Problem **1**(a):

Let's start with the transverse polarizations — $\lambda = \pm 1$ in the helicity basis — For which the polarization 3-vectors $\mathbf{e}_{\mathbf{k},\lambda}$ are $\perp$ $\mathbf{k}$ and hence $\perp$ to the boost velocity $\mathbf{v} = \mathbf{k}/\omega_{\mathbf{k}}$ to the moving particle's frame. Consequently, the 4-vector $(0, \mathbf{e}_{\mathbf{k},\lambda})$ is invariant under the Lorentz boosts in this direction, thus

$$B^\mu_{\ \nu}(0, \mathbf{e}_{\mathbf{k},\lambda})^\nu \;=\; \text{same}(0, \mathbf{e}_{\mathbf{k},\lambda})^\nu \;=\; \text{by the top eq. (2)} \;=\; f^\mu_{\mathbf{k},\lambda}\,. \tag{S.1}$$

OOH, for the longitudinal polarization $\lambda = 0$, the polarization vector $\mathbf{e}_{\mathbf{k},\lambda}$ points in the direction of $\mathbf{k}$ and hence of the Lorentz boost. Consequently, in 2 dimensions spanning the time and the boost direction, the boosted 4-vector becomes

$$\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} \beta\gamma \\ \gamma \end{pmatrix}, \tag{S.2}$$

hence in 4D notations

$$\big(\text{boosted } (0, \mathbf{e}_{\mathbf{k},\lambda})\big)^0 \;=\; \beta\gamma, \qquad \big(\text{boosted } (0, \mathbf{e}_{\mathbf{k},\lambda})\big)^i \;=\; \gamma \mathbf{n}^i_{\mathbf{k}}\,. \tag{S.3}$$

For the boost in question, the velocity is $\mathbf{k}/\omega_{\mathbf{k}}$, hence

$$\gamma \;=\; \frac{\omega_{\mathbf{k}}}{m}\,, \qquad \beta\gamma \;=\; \frac{|\mathbf{k}|}{m}\,, \tag{S.4}$$

and therefore

$$\text{for } \lambda = 0: \quad \big(\text{boosted } (0, \mathbf{e}_{\mathbf{k},\lambda})\big) \;=\; \left(\frac{|\mathbf{k}|}{m}\,,\, \frac{\omega_{\mathbf{k}}\mathbf{n}_{\mathbf{k}}}{m}\right), \tag{S.5}$$

exactly as the $f^\mu_{\mathbf{k},\lambda}$ on second line of eq. (2). Thus altogether, for all 3 polarizations

$$f^\mu_{\mathbf{k},\lambda} \;=\; \big(\text{boosted } (0, \mathbf{e}_{\mathbf{k},\lambda})\big)^\mu. \tag{S.6}$$

Quod erat demonstrandum.

Now consider the normalization equations (4). In the rest frame of the particle — where $f^{\mu}_{\mathbf{k},\lambda} = (0, \mathbf{e}_{\mathbf{k},\lambda})$ — the product of two polarization vectors (for the same $\mathbf{k}$ but different $\lambda' \neq \lambda$) is obviously

$$(0, \mathbf{e}_{\mathbf{k},\lambda})^* \cdot (0, \mathbf{e}'_{\mathbf{k},\lambda}) \;=\; 0 \;-\; \mathbf{e}^*_{\mathbf{k},\lambda} \cdot \mathbf{e}_{\mathbf{k},\lambda'} \;=\; -\delta_{\lambda,\lambda'} \tag{S.7}$$

since the 3 complex unit vectors $\mathbf{e}_{\mathbf{k},\lambda}$ form an orthonormal basis.

As we just saw, the polarization vectors $f^{\mu}_{\mathbf{k},\lambda}$ for $\mathbf{k} \neq 0$ obtain by Lorentz boosting the $(0, \mathbf{e}_{\mathbf{k},\lambda})$ 4-vectors to the moving particle's frame. But since $f^{\mu}_{\mathbf{k},\lambda}$ and $f^{\mu}_{\mathbf{k},\lambda'}$ have the same $\mathbf{k}$, they are both Lorentz-boosted by the same velocity, so their scalar product remains the same:

$$g_{\mu\nu} f^{\mu *}_{\mathbf{k},\lambda} f^{\nu}_{\mathbf{k},\lambda'} \;=\; (0, \mathbf{e}_{\mathbf{k},\lambda})^* \cdot (0, \mathbf{e}'_{\mathbf{k},\lambda}) \;=\; -\delta_{\lambda,\lambda'} \,. \tag{S.8}$$

Quod erat demonstrandum.

