Problem **1**(a):

In the previous homework set#2, problem 4(d), we saw that (in the Schrödinger picture and in the box normalization)

$$\begin{aligned}\hat{E}_{\mathbf{k},\lambda} &= \frac{i\sqrt{\omega_{\mathbf{k}}}}{\sqrt{2C_{\mathbf{k},\lambda}}}\left(\hat{a}_{\mathbf{k},\lambda}+\hat{a}^{\dagger}_{-\mathbf{k},\lambda}\right),\\ \hat{A}_{\mathbf{k},\lambda} &= \frac{\sqrt{C_{\mathbf{k},\lambda}}}{\sqrt{2\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k},\lambda}-\hat{a}^{\dagger}_{-\mathbf{k},\lambda}\right),\end{aligned} \tag{S2.58}$$

cf. eq. (58) on page 12 of the solutions to set#2. Consequently, the vector field $\hat{\mathbf{A}}(\mathbf{x})$ expands into the creation and annihilation operators as

$$\begin{aligned}\hat{\mathbf{A}}(\mathbf{x}) &= \sum_{\mathbf{k},\lambda} L^{-3/2}e^{i\mathbf{kx}}\mathbf{e}_{\lambda}(\mathbf{k})\hat{A}_{\mathbf{k},\lambda}\\ &= \sum_{\mathbf{k},\lambda}\frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\,e^{i\mathbf{kx}}\mathbf{ek},\lambda\big(\hat{a}_{\mathbf{k},\lambda}-\hat{a}^{\dagger}_{-\mathbf{k},\lambda}\big)\\ &= \sum_{\mathbf{k},\lambda}\frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\,e^{i\mathbf{kx}}\mathbf{e}_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda} \;-\; \sum_{\mathbf{k},\lambda}\frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\,e^{i\mathbf{kx}}\mathbf{e}_{\mathbf{k},\lambda}\,\hat{a}^{\dagger}_{-\mathbf{k},\lambda}\\ &\qquad\langle\!\langle\,\text{change } \mathbf{k}\to-\mathbf{k} \text{ in the second sum but not the first sum}\,\rangle\!\rangle\\ &= \sum_{\mathbf{k},\lambda}\frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\,e^{i\mathbf{kx}}\mathbf{e}_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda} \;-\; \sum_{\mathbf{k},\lambda}\frac{\sqrt{C_{-\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{-\mathbf{k}}}}\,e^{-i\mathbf{kx}}\mathbf{e}_{-\mathbf{k},\lambda}\,\hat{a}^{\dagger}_{\mathbf{k},\lambda}\\ &= \sum_{\mathbf{k},\lambda}\frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\,e^{i\mathbf{kx}}\mathbf{e}_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda} \;-\; \sum_{\mathbf{k},\lambda}\frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\,e^{-i\mathbf{kx}}(-\mathbf{e}_{\mathbf{k},\lambda})^{*}\,\hat{a}^{\dagger}_{\mathbf{k},\lambda}\\ &= \sum_{\mathbf{k},\lambda}\frac{\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_{\mathbf{k}}}}\Big(e^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + e^{-\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^{*}_{\mathbf{k},\lambda}\hat{a}^{\dagger}_{\mathbf{k},\lambda}\Big).\end{aligned} \tag{S.1}$$

Next, let's go to the infinite volume limit and hence continuous momenta. In the infinite-space but non-relativistic normalization, eq. (S.1) becomes

$$\hat{\mathbf{A}}(\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda}\frac{\sqrt{C_{\mathbf{k},\lambda}}}{\sqrt{2\omega_{\mathbf{k}}}}\Big(e^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + e^{-\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^{*}_{\mathbf{k},\lambda}\hat{a}^{\dagger}_{\mathbf{k},\lambda}\Big)^{\text{non rel.}}. \tag{S.2}$$

Finally, in the relativistic normalization

$$\big(\hat{a}_{\mathbf{k},\lambda}\big)^{\rm rel} \;=\; \sqrt{2\omega_{\mathbf{k}}}\,\big(\hat{a}_{\mathbf{k},\lambda}\big)^{\rm non\,rel} \quad\Longrightarrow\quad \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\big(\hat{a}_{\mathbf{k},\lambda}\big)^{\rm non\,rel} \;=\; \frac{1}{2\omega_{\mathbf{k}}}\big(\hat{a}_{\mathbf{k},\lambda}\big)^{\rm rel} \tag{S.3}$$

and likewise for the creation operators $\hat{a}^\dagger_{\mathbf{k},\lambda}$. Consequently, eq. (S.2) becomes

$$\hat{\mathbf{A}}(\mathbf{x}) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\Big(e^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \,+\, e^{-\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^*_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda}\Big)^{\rm rel.}. \tag{1}$$

