\mathbf{k},\lambda}}\,, \tag{S.50}$$

thus the Hamiltonian (S.49) amounts to

$$\hat{H} = \sum_{\mathbf{k},\lambda} \left(\frac{C_{\mathbf{k},\lambda}}{2} \times \hat{E}^\dagger_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda} + \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}} \times \hat{A}^\dagger_{\mathbf{k},\lambda} \hat{A}_{\mathbf{k},\lambda} \right), \tag{S.51}$$

precisely as in eq. (19.a).

<u>Problem **4**(c)</u>:

Given the definitions (20) of the creation and the annihilation operators and the commutation relations eqs. (S.32) and (S.34) of the field modes $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$, we obtain

$$\begin{aligned} [\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}] &= -i\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} \left([\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] = (+i)\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}\delta_{\lambda,\lambda'} \right) \\ &\quad - i\sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} \left([\hat{E}_{\mathbf{k},\lambda}, \hat{A}_{\mathbf{k}',\lambda'}] = (-i)\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}\delta_{\lambda,\lambda'} \right) \\ &= \delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}\delta_{\lambda,\lambda'} \times \left(\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} - \sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} \right) \\ &= 0 \end{aligned} \tag{S.52}$$

because for $\mathbf{k}' + \mathbf{k} = 0$ and $\lambda' = \lambda$ we have $\omega_{\mathbf{k}'} = \omega_{\mathbf{k}}$ and $C_{\mathbf{k}',\lambda'} = C_{\mathbf{k},\lambda}$.

Likewise, $[\hat{a}^\dagger_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}] = 0$.

On the other hand,

$$\begin{aligned} [\hat{a}_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}] &= -i\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} \left([\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] = (+i)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \right) \\ &\quad + i\sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} \left([\hat{E}_{\mathbf{k},\lambda}, \hat{A}_{\mathbf{k}',\lambda'}] = (-i)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \right) \\ &= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \times \left(\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} + \sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} \right) \\ &= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \times 1 \end{aligned} \tag{S.53}$$

because for $\mathbf{k}' = \mathbf{k}$ and $\lambda' = \lambda$

$$\sqrt{\frac{C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}{4C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}} + \sqrt{\frac{C_{\mathbf{k},\lambda}\omega_{\mathbf{k}'}}{4C_{\mathbf{k}',\lambda'}\omega_{\mathbf{k}}}} = 1. \tag{S.54}$$

Quod erat demonstrandum.

Problem **4**(d):

Lets compare the first eq. (20) for the annihilation operator and the second eq. (20) for the creation operators for the two modes with opposite momenta $\pm\mathbf{k}$. For the annihilation operator we have

$$\sqrt{2\omega_{\mathbf{k}}C_{\mathbf{k},\lambda}}\hat{a}_{\mathbf{k},\lambda} = \omega_{\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} - iC_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \tag{S.55}$$

while for the creation operator

$$\sqrt{2\omega_{-\mathbf{k}}C_{-\mathbf{k},\lambda}}\hat{a}^{\dagger}_{-\mathbf{k},\lambda} = \omega_{-\mathbf{k}}\hat{A}^{\dagger}_{-\mathbf{k},\lambda} + iC_{-\mathbf{k},\lambda}\hat{E}^{\dagger}_{-\mathbf{k},\lambda} = -\omega_{-\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} - iC_{-\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \tag{S.56}$$

and hence

$$\sqrt{2\omega_{\mathbf{k}}C_{\mathbf{k},\lambda}}\hat{a}^{\dagger}_{-\mathbf{k},\lambda} = -\omega_{\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} - iC_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \tag{S.57}$$

because $\omega_{-\mathbf{k}} = \omega_{\mathbf{k}}$ and $C_{-\mathbf{k},\lambda} = C_{\mathbf{k},\lambda}$. Eqs. (S.55) and (S.57) involve the same field-mode operators $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$, so adding and subtracting them from each other gives us

$$\begin{aligned} \hat{E}_{\mathbf{k},\lambda} &= \frac{i\sqrt{\omega_{\mathbf{k}}}}{\sqrt{2C_{\mathbf{k},\lambda}}}\big(\hat{a}_{\mathbf{k},\lambda} + \hat{a}^{\dagger}_{-\mathbf{k},\lambda}\big), \\ \hat{A}_{\mathbf{k},\lambda} &= \frac{\sqrt{C_{\mathbf{k},\lambda}}}{\sqrt{2\omega_{\mathbf{k},\lambda}}}\big(\hat{a}_{\mathbf{k},\lambda} - \hat{a}^{\dagger}_{-\mathbf{k},\lambda}\big). \end{aligned} \tag{S.58}$$