Problem **1**(b):

We may prove Lemma 2 by a direct calculation using eqs. (2) for the $f^{\mu}_{\mathbf{k},\lambda}$, but it's easier to use Lemma 1 rephrased as

$$f^{\mu}_{\mathbf{k},\lambda} \;=\; B^{\mu}_{\ \nu}\big(e^{\nu}_{\mathbf{k},\lambda} \;=\; (0, \mathbf{e}_{\mathbf{k},\lambda})^{\nu}\big) \quad \text{for the same boost } B^{\mu}_{\ \nu} \text{ as} \quad k^{\mu} \;=\; B^{\mu}_{\ \nu}\big(k^{\nu}_{\text{rest}} \;=\; (m, \mathbf{0})^{\nu}\big). \tag{S.9}$$

Consequently, it's enough to prove that in the rest frame of the quantum

$$\sum_{\lambda} e^{\mu}_{\mathbf{k},\lambda} e^{*\nu}_{\mathbf{k},\lambda} \;=\; -g^{\mu\nu} \;+\; \frac{k^{\mu}_{\text{rest}} k^{\nu}_{\text{rest}}}{m^2} \,. \tag{S.10}$$

Indeed, eq. (12) immediately follows from eqs. (S.10) via the Lorentz boost (S.9) of both sides of the equation.

So let's verify eq. (S.10). On its LHS, the three 3-vectors $\mathbf{e}_{\lambda}(\mathbf{k})$ for $\lambda = -1, 0, +1$ form an orthonormal basis of the 3D space, which means

$$\mathbf{e}_\lambda(\mathbf{k}) \cdot \mathbf{e}^*_{\lambda'}(\mathbf{k}) \;=\; \delta_{\lambda,\lambda'} \tag{S.11}$$

and also

$$\sum_\lambda e^i_\lambda(\mathbf{k}) e^{*j}_\lambda(\mathbf{k}) \;=\; \delta^{ij}\,. \tag{S.12}$$

In terms of the purely-spatial 4-vectors $e^\mu_{\mathbf{k},\lambda} = (0, \mathbf{e}_{\mathbf{k},\lambda})$, eq. (S.12) becomes

$$\sum_\lambda e^\mu_{\mathbf{k},\lambda} e^{*\nu}_{\mathbf{k},\lambda} \;=\; \begin{cases} 1 & \text{for } \mu=\nu=1,2,3, \\ 0 & \text{for all other } \mu,\nu, \end{cases} \tag{S.13}$$

or in matrix form

$$\sum_\lambda e^\mu_{\mathbf{k},\lambda} e^{*\nu}_{\mathbf{k},\lambda} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{S.14}$$

At the same time, on the RHS of eq. (S.10)

$$\frac{k^\mu_{\text{rest}} k^\nu_{\text{rest}}}{m^2} \;=\; \begin{cases} 1 & \text{for } \mu=\nu=0, \\ 0 & \text{otherwise,} \end{cases} \tag{S.15}$$

and therefore

$$-g^{\mu\nu} \;+\; \frac{k^\mu_{\text{rest}} k^\nu_{\text{rest}}}{m^2} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{S.16}$$

By inspection, this is the same matrix as in eq. (S.14), which verifies

$$\sum_\lambda e^\mu_{\mathbf{k},\lambda} e^{*\nu}_{\mathbf{k},\lambda} \;-\; g^{\mu\nu} \;+\; \frac{k^\mu_{\text{rest}} k^\nu_{\text{rest}}}{m^2} \tag{S.10}$$

and hence the Lemma 2

$$\sum_\lambda f^\mu_{\mathbf{k},\lambda} f^{*\nu}_{\mathbf{k},\lambda} \;=\; -g^{\mu\nu} \;+\; \frac{k^\mu k^\nu}{m^2}\,. \tag{5}$$

Problem **1**(c):

In the operator product $\hat{A}^\mu(x)\hat{A}^\nu(y)$, both factors are linear combinations of the creation and annihilation operators according to the expansion (1). Therefore, the product comprises terms of the form $\hat{a}\hat{a}$, $\hat{a}^\dagger\hat{a}^\dagger$, $\hat{a}^\dagger\hat{a}$, and $\hat{a}\hat{a}^\dagger$. The first three kinds of terms have zero zero matrix elements between vacuum states,

$$\langle 0|\,\hat{a}\hat{a}\,|0\rangle \;=\; \langle 0|\,\hat{a}^\dagger\hat{a}^\dagger\,|0\rangle \;=\; \langle 0|\,\hat{a}^\dagger\hat{a}\,|0\rangle \;=\; 0, \tag{S.17}$$

while for the fourth kind

$$\langle 0|\,\hat{a}_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k}',\lambda'}\,|0\rangle \;= 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\,. \tag{S.18}$$