Eq. (1) applies in the Schrödinger picture where all the operators — including the creation and annihilation operators for all the models as well as the quantum fields — are time-independent. In the Heisenberg picture, all operators become time-dependent, but the algebraic relation between different operators *at equal times* remain the same as in the Schrödinger picture, thus

$$\hat{\mathbf{A}}(\mathbf{x},t) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\,\Big(e^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}(t) \,+\, e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^*_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda}(t)\Big)\,. \tag{2}$$

Quod erat demonstrandum

Problem **1**(b):

In the Heisenberg picture, the time-dependence of the quantum field $\hat{\mathbf{A}}(\mathbf{x},t)$ follows from the time-dependence of the creation and annihilation operators,

$$i\frac{d}{dt}\,\hat{a}_{\mathbf{k},\lambda}(t) \;=\; [\hat{a}_{\mathbf{k},\lambda}(t),\hat{H}], \qquad i\frac{d}{dt}\,\hat{a}^\dagger_{\mathbf{k},\lambda}(t) \;=\; [\hat{a}^\dagger_{\mathbf{k},\lambda}(t),\hat{H}]. \tag{S.4}$$

In the previous homework (problem 4(d)) we wrote the Hamiltonian of the free vector field as

$$\hat{H} \;=\; \sum_{\mathbf{k},\lambda}\omega_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \;+\; \text{const} \tag{S.5}$$

for the box normalization of creation and annihilation operators. In the infinite-space relativistic normalization of the operators, this Hamiltonian becomes

$$\hat{H} \;=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \omega_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \;+\; \mathrm{const} = \int \frac{\mathrm{d}^3\mathbf{k}}{2(2\pi)^3}\sum_\lambda \hat{\mathrm{a}}^\dagger_{\mathbf{k},\lambda}\hat{\mathrm{a}}_{\mathbf{k},\lambda} \;+\; \mathrm{const}. \tag{S.6}$$

Consequently,

$$\begin{aligned}
\big[\hat{a}_{\mathbf{k},\lambda},\hat{H}\big] \;&=\int\!\frac{d^3\mathbf{k}'}{2(2\pi)^3}\sum_{\lambda'}\Big(\big[\hat{a}_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\hat{a}_{\mathbf{k}',\lambda'}\big] \;=\; \big[\hat{a}_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\big]\hat{a}_{\mathbf{k}',\lambda'} \;+\;0\Big)\\
&=\int\!\frac{d^3\mathbf{k}'}{2(2\pi)^3}\sum_{\lambda'}\Big(+2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\Big)\hat{a}_{\mathbf{k}',\lambda'}\\
&=\;+\omega_{\mathbf{k}}\hat{a}_{\mathbf{k},\lambda}\,,\\
\big[\hat{a}^\dagger_{\mathbf{k},\lambda},\hat{H}\big] \;&=\int\!\frac{d^3\mathbf{k}'}{2(2\pi)^3}\sum_{\lambda'}\Big(\big[\hat{a}^\dagger_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\hat{a}_{\mathbf{k}',\lambda'}\big] \;=\; 0\;+\;\hat{a}^\dagger_{\mathbf{k}',\lambda'}\big[\hat{a}^\dagger_{\mathbf{k},\lambda},\hat{a}_{\mathbf{k}',\lambda'}\big]\Big)\\
&=\int\!\frac{d^3\mathbf{k}'}{2(2\pi)^3}\sum_{\lambda'}\hat{a}^\dagger_{\mathbf{k}',\lambda'}\Big(-2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\Big)\\
&=\;-\omega_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k},\lambda}\,,
\end{aligned} \tag{S.7}$$

similar to the scalar creation and annihilation operators we have studied in class. Plugging these relations into the Heisenberg equations (S.4) for the creation and annihilation operators, we get

$$\frac{d}{dt}\,\hat{a}_{\mathbf{k},\lambda}(t) \;=\; -i\omega_{\mathbf{k}}\hat{a}_{\mathbf{k},\lambda}(t), \quad \frac{d}{dt}\,\hat{a}^\dagger_{\mathbf{k},\lambda}(t) \;=\; +i\omega_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k},\lambda}(t), \tag{S.8}$$

which give us the time dependence of the creation/annihilation operators:

$$\hat{a}_{\mathbf{k},\lambda}(t) \;=\; \exp(-i\omega_{\mathbf{k}}t)\times\hat{a}_{\mathbf{k},\lambda}(0), \quad \hat{a}^\dagger_{\mathbf{k},\lambda}(t) \;=\; \exp(+i\omega_{\mathbf{k}}t)\times\hat{a}^\dagger_{\mathbf{k},\lambda}(0). \tag{S.9}$$