Now let's plug eqs. (S.58) into the Hamiltonian (19). For the first term inside the integral/sum, we have

$$\begin{aligned} \frac{C_{\mathbf{k},\lambda}}{2}\,\hat{E}^{\dagger}_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} &= \frac{\omega_{\mathbf{k}}}{4}\big(\hat{a}^{\dagger}_{\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda}\big)\big(\hat{a}_{\mathbf{k},\lambda} + \hat{a}^{\dagger}_{-\mathbf{k},\lambda}\big) \\ &= \frac{\omega_{\mathbf{k}}}{4}\big(\hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}^{\dagger}_{-\mathbf{k},\lambda} + \hat{a}_{-\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-kl}\hat{a}^{\dagger}_{-kl}\big), \end{aligned} \tag{S.59}$$

while for the second term we have

$$\begin{aligned} \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}\,\hat{A}^{\dagger}_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda} &= \frac{\omega_{\mathbf{k}}}{4}\big(\hat{a}^{\dagger}_{\mathbf{k},\lambda} - \hat{a}_{-\mathbf{k},\lambda}\big)\big(\hat{a}_{\mathbf{k},\lambda} - \hat{a}^{\dagger}_{-\mathbf{k},\lambda}\big) \\ &= \frac{\omega_{\mathbf{k}}}{4}\big(\hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} - \hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}^{\dagger}_{-\mathbf{k},\lambda} - \hat{a}_{-\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{-kl}\hat{a}^{\dagger}_{-kl}\big), \end{aligned} \tag{S.60}$$

so altogether

$$\frac{C_{\mathbf{k},\lambda}}{2}\,\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\;+\;\frac{\omega_{\mathbf{k}}^2}{2C_{\mathbf{k},\lambda}}\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\;=\;\frac{\omega_{\mathbf{k}}}{4}\big(2\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}+2\hat{a}_{-\mathbf{k},\lambda}\hat{a}^\dagger_{-\mathbf{k},\lambda}\big)\\ =\;\frac{\omega_{\mathbf{k}}}{2}\big(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}+\hat{a}^\dagger_{-\mathbf{k},\lambda}\hat{a}_{-\mathbf{k},\lambda}+1\big). \tag{S.61}$$

Consequently, the net Hamiltonian (19) amounts to

$$\begin{aligned}\hat{H}\;&=\;\sum_{\mathbf{k},\lambda}\left(\frac{C_{\mathbf{k},\lambda}}{2}\,\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\;+\;\frac{\omega_{\mathbf{k}}^2}{2C_{\mathbf{k},\lambda}}\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\right)\\ &=\;\sum_{\mathbf{k},\lambda}\frac{\omega_{\mathbf{k}}}{2}\big(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}+\hat{a}^\dagger_{-\mathbf{k},\lambda}\hat{a}_{-\mathbf{k},\lambda}+1\big)\\ &=\;\frac{1}{2}\sum_{\mathbf{k},\lambda}\omega_{\mathbf{k}}(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}+\tfrac{1}{2})\;+\;\frac{1}{2}\sum_{\mathbf{k},\lambda}\omega_{\mathbf{k}}(\hat{a}^\dagger_{-\mathbf{k},\lambda}\hat{a}_{-\mathbf{k},\lambda}+\tfrac{1}{2})\\ &\qquad\langle\!\langle\text{ renaming the summation variable in the second term only from }\mathbf{k}\text{ to }-\mathbf{k}\,\rangle\!\rangle\\ &=\;\frac{1}{2}\sum_{\mathbf{k},\lambda}\omega_{\mathbf{k}}(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}+\tfrac{1}{2})\;+\;\frac{1}{2}\sum_{-\mathbf{k},\lambda}\omega_{-\mathbf{k}}(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}+\tfrac{1}{2})\\ &\qquad\langle\!\langle\text{ by }\mathbf{k}\leftrightarrow-\mathbf{k}\text{ symmetry of the summation range and thanks to }\omega_{-\mathbf{k}}=\omega_{\mathbf{k}}\,\rangle\!\rangle\\ &\qquad\langle\!\langle\text{ the two terms here are equal}\,\rangle\!\rangle\\ &=\;\sum_{\mathbf{k},\lambda}\omega_{\mathbf{k}}(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}+\tfrac{1}{2}).\end{aligned} \tag{S.62}$$

Problem **4**(e):

As explained in class — *cf.* my notes on identical bosons, — Hamiltonians of the form

$$\hat{H}\;=\;\sum_\alpha\omega_\alpha\hat{a}^\dagger_\alpha\hat{a}_\alpha\;+\;\text{const} \tag{S.63}$$

for some family of creation and the annihilation operators $\hat{a}^\dagger_\alpha$ and $\hat{a}_\alpha$ (obeying the bosonic commutation relations) describe theories of arbitrary numbers of identical bosons. Specifically, each boson has an independent 1-particle Hamiltonian with eigenstates $|\alpha\rangle$ and eigenvalues ω_α labeled by the same index α as in the sum (S.63).

In our case of the Hamiltonian (21), the 'index' α becomes $(\mathbf{k}, \lambda)$ where the momentum $\mathbf{k}$ has a discrete but dense spectrum corresponding to a particle living in a large L^3 box with periodic boundary conditions, while the helicity λ takes 3 discrete values $-1, 0, +1$. Physically, this means that the Hamiltonian (21) describes the Fock space of identical bosons, where each boson has a basis of 1-particle states $|\mathbf{k}, \lambda\rangle$ parametrized by the in-box momentum $\mathbf{k}$ and the helicity λ, with single-particle energies

$$\omega_{\mathbf{k}} \;=\; +\sqrt{\mathbf{k}^2 + m^2} \tag{S.64}$$

(in the $\hbar = c = 1$ units). Thus, each boson is a free relativistic particle of mass m. Moreover, for each particular momentum $\mathbf{k}$, the boson has 3 degenerate states with different helicities $\lambda = -1, 0, +1$. *Quod erat demonstrandum.*

PS: For the purpose of this problem, λ is just a label distinguishing the 3 distinct polarization states of the relativistic boson. Later in class we shall see that λ happens to be the boson's *helicity*, that is the component of its spin along the direction of it's motion.