Consequently,

$$\begin{aligned}
\langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;&= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_\lambda\int\!\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{2\omega'_{\mathbf{k}}}\sum_{\lambda'} e^{-ikx+ik'y} f^\mu_{\mathbf{k},\lambda}f^{*\nu}_{\mathbf{k}',\lambda'}\,\times \\
&\qquad\qquad \times\left(\langle 0|\,\hat{a}_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k}',\lambda'}\,|0\rangle \;=\; 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\right) \\
&= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_\lambda\left[e^{-ik(x-y)}\,f^\mu_{\mathbf{k},\lambda}f^{*\nu}_{\mathbf{k},\lambda}\right]_{k^0=+\omega_{\mathbf{k}}} \\
&= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left[\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)e^{-ik(x-y)}\right]_{k^0=+\omega_{\mathbf{k}}} \\
&= \left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left[e^{-ik(x-y)}\right]_{k^0=+\omega_{\mathbf{k}}} \\
&\equiv \left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)D(x-y),
\end{aligned} \tag{S.19}$$

precisely as in eq. (6).

Problem **1**(d):

Let $\hat{\Phi}(x)$ be a free scalar field of the same mass m as the vector field in question. In class we saw that

$$G_F^{\text{scalar}}(x-y) \stackrel{\text{def}}{=} \langle 0|\,\mathbf{T}\hat\Phi(x)\hat\Phi(y)\,|0\rangle = \begin{cases} D(x-y) & \text{when } x^0>y^0,\\ D(y-x) & \text{when } x^0<y^0.\end{cases} \tag{S.20}$$

We also saw that

$$\partial^0\partial^0 G_F^{\text{scalar}}(x-y) = \langle 0|\,\mathbf{T}\big(\partial^0\partial^0\hat\Phi(x)\big)\hat\Phi(y)\,|0\rangle - i\delta^{(4)}(x-y) \tag{S.21}$$

while the single time derivative of the propagator does not have a δ–singularity. The space derivatives also do not produce δ–singularities, thus for the second-order differential operator

$$\mathcal{Z}^{\mu\nu} \stackrel{\text{def}}{=} -g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2} \tag{S.22}$$

we have

$$\mathcal{Z}_x^{\mu\nu} G_F^{\text{scalar}}(x-y) = \langle 0|\,\mathbf{T}\big(\mathcal{Z}^{\mu\nu}\hat\Phi(x)\big)\hat\Phi(y)\,|0\rangle + \frac{i}{m^2}\,\delta^{\mu,0}\delta^{\nu,0}\delta^{(4)}(x-y). \tag{S.23}$$

For the first term on the RHS here, we have two possibilities, depending on the sign of x^0-y^0. When $x^0>y^0$,

$$\begin{aligned}\langle 0|\,\mathbf{T}\big(\mathcal{Z}^{\mu\nu}\hat\Phi(x)\big)\hat\Phi(y)\,|0\rangle &= \langle 0|\,\big(\mathcal{Z}^{\mu\nu}\hat\Phi(x)\big)\hat\Phi(y)\,|0\rangle\\ &= \mathcal{Z}_x^{\mu\nu}\,\langle 0|\,\hat\Phi(x)\hat\Phi(y)\,|0\rangle\\ &= \mathcal{Z}_x^{\mu\nu} D(x-y)\\ &= \langle 0|\,\hat A^\mu(x)\hat A^\nu(y)\,|0\rangle\end{aligned} \tag{S.24}$$

where the last equality follows from eq. (6). Likewise, when $x^0<y^0$ we have

$$\begin{aligned}\langle 0|\,\mathbf{T}\big(\mathcal{Z}^{\mu\nu}\hat\Phi(x)\big)\hat\Phi(y)\,|0\rangle &= \langle 0|\,\hat\Phi(y)\big(\mathcal{Z}^{\mu\nu}\hat\Phi(x)\big)\,|0\rangle\\ &= \mathcal{Z}_x^{\mu\nu}\,\langle 0|\,\hat\Phi(y)\hat\Phi(x)\,|0\rangle\\ &= \mathcal{Z}_x^{\mu\nu} D(y-x)\\ &= \mathcal{Z}_y^{\nu\mu} D(y-x)\\ &= \langle 0|\,\hat A^\nu(y)\hat A^\mu(x)\,|0\rangle\,.\end{aligned} \tag{S.25}$$