Consequently, substituting this time dependence into eq. (2) for the vector field, we arrive at

$$\begin{aligned}
\hat{\mathbf{A}}(\mathbf{x},t) \;&=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda\sqrt{C_{\mathbf{k},\lambda}}\,\Big(e^{+i\mathbf{k}\mathbf{x}-i\omega_{\mathbf{k}}t}\mathbf{e}_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda}(0)\;+\;e^{-i\mathbf{k}\mathbf{x}+i\omega_{\mathbf{k}}t}\mathbf{e}^*_{\mathbf{k},\lambda}\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\Big)\\
&=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda\sqrt{C_{\mathbf{k},\lambda}}\,\Big(e^{-ik_\mu x^\mu}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(0)\;+\;e^{+ik_\mu x^\mu}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\Big)^{k^0=+\omega_{\mathbf{k}}}
\end{aligned} \tag{S.10}$$

precisely as in eq. (3).

Problem $\mathbf{1}$(c):

Proceeding exactly as in part (a), we find that in the Schrödinger picture and the box normalization of the creation and annihilation operators, the electric field becomes

$$\begin{aligned}
\hat{\mathbf{E}}(\mathbf{x}) &= \sum_{\mathbf{k},\lambda} L^{-3/2} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k}) \hat{E}_{\mathbf{k},\lambda} \\
&= \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_\mathbf{k}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}\mathbf{k},\lambda \big(\hat{a}_{\mathbf{k},\lambda} + \hat{a}^\dagger_{-\mathbf{k},\lambda}\big) \\
&= \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_\mathbf{k}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\, \hat{a}_{\mathbf{k},\lambda} + \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_\mathbf{k}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\, \hat{a}^\dagger_{-\mathbf{k},\lambda} \\
&\qquad \langle\!\langle \text{ change } \mathbf{k} \to -\mathbf{k} \text{ in the second sum but not the first sum } \rangle\!\rangle \qquad \text{(S.11)} \\
&= \sum_{\mathbf{k},\lambda} \frac{i\sqrt{C_{\mathbf{k},\lambda}}}{L^{3/2}\sqrt{2\omega_\mathbf{k}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\, \hat{a}_{\mathbf{k},\lambda} + \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_{-\mathbf{k}}}}{L^{3/2}\sqrt{2C_{-\mathbf{k},\lambda}}} e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}_{-\mathbf{k},\lambda}\, \hat{a}^\dagger_{\mathbf{k},\lambda} \\
&= \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_\mathbf{k}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\, \hat{a}_{\mathbf{k},\lambda} + \sum_{\mathbf{k},\lambda} \frac{i\sqrt{\omega_\mathbf{k}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} e^{-i\mathbf{k}\mathbf{x}} (-\mathbf{e}_{\mathbf{k},\lambda})^*\, \hat{a}^\dagger_{\mathbf{k},\lambda} \\
&= \sum_{\mathbf{k},\lambda} \frac{\sqrt{\omega_\mathbf{k}}}{L^{3/2}\sqrt{2C_{\mathbf{k},\lambda}}} \Big(ie^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} - ie^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}^*_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda} \Big).
\end{aligned}$$

Going to the infinite space limit and then to the relativistic normalization of the operators, we turn this formula to

$$\begin{aligned}
\hat{\mathbf{E}}(\mathbf{x}) &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda \frac{\sqrt{\omega_\mathbf{k}}}{\sqrt{2C_{\mathbf{k},\lambda}}} \Big(ie^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} - ie^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}^*_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda} \Big)^{\text{non rel}} \\
&= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda \frac{1}{2\sqrt{C_{\mathbf{k},\lambda}}} \Big(ie^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} - ie^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}^*_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda} \Big)^{\text{rel}} \qquad \text{(S.12)} \\
&= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega\mathbf{k}} \sum_\lambda \frac{\omega_\mathbf{k}}{2\sqrt{C_{\mathbf{k},\lambda}}} \Big(ie^{+i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} - ie^{-\mathbf{k}\cdot\mathbf{x}} \mathbf{e}^*_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda} \Big)^{\text{rel}}.
\end{aligned}$$