Altogether, we may summarize both cases as

$$\langle 0|\,\mathbf{T}\big(\mathcal{Z}^{\mu\nu}\hat{\Phi}(x)\big)\hat{\Phi}(y)\,|0\rangle \;=\; \langle 0|\,\mathbf{T}\hat{A}^{\mu}(x)\hat{A}^{\nu}(y)\,|0\rangle\,. \tag{S.26}$$

Plugging eq. (S.26) into eq. (S.23) and spelling out the $\mathcal{Z}^{\mu\nu}$ operator, we immediately arrive at

$$\left(-g^{\mu\nu}\;-\;\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)G_F^{\text{scalar}}(x-y) \;=\; \langle 0|\,\mathbf{T}\hat{A}^{\mu}(x)\hat{A}^{\nu}(y)\,|0\rangle + \frac{i}{m^2}\,\delta^{\mu,0}\delta^{\nu,0}\delta^{(4)}(x-y). \tag{8}$$

Quod erat demonstrandum.

Problem **1**(e):

In light of eqs. (8), (9), and (10), we can immediately relate the vector field's Feynman propagator to the scalar propagator of the same mass,

$$G_F^{\mu\nu}(x-y) \;=\; \left(-g^{\mu\nu}\;-\;\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)G_F^{\text{scalar}}(x-y). \tag{S.27}$$

Meanwhile, in class we have derived the momentum-integral formula for the scalar Feynman propagator,

$$G_F^{\text{scalar}}(x-y) \;=\int\!\frac{d^4k}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\,. \tag{S.28}$$

Plugging this formula into the relation (S.27), we obtain

$$\begin{aligned}
G_F^{\mu\nu}(x-y) \;&=\; \left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)G_F^{\text{scalar}}(x-y)\\
&=\; \left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)\int\!\frac{d^4k}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\\
&=\int\!\frac{d^4k}{(2\pi)^4}\left(-g^{\mu\nu}\;+\;\frac{k^\mu k^\nu}{m^2}\right)\times\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\,.
\end{aligned} \tag{11}$$

Quod erat demonstrandum.

Problem **1**(f):

The free massive vector field has classical Lagrangian density

$$
\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}\,F_{\nu\mu}F^{\nu\mu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \\
&= -\tfrac{1}{4}(\partial_\nu A_\mu \,-\, \partial_\mu A_\nu)(\partial^\nu A^\mu \,-\, \partial^\mu A^\nu) \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \\
&= -\tfrac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) \;+\; \tfrac{1}{2}(\partial_\nu A_\mu)(\partial^\mu A^\nu) + \tfrac{1}{2}m^2\,A_\mu A^\mu \\
&= \tfrac{1}{2}\partial_\nu(A_\mu\partial^\nu A^\mu) \;+\; \tfrac{1}{2}A_\mu\partial^2 A^\mu \;-\; \tfrac{1}{2}\partial_\nu(A_\mu\partial^\mu A^\nu) \;-\; \tfrac{1}{2}A_\mu\partial_\nu\partial^\mu A^\nu \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \\
&= \partial_\nu(\text{stuff}) \;+\; \tfrac{1}{2}A_\mu\mathcal{D}^{\mu\nu}A_\nu\,,
\end{aligned}
\tag{S.29}
$$

where

$$
\mathcal{D}^{\mu\nu} \;=\; (\partial^2+m^2)g^{\mu\nu} \;-\; \partial^\mu\partial^\nu. \tag{13}
$$

When we integrate $\int d^4x$ the Lagrangian density (S.29) to obtain the classical action, the total derivative term on the RHS integrates to zero, which leaves us with

$$
S \;=\; \int d^4x\,\mathcal{L} \;=\; \tfrac{1}{2}\int d^4x\,A_\mu\mathcal{D}^{\mu\nu}A_\mu\,, \tag{S.30}
$$

precisely as in eq. (12).