In the Heisenberg picture, this formula becomes

$$
\begin{aligned}
\hat{\mathbf{E}}(\mathbf{x},t) &= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\sum_\lambda \frac{\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},\lambda}}}\left(ie^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\hat{a}_{\mathbf{k},\lambda}(t)\,-\,ie^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\hat{a}^\dagger_{\mathbf{k},\lambda}(t)\right)\\
&\qquad\langle\!\langle\text{ using part (b) }\rangle\!\rangle\\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\sum_\lambda \frac{\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},\lambda}}}\left(ie^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})e^{-i\omega_{\mathbf{k}}t}\hat{a}_{\mathbf{k},\lambda}\,-\,ie^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})e^{+i\omega_{\mathbf{k}}t}\hat{a}^\dagger_{\mathbf{k},\lambda}\right)\\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\sum_\lambda \frac{\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},\lambda}}}\left(ie^{-ik_\mu x^\mu}\mathbf{e}_\lambda(\mathbf{k})\hat{a}_{\mathbf{k},\lambda}\,-\,ie^{+ik_\mu x^\mu}\mathbf{e}^*_\lambda(\mathbf{k})\hat{a}^\dagger_{\mathbf{k},\lambda}\right)^{k^0=+\omega_{\mathbf{k}}}.
\end{aligned}
\tag{S.13}
$$

Finally, the $\hat{A}^0(x)$ field obtains from the electric field and eq. (4). Applying this identity to eq. (S.13), we obtain

$$
\begin{aligned}
\hat{A}^0(\mathbf{x},t) &= \frac{-1}{m^2}\,\nabla\cdot\hat{\mathbf{E}}(\mathbf{x},t)\\
&= \frac{-1}{m^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda\frac{i\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},\lambda}}}\bigg(e^{-i\omega_{\mathbf{k}}t}\times\left(\nabla e^{+i\mathbf{kx}}\cdot\mathbf{e}_\lambda(\mathbf{k})\right)\times\hat{a}_{\mathbf{k},\lambda}\\
&\qquad\qquad -\;e^{+i\omega_{\mathbf{k}}t}\times\left(\nabla e^{-i\mathbf{kx}}\cdot e^*_\lambda(\mathbf{k})\right)\times\hat{a}^\dagger_{\mathbf{k},\lambda}\bigg).
\end{aligned}
\tag{S.14}
$$

In this formula

$$
\nabla e^{+i\mathbf{kx}}\cdot\mathbf{e}_\lambda(\mathbf{k}) \;=\; e^{+i\mathbf{kx}}\big(i\mathbf{k}\cdot\mathbf{e}_\lambda(\mathbf{k})\big) \;=\; e^{+i\mathbf{kx}}\times i|\mathbf{k}|\delta_{\lambda,0}
\tag{S.15}
$$

in the helicity basis, and likewise

$$
\nabla e^{-i\mathbf{kx}}\cdot e^*_\lambda(\mathbf{k}) \;=\; e^{-i\mathbf{kx}}\times(-i)|\mathbf{k}|\delta_{\lambda,0}\,.
\tag{S.16}
$$

Thus, only the $\lambda=0$ modes contribute to the scalar potential (S.14). Specifically,

$$
\hat{A}^0(\mathbf{x},t) \;=\; \frac{-1}{m^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\times\frac{i\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},0}}}\times i|\mathbf{k}|\left(e^{-i\omega_{\mathbf{k}}t+i\mathbf{kx}}\times\hat{a}_{\mathbf{k},0}\;+\;e^{+i\omega_{\mathbf{k}}t-i\mathbf{kx}}\times\hat{a}^\dagger_{\mathbf{k},0}\right)
\tag{S.17}
$$

where

$$
\sqrt{C_{\mathbf{k},0}} \;=\; \frac{\omega_{\mathbf{k}}}{m} \quad\Longrightarrow\quad \frac{-1}{m^2}\times\frac{i\omega_{\mathbf{k}}}{\sqrt{C_{\mathbf{k},0}}}\times i|\mathbf{k}| \;=\; +\frac{|\mathbf{k}|}{m}\,.
$$