Now let's verify that the Feynman propagator (11) for the massive vector field is a Green's function of the differential operator (13). Using eq. (S.27) for the vector propagator in terms of the scalar propagator and acting on it with the operator $\mathcal{D}^{\mu\nu}$, we obtain

$$
\begin{aligned}
\mathcal{D}^{\mu\nu}G^F_{\nu\lambda}(x-y) &= \Big((\partial^2+m^2)\,g^{\mu\nu} \;-\; \partial^\mu\partial^\nu\Big)\left(-g_{\nu\lambda} \;-\; \frac{1}{m^2}\,\partial_\nu\partial_\lambda\right)G_F^{\text{scalar}}(x-y) \\
&= \left(\begin{aligned} &-\delta^\mu_\lambda\times(\partial^2+m^2) \quad +\,\partial^\mu\partial_\lambda \\ &-\;\frac{\partial^2+m^2}{m^2}\times\partial^\mu\partial_\lambda \;+\;\partial^\mu\times\frac{\partial^2}{m^2}\times\partial_\lambda \end{aligned}\right)\times G_F^{\text{scalar}}(x-y) \\
&= \Big(-\delta^\mu_\lambda\times(\partial^2+m^2) \;+\; 0\times\partial^\mu\partial_\lambda\Big)\times G_F^{\text{scalar}}(x-y) \\
&= -\delta^\mu_\lambda\times\Big((\partial^2+m^2)G_F^{\text{scalar}}(x-y) \;=\; -i\delta^{(4)}(x-y)\Big) \\
&= +i\delta^\mu_\lambda\delta^{(4)}(x-y)
\end{aligned}
\tag{S.31}
$$

in accordance with eq. (14). This means that the Feynman propagator of the massive vector fields is indeed a Green's function of the differential operator (13).

Problem **2**(a):

For the classocal Lagrangian density (15),

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_a)} = \partial^\mu\Phi_a\,, \quad \frac{\partial\mathcal{L}}{\partial\Phi_a} = -m^2\Phi_a - \frac{\lambda}{6}\left(\sum_c \Phi_c^2\right)\times\Phi_a\,, \tag{S.32}$$

hence Euler–Lagrange field equations

$$\forall a: \quad \partial^2\Phi_a + \left(m^2 + \frac{\lambda}{6}\sum_c \Phi_c^2\right)\times\Phi_a = 0. \tag{S.33}$$

For convenience, let me rewrite this equation as

$$\forall a: \quad \partial^2\Phi(x) = -T(x)\Phi_a(x) = 0 \tag{S.34}$$

where

$$T(x) \stackrel{\rm def}{=} m^2 + \frac{\lambda}{6}\sum_c \Phi_c^2(x), \quad \text{same for all } a. \tag{S.35}$$

Now consider the currents (16) and their divergences:

$$\begin{aligned}
\partial_\mu J^\mu_{ab} &= \partial_\mu\Big(\Phi_a\partial^\mu\Phi_b - \Phi_b\partial^\mu\Phi_a\Big) \\
&= \cancel{\partial_\mu\Phi_a\times\partial^\mu\Phi_b} + \Phi_a\times\partial^2\Phi_b - \cancel{\partial_\mu\Phi_b\times\partial^\mu\Phi_a} - \Phi_b\times\partial^2\Phi_a \\
&= \Phi_a\times\partial^2\Phi_b - \Phi_b\times\partial^2\Phi_c \\
&\qquad\langle\!\langle \text{ by the Euler–Lagrange eqs. (S.34) } \rangle\!\rangle \\
&= \Phi_a\times\big(-T\Phi_b\big) - \Phi_b\times\big(-T\Phi_a\big) \\
&\qquad\langle\!\langle \text{ for the same } T \text{ as in eq. (S.35) in both terms } \rangle\!\rangle \\
&= T\times\big(-\Phi_a\Phi_b + \Phi_b\Phi_a\big) = 0.
\end{aligned} \tag{S.36}$$

Thus, as long as the field obey their Euler–Lagrange equations, all the currents (16) are conserved. *Quod erat demonstrandum.*

Problem $\mathbf{2}$(b):

Classically, for each scalar field $\Phi_a(\mathbf{x},t)$ there is a canonically conjugate field

$$\Pi_a(\mathbf{x},t) \;=\; \frac{\delta L}{\delta\dot\Phi_a(\mathbf{x})}\bigg|_t \;=\; \dot\Phi_a(\mathbf{x},t). \tag{S.37}$$

Consequently, the classical Hamiltonian density is

$$\mathcal{H} \;=\; \sum_a \Pi_a\dot\Phi_a \;-\; \mathcal{L} \;=\; \frac{1}{2}\sum_a \Pi_a^2 \;+\; \frac{1}{2}\sum_a\big(\nabla\Phi_a\big)^2 \;+\; \frac{m^2}{2}\sum_a\Phi_a^2 \;+\; \frac{\lambda}{24}\left(\sum_a\Phi_a^2\right)^2 \tag{S.38}$$

while the Poisson brackets involve $\sum_a$ as well as $\int d^3\mathbf{x}$:

$$[[A,B]] \;=\; \int\! d^3\mathbf{x}\sum_a\left(\frac{\delta A}{\delta\Phi_a(x)}\frac{\delta B}{\delta\Pi_a(x)} \;-\; \frac{\delta A}{\delta\Pi_a(x)}\frac{\delta B}{\delta\Phi_a(x)}\right). \tag{S.39}$$