Thus altogether,

$$
\hat{A}^0(\mathbf{x},t) \;=\int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\times\frac{|\mathbf{k}|}{m}\times\left(e^{-ik_\mu x^\mu}\times\hat{a}_{\mathbf{k},0}\;+\;e^{+ik_\mu x^\mu}\times\hat{a}^\dagger_{\mathbf{k},0}\right)^{k^0=+\omega_{\mathbf{k}}}. \qquad (\text{S.18}).
$$

Problem $\mathbf{1}$(d):

In light of obvious similarity between eqs. (2) and (S.18), we may immediately combine them into

$$\hat{A}^\mu(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_\mathbf{k}}\sum_\lambda\Big(e^{-ikx}\,f^\mu(\mathbf{k},\lambda)\times\hat{a}_{\mathbf{k},0}(0) + e^{+ikx}\,f^{\mu*}(\mathbf{k},\lambda)\times\hat{a}^\dagger_{\mathbf{k},0}(0)\Big)_{k^0=+\omega_\mathbf{k}} \tag{S.19}$$

— precisely as in eq. (5) — for

$$\mathbf{f}(\mathbf{k},\lambda) = \sqrt{C_{\mathbf{k},\lambda}}\,\mathbf{e}_\lambda(\mathbf{k}) \quad\text{and}\quad f^0(\mathbf{k},\lambda) = \frac{|\mathbf{k}|}{m}\,\delta_{\lambda,0}\,. \tag{S.20}$$

Specifically,

$$\text{for } \lambda=\pm1,\quad \mathbf{f} = \mathbf{e}_\lambda(\mathbf{k}),\quad f^0 = 0, \tag{S.21}$$

while

$$\text{for } \lambda=0,\quad \mathbf{f} = \frac{\omega_\mathbf{k}}{m}\,\frac{\mathbf{k}}{|\mathbf{k}|}\,,\quad f^0 = \frac{|\mathbf{k}|}{m}\,. \tag{S.22}$$

Problem $\mathbf{1}$(e):

It is easy to see that the polarization 4-vectors $f^\mu(\mathbf{k},\lambda)$ obey $k_\mu f^\mu(\mathbf{k},\lambda)=0$. Indeed, for the transverse polarizations $\lambda=\pm1$, eq. (S.21) tells us that $f^0=0$ while the space part $\mathbf{f}=\mathbf{e}(\lambda=\pm1)$ is transverse to the 3-vector $\mathbf{k}$, hence $k_\mu f^\mu=0$. As to the longitudinal polarization $\lambda=0$, in light of eq. (S.22)

$$k_\mu f^\mu = \omega_\mathbf{k} f^0 - \mathbf{k}\cdot\mathbf{f} = \omega_\mathbf{k}\,\frac{|\mathbf{k}|}{m} - \mathbf{k}\cdot\left(\frac{\omega_\mathbf{k}}{m}\,\frac{\mathbf{k}}{|\mathbf{k}|}\right) = \frac{\omega_\mathbf{k}|\mathbf{k}|}{m} - \frac{\omega_\mathbf{k}\mathbf{k}^2}{m|\mathbf{k}|} = 0. \tag{S.23}$$

Consequently, the quantum field $\hat{A}^\mu(x)$ obeys the classical equation $\partial_\mu\hat{A}^\mu=0$. Indeed, each plane wave factor in the expansion (24) obeys this equation:

$$\partial_\mu\big(e^{-ikx}f^\mu(\mathbf{k},\lambda)\big) = e^{-ikx}\times(-ik_\mu)f^\mu(\mathbf{k},\lambda) = 0 \qquad \langle\!\langle\,\text{because } k_\mu f^\mu(\mathbf{k},\lambda)=0\,\rangle\!\rangle, \tag{S.24}$$

and likewise

$$\partial_\mu\big(e^{+ikx}f^{\mu*}(\mathbf{k},\lambda)\big) = e^{+ikx}\times(+ik_\mu)f^{\mu*}(\mathbf{k},\lambda) = 0 \tag{S.25}$$

because $k_\mu f^{\mu*} = (k_\mu f^\mu)^* = 0$. And therefore, by linearity, the whole quantum field obeys this equation,

$$\partial_\mu \hat{A}^\mu(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\sum_\lambda \Big(\partial_\mu\big(e^{-ikx}f^\mu(\mathbf{k},\lambda)\big)\hat{a}_{\mathbf{k},\lambda} \,+\, \partial_\mu(\big(e^{+ikx}f^{\mu*}(\mathbf{k},\lambda)\big)\hat{a}^\dagger_{\mathbf{k},\lambda}\Big) \;=\; 0. \tag{S.26}$$

Likewise, thanks to $k_0 = +\omega_{\mathbf{k}}$ — and hence $k_\mu k^\mu = \omega_{\mathbf{k}}^2 - \mathbf{k}^2 = m^2$ — in every plane wave factor in the decomposition (24) of the quantum field, every plane wave factor obeys the Klein–Gordon equation

$$\begin{aligned} (\partial^2+m^2)\big(e^{-ikx}f^\mu(\mathbf{k},\lambda)\hat{a}_{\mathbf{k},\lambda}\big) \;&=\; (-k^2+m^2)\big(e^{-ikx}f^\mu(\mathbf{k},\lambda)\hat{a}_{\mathbf{k},\lambda}\big) \;=\; 0,\\ (\partial^2+m^2)\big(e^{+ikx}f^{\mu*}(\mathbf{k},\lambda)\hat{a}^\dagger_{\mathbf{k},\lambda}\big) \;&=\; (-k^2+m^2)\big(e^{+ikx}f^{\mu*}(\mathbf{k},\lambda)\hat{a}^\dagger_{\mathbf{k},\lambda}\big) \;=\; 0. \end{aligned} \tag{S.27}$$

Hence, by linearity, the whole quantum field obeys this equation,

$$\begin{aligned} &(\partial^2+m^2)\hat{A}^\mu(x) \;=\\ &\quad=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\sum_\lambda \Big((\partial^2+m^2)\big(e^{-ikx}f^\mu(\mathbf{k},\lambda)\big)\hat{a}_{\mathbf{k},\lambda} \,+\, (\partial^2+m^2)(\big(e^{+ikx}f^{\mu*}(\mathbf{k},\lambda)\big)\hat{a}^\dagger_{\mathbf{k},\lambda}\Big)\\ &\quad=\; 0. \end{aligned} \tag{S.28}$$

Problem **2**(a):

The Hamiltonian (7) of a free relativistic particle — and hence the evolution operator $\exp(-it\hat{H})$ — are functions of the momentum operator $\hat{\mathbf{p}}$, so they diagonalize in the momentum basis. In the non-relativistic normalization of $|\mathbf{k}\rangle$ states,

$$\begin{aligned} \hat{H} \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\;|\mathbf{k}\rangle\,\omega(\mathbf{k})\,\langle\mathbf{k}|\,,\\ \exp(-i\hat{H}t) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\;|\mathbf{k}\rangle\exp\big(-it\omega(\mathbf{k})\big)\,\langle\mathbf{k}|\,, \end{aligned} \tag{S.29}$$

where $t = y^0 - x^0$ and $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2+M^2}$. In the same non-relativistic normalization

$\langle \mathbf{x}|\mathbf{k}\rangle = \exp(i\mathbf{k}\cdot\mathbf{k})$, therefore

$$\begin{aligned} U(x\to y) \;\equiv\; \langle\mathbf{y}|\exp(-i\hat{H}t)\,|\mathbf{x}\rangle \;&=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\langle\mathbf{y}|\mathbf{k}\rangle\exp\bigl(-it\omega(\mathbf{k})\bigr)\,\langle\mathbf{k}|\mathbf{x}\rangle \\ &=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\exp\left(i(\mathbf{y}-\mathbf{x})\cdot\mathbf{k}\,-\,it\omega(\mathbf{k})\right). \end{aligned} \tag{S.30}$$

precisely as in eq. (9).

By the way, this non-relativistically normalized evolution kernel $U(x\to y)$ is related to the

$$D(x-y)\;\stackrel{\text{def}}{=}\int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\,\exp\bigl(i(\mathbf{y}-\mathbf{x})\cdot\mathbf{k}\,-\,i(y^0-x^0)\omega_{\mathbf{k}}\bigr) \tag{S.31}$$

we have used in class (in the context of relativistic causality and also of Feynman propagators) as

$$U(x\to y)\;=\;2i\frac{\partial}{\partial t}D(y-x)\qquad\langle\!\langle\,\text{where }t=y^0-x^0\,\rangle\!\rangle. \tag{S.32}$$