In particular,

$$\big[\big[\Phi_a(\mathbf{x}),\Phi_b(\mathbf{y})\big]\big] \;=\; 0, \quad \big[\big[\Pi_a(\mathbf{x}),\Pi_b(\mathbf{y})\big]\big] \;=\; 0, \quad \big[\big[\Phi_a(\mathbf{x}),\Pi_b(\mathbf{y})\big]\big] \;=\; \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{S.40}$$

Consequently, in the quantum theory the corresponding quantum fields $\hat\Phi_a(\mathbf{x},t)$ and $\hat\Pi_a(\mathbf{x},t)$ obey similar *equal-time commutation relations:*

$$\begin{aligned}
\big[\hat\Phi_a(\mathbf{x},t),\hat\Phi_b(\mathbf{y},\text{same }t)\big] \;&=\; 0,\\
\big[\hat\Pi_a(\mathbf{x},t),\hat\Pi_b(\mathbf{y},\text{same }t)\big] \;&=\; 0,\\
\big[\hat\Phi_a(\mathbf{x},t),\hat\Pi_b(\mathbf{y},\text{same }t)\big] \;&=\; i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}).
\end{aligned} \tag{S.41}$$

And the Hamiltonian operator of the quantum theory follows from the classical Hamiltonian (S.38):

$$\begin{aligned}
\hat H \;&=\; \int\! d^3\mathbf{x}\,\widehat{\mathcal{H}}(\mathbf{x},t) \quad \text{where}\\
\widehat{\mathcal{H}}(\mathbf{x},t) \;&=\; \frac{1}{2}\sum_a\hat\Pi_a^2(\mathbf{x},t) \;+\; \frac{1}{2}\sum_a\big(\nabla\hat\Phi_a(\mathbf{x},t)\big)^2\\
&\quad+\; \frac{m^2}{2}\sum_a\hat\Phi_a^2(\mathbf{x},t) \;+\; \frac{\lambda}{24}\left(\sum_a\hat\Phi_a^2(\mathbf{x},t)\right)^2.
\end{aligned} \tag{S.42}$$

Problem **2**(c):

Applying the Leibniz rule to the equal-time commutators (S.41), we have

$$\begin{aligned}\left[\hat{\Phi}_a(\mathbf{y},t)\hat{\Pi}_b(\mathbf{y},t),\hat{\Phi}_c(\mathbf{x},t)\right] &= \hat{\Phi}_a(\mathbf{y},t)\left[\hat{\Pi}_b(\mathbf{y},t),\hat{\Phi}_c(\mathbf{x},t)\right] + \left[\hat{\Phi}_a(\mathbf{y},t),\hat{\Phi}_c(\mathbf{x},t)\right]\hat{\Pi}_b(\mathbf{y},t)\\ &= \hat{\Phi}_a(\mathbf{y},t)\times(-i)\delta_{bc}\delta^{(3)}(\mathbf{y}-\mathbf{x}) + 0\times\hat{\Pi}_b(\mathbf{y},t)\\ &= -i\delta_{bc}\hat{\Phi}_a(\mathbf{y})\times\delta^{(3)}(\mathbf{y}-\mathbf{x})\end{aligned} \tag{S.43}$$

and likewise

$$\left[\hat{\Phi}_b(\mathbf{y},t)\hat{\Pi}_a(\mathbf{y},t),\hat{\Phi}_c(\mathbf{x},t)\right] = -i\delta_{ac}\hat{\Phi}_b(\mathbf{y},t)\times\delta^{(3)}(\mathbf{y}-\mathbf{x}). \tag{S.44}$$

Hence, for the net charge operator $\hat{Q}_{ab}$ as in eq. (18),

$$\begin{aligned}\left[\hat{Q}_{ab}(t),\hat{\Phi}_c(\mathbf{x},t)\right] &= \int d^3\mathbf{y}\left[\hat{\Phi}_a(\mathbf{y},t)\hat{\Pi}_b(\mathbf{y},t)-\hat{\Phi}_b(\mathbf{y},t)\hat{\Pi}_a(\mathbf{y},t),\hat{\Phi}_c(\mathbf{x},t)\right]\\ &= \int d^3\mathbf{y}\left(-i\delta_{bc}\hat{\Phi}_a(\mathbf{y},t) + i\delta_{ac}\hat{\Phi}_b(\mathbf{y},t)\right)\times\delta^{(3)}(\mathbf{y}-\mathbf{x})\\ &= -i\delta_{bc}\hat{\Phi}_a(\mathbf{x},t) + i\delta_{ac}\hat{\Phi}_b(\mathbf{x},t).\end{aligned} \tag{S.45}$$