Indeed,

$$2i\frac{\partial}{\partial t}\exp\left(i(\mathbf{y}-\mathbf{x})\cdot\mathbf{k}\,-\,it\omega_{\mathbf{k}}\right)\;=\;2\omega_b k\times\exp\left(i(\mathbf{y}-\mathbf{x})\cdot\mathbf{k}\,-\,it\omega_{\mathbf{k}}\right) \tag{S.33}$$

where the $2\omega_{\mathbf{k}}$ factor cancels the similar denominator of the relativistic measure, thus

$$2i\frac{\partial}{\partial t}D(y-x)\;=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\exp\left(i(\mathbf{y}-\mathbf{x})\cdot\mathbf{k}\,-\,it\omega(\mathbf{k})\right)\;=\;U(x\to y). \tag{S.34}$$

Now let's simplify the 3D integral on the RHS of eq. (9) by integrating over the directions of the $\mathbf{k}$ vector. In spherical coordinates (k,θ,ϕ) where $k=|\mathbf{k}|$ and θ is the angle between $\mathbf{k}$ and $\mathbf{y}-\mathbf{x}$,

$$d^3\mathbf{k}\;=\;dk\,k^2\,d\cos\theta\,d\phi,\qquad \mathbf{k}\cdot(\mathbf{y}-\mathbf{x})\;=\;rk\,\cos\theta, \tag{S.35}$$

hence

$$\iint d\cos\theta\,d\phi\,e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\;=\;2\pi\int_{-1}^{+1}d\cos\theta\,e^{ikr\times\cos\theta}\;=\;\frac{2\pi}{irk}\left(e^{+irk}\,-\,e^{-irk}\right). \tag{S.36}$$

Consequently, for any symmetric function $f(k) = f(-k)$ we have

$$\begin{aligned}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}\times f(|\mathbf{k}|) &= \frac{1}{4\pi^2}\int\limits_0^\infty dk\,k^2\,\frac{1}{irk}\left(e^{+irk}\,-\,e^{-irk}\right)\times f(k)\\ &= \frac{1}{4\pi^2 i\,r}\int\limits_0^\infty dk\,k\,\left(e^{+irk}\times f(k)\,-\,e^{-irk}\times\bigl(f(k)=f(-k)\bigr)\right)\\ &= \frac{1}{4\pi^2 i\,r}\int\limits_{-\infty}^{+\infty} dk\,k\,e^{+irk}\times f(k).\end{aligned}\tag{S.37}$$

In particular, for the $f(k) = e^{-it\omega(k)} = f(-k)$ we have

$$U(x\to y) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}\times e^{-it\omega(\mathbf{k})} \;=\; \frac{1}{4\pi^2 i\,r}\int\limits_{-\infty}^{+\infty} dk\,k\,e^{+irk}\times e^{-it\omega(\mathbf{k})}. \tag{S.38}$$

precisely as in eq. (11). *Quod erat demonstrandum*

Problem **2**(c):

As explained in my notes on the *saddle point method*, integrals of the form

$$I \;=\; \int\limits_\Gamma dz\,f(z)\,e^{Ag(z)} \tag{S.39}$$

in the large A limit become

$$I \;=\; e^{Ag(z_0)}\times\frac{\sqrt{2\pi}\eta\,f(z_0)}{\sqrt{-\eta^2 Ag''(z_0)}}\times\left(1\,+\,O(A^{-1})\right). \tag{S.40}$$

In general, f and g are complex analytic functions of a complex variable z which is integrated over some contour Γ; quite often Γ is the real axis, but one should allow for its deformation in the complex plane. In eq. (S.40), z_0 is a saddle point of $g(z)$ where the derivative $g'(z_0) = 0$; this saddle point does not have to lie on the original integration contour Γ — if it does not, we deform the contour $\Gamma\to\Gamma'$ so that Γ' does go through the z_0. If several saddle points are present near the contour Γ, the point with the largest $\operatorname{Re} g$ dominates the integral. Finally, η is the direction dz of the Γ' at the saddle point z_0; it should be chosen such that $\operatorname{Re}(-\eta^2 g''(z_0)) \geq 0$, which assures that Γ' crosses z_0 as a mountain path, from a valley to the lowest crossing point to another valley.