Similarly,

$$\begin{aligned}\left[\hat{\Phi}_a(\mathbf{y},t)\hat{\Pi}_b(\mathbf{y},t),\hat{\Pi}_c(\mathbf{x},t)\right] &= \hat{\Phi}_a(\mathbf{y},t)\left[\hat{\Pi}_b(\mathbf{y},t),\hat{\Pi}_c(\mathbf{x},t)\right] + \left[\hat{\Phi}_a(\mathbf{y},t),\hat{\Pi}_c(\mathbf{x},t)\right]\hat{\Pi}_b(\mathbf{y},t)\\ &= \hat{\Phi}_a(\mathbf{y},t)\times 0 + i\delta_{ac}\delta^{(3)}(\mathbf{y}-\mathbf{x})\times\hat{\Pi}_b(\mathbf{y},t)\\ &= +i\delta_{ac}\hat{\Pi}_b(\mathbf{y},t)\times\delta^{(3)}(\mathbf{y}-\mathbf{x})\end{aligned} \tag{S.46}$$

and likewise

$$\left[\hat{\Phi}_b(\mathbf{y},t)\hat{\Pi}_a(\mathbf{y},t),\hat{\Pi}_c(\mathbf{x},t)\right] = +i\delta_{bc}\hat{\Pi}_a(\mathbf{y},t)\times\delta^{(3)}(\mathbf{y}-\mathbf{x}), \tag{S.47}$$

hence

$$\begin{aligned}\left[\hat{Q}_{ab}(t),\hat{\Pi}_c(\mathbf{x})\right] &= \int d^3\mathbf{y}\left[\hat{\Phi}_a(\mathbf{y},t)\hat{\Pi}_b(\mathbf{y},t)-\hat{\Phi}_b(\mathbf{y},t)\hat{\Pi}_a(\mathbf{y},t),\hat{\Pi}_c(\mathbf{x},t)\right]\\ &= \int d^3\mathbf{y}\left(+i\delta_{ac}\hat{\Pi}_b(\mathbf{y},t) - i\delta_{bc}\hat{\Pi}_a(\mathbf{y},t)\right)\times\delta^{(3)}(\mathbf{y}-\mathbf{x})\\ &= -i\delta_{bc}\hat{\Pi}_a(\mathbf{x},t) + i\delta_{ac}\hat{\Pi}_b(\mathbf{x},t).\end{aligned} \tag{S.48}$$

Quod erat demonstrandum.

Problem **2**(d):

The Hamiltonian operator (S.42) is $SO(N)$ invariant — in fact, each of the 4 terms comprising the Hamiltonian density $\widehat{\mathcal{H}}(\mathbf{x},t)$ is separately $SO(N)$ invariant — and that makes them commute with all the $\hat{Q}_{ab}$ charges. Indeed, suppose some N operators $\hat{V}_c$ satisfy commutation relations similar to eqs. (19), namely

$$\left[\hat{Q}_{ab}(t),\hat{V}_c(\text{same }t)\right] \;=\; -i\delta_{bc}\hat{V}_a(t) \;+\; i\delta_{ac}\hat{V}_b(t)\,; \tag{S.49}$$

then the $\sum_c \hat{V}_c^2$ operator commutes with all the charges $\hat{Q}_{ab}$ (at equal times). Here is the proof:

$$\begin{aligned}
\left[\hat{Q}_{ab},\sum\nolimits_c \hat{V}_c^2\right] &= \sum_c \left[\hat{Q}_{ab},\hat{V}_c^2\right] \;=\; \sum_c \left\{\hat{V}_c,\left[\hat{Q}_{ab},\hat{V}_c\right]\right\}\\
&= \sum_c \left\{\hat{V}_c,\left(-i\delta_{bc}\hat{V}_a \;+\; i\delta_{ac}\hat{V}_b\right)\right\}\\
&= -i\left\{\hat{V}_b,\hat{V}_a\right\} \;+\; i\left\{\hat{V}_a,\hat{V}_b\right\}\\
&= 0.
\end{aligned} \tag{S.50}$$