For the integral (11) at hand, we identify

$$A \;=\; t, \qquad g(k) \;=\; i\frac{r}{t}\,k \;-\; i\omega(k), \qquad f(k) \;=\; \frac{k}{4\pi^2 i\, r}\,. \tag{S.41}$$

The saddle point in the k plane follows from

$$\frac{dg}{dk} \;\equiv\; i\frac{r}{t} \;-\; i\frac{d\omega}{dk} \;\equiv\; i\frac{r}{t} \;-\; i\frac{k}{\omega} \;=\; 0. \tag{S.42}$$

For $r < t$ this equation has a real solution, namely

$$k_0 \;=\; M \times \frac{r}{\sqrt{t^2 - r^2}}\,, \qquad \omega(k_0) \;=\; M \times \frac{t}{\sqrt{t^2 - r^2}}\,. \tag{S.43}$$

At this point

$$\begin{aligned} Ag(k_0) \;&=\; irk_0 \;-\; it\omega(k_0) \;=\; iM\,\frac{r^2 - t^2}{\sqrt{t^2 - r^2}} \;=\; -iM \times \sqrt{t^2 - r^2}\,, \\ f(k_0) \;&=\; \frac{-iM}{4\pi^2}\,\frac{1}{\sqrt{t^2 - r^2}}\,, \\ Ag''(k_0) \;&\equiv\; \frac{-it\,M^2}{\omega^3(k_0)} \;=\; \frac{(t^2 - r^2)^{3/2}}{iMt^2}\,, \end{aligned} \tag{S.44}$$

and the direction of the integration contour at k_0 should be in the fourth quadrant of the complex plane, $\arg(\eta)$ between 0 and $-\pi/2$; the real-axis contour is marginally OK. Substituting all these data into eq. (S.40) gives us

$$\frac{\sqrt{2\pi}\eta\, f(z_0)}{\sqrt{-\eta^2 A g''(z_0)}} \;=\; \left(\frac{-iM}{2\pi}\right)^{3/2} \times \frac{t}{(t^2 - r^2)^{5/4}} \tag{S.45}$$

and therefore

$$U(x \to y) \;=\; \exp\left(-iM\sqrt{t^2 - r^2}\right) \times \left(\frac{-iM}{2\pi}\right)^{3/2} \times \frac{t}{(t^2 - r^2)^{5/4}} \times \left(1 + O\left(\frac{1}{M\sqrt{t^2 - r^2}}\right)\right) \tag{S.46}$$

in accordance with eq. (13).

Problem **2**(d):

Again, we use the saddle point method to evaluate the integral (11). That is, we identify A, $g(k)$, and $f(k)$ according to eq. (S.41), and solve eq. (S.42) to find the saddle point. But this time, for $r > t$ the saddle point is imaginary

$$k_0 = \frac{iMr}{\sqrt{r^2 - t^2}}, \qquad \omega(k_0) = \frac{iMt}{\sqrt{r^2 - t^2}}, \tag{S.47}$$

so the integration contour must be deformed away from the real axis. At the saddle point (S.47),

$$\begin{aligned} Ag(k_0) &= irk_0 - it\omega(k_0) = -M \times \sqrt{r^2 - t^2}, \\ f(k_0) &= \frac{M}{4\pi^2} \frac{1}{\sqrt{r^2 - t^2}}, \\ Ag''(k_0) &\equiv \frac{-it\,M^2}{\omega^3(k_0)} = +\frac{(r^2 - t^2)^{3/2}}{Mt^2}, \end{aligned} \tag{S.48}$$

all being real, and the deformed contour should cross k_0 in the imaginary direction, $\arg(\eta) = \frac{\pi}{2} \pm \frac{\pi}{4}$. Consequently, in eq. (S.40)

$$\frac{\sqrt{\pi}\eta\, f(z_0)}{\sqrt{-\eta^2 Ag''(z_0)}} = \frac{+iM^{3/2}}{(2\pi)^{3/2}} \times \frac{t}{(r^2 - t^2)^{5/4}} \tag{S.49}$$

and therefore

$$U(\mathbf{x}-\mathbf{y};t) = \exp\left(-M\sqrt{r^2 - t^2}\right) \times \frac{iM^{3/2}}{(2\pi)^{3/2}} \times \frac{t}{(r^2 - t^2)^{5/4}} \times \left(1 + O\left(\frac{1}{M\sqrt{r^2 - t^2}}\right)\right) \tag{S.50}$$

in accordance with eq. (14).

Note that the exponential factor here decays as one goes further outside the future light cone. In other words, the probability of a relativistic particle moving faster than light is exponentially small. But tiny as it is, this probability does not vanish, and this violates the relativistic causality.