In particular, letting $\hat{V}_c=\hat{\Pi}_c(\mathbf{x})$, or $\hat{V}_c=\hat{\Phi}_c(\mathbf{x})$, or $\hat{V}_c=\nabla\hat{\Phi}_c(\mathbf{x})$ — which also satisfy

$$\left[\hat{Q}_{ab}(t),\nabla\hat{\Phi}_c(\mathbf{x},t)\right] \;=\; \nabla\left[\hat{Q}_{ab}(t),\hat{\Phi}_c(\mathbf{x},t)\right] \;=\; -i\delta_{bc}\nabla\hat{\Phi}_a(\mathbf{x},t) \;+\; i\delta_{ac}\nabla\hat{\Phi}_b(\mathbf{x},t) \tag{S.51}$$

— we immediately obtain

$$\begin{aligned}
\left[\hat{Q}_{ab}(t),\sum\nolimits_c \hat{\Pi}_c^2(\mathbf{x},t)\right] &= 0,\\
\left[\hat{Q}_{ab}(t),\sum\nolimits_c \nabla\hat{\Phi}_c^2(\mathbf{x},t)\right] &= 0,\\
\left[\hat{Q}_{ab}(t),\sum\nolimits_c \hat{\Phi}_c^2(\mathbf{x},t)\right] &= 0,
\end{aligned} \tag{S.52}$$

hence also

$$\left[\hat{Q}_{ab}(t),\left(\sum\nolimits_c \hat{\Phi}_c^2(\mathbf{x},t)\right)^2\right] \;=\; 0, \tag{S.53}$$

and therefore $\left[\hat{Q}_{ab}(t),\hat{H}\right] \;=\; 0.$ *Quod erat demonstrandum.*

Problem **2**(e):

The commutations relations (20) between the charges follow from expanding $\hat{Q}_{cd}$ into quantum fields according to eq. (18) and then using the commutators (19) of those fields with the $\hat{Q}_{ab}$ charge:

$$
\begin{aligned}
\left[\hat{Q}_{ab}, \hat{Q}_{cd}\right] &= \left[\hat{Q}_{ab}, \int d^3\mathbf{x} \left(\hat{\Phi}_c(\mathbf{x})\hat{\Pi}_d(\mathbf{x}) \;-\; \hat{\Phi}_d(\mathbf{x})\hat{\Pi}_c(\mathbf{x})\right)\right] \\
&= \int d^3\mathbf{x} \left[\hat{Q}_{ab}, \left(\hat{\Phi}_c(\mathbf{x})\hat{\Pi}_d(\mathbf{x}) \;-\; \hat{\Phi}_d(\mathbf{x})\hat{\Pi}_c(\mathbf{x})\right)\right] \\
&= \int d^3\mathbf{x} \biggl(\hat{\Phi}_c(\mathbf{x}) \left[\hat{Q}_{ab}, \hat{\Pi}_d(\mathbf{x})\right] \;+\; \left[\hat{Q}_{ab}, \hat{\Phi}_c(\mathbf{x})\right] \hat{\Pi}_d(\mathbf{x}) \\
&\qquad\qquad -\; \hat{\Phi}_d(\mathbf{x}) \left[\hat{Q}_{ab}, \hat{\Pi}_c(\mathbf{x})\right] \;-\; \left[\hat{Q}_{ab}, \hat{\Phi}_d(\mathbf{x})\right] \hat{\Pi}_c(\mathbf{x}) \biggr) \\
&= \int d^3\mathbf{x} \biggl(\hat{\Phi}_c \Bigl(-i\delta_{bd}\hat{\Pi}_a \;+\; i\delta_{ad}\hat{\Pi}_b\Bigr) \;+\; \Bigl(-i\delta_{bc}\hat{\Phi}_a \;+\; i\delta_{ac}\hat{\Phi}_b\Bigr)\hat{\Pi}_d \\
&\qquad\qquad -\; \hat{\Phi}_d \Bigl(-i\delta_{bc}\hat{\Pi}_a \;+\; i\delta_{ac}\hat{\Pi}_b\Bigr) \;-\; \Bigl(-i\delta_{bd}\hat{\Phi}_a \;+\; i\delta_{ad}\hat{\Phi}_b\Bigr)\hat{\Pi}_c \biggr) @\mathbf{x} \\
&= \; -i\delta_{bd} \times \int d^3\mathbf{x} \left(\hat{\Phi}_c\hat{\Pi}_a \;-\; \hat{\Phi}_a\hat{\Pi}_c\right) @\mathbf{x} \;+\; i\delta_{ad} \times \int d^3\mathbf{x} \left(\hat{\Phi}_c\hat{\Pi}_b \;-\; \hat{\Phi}_b\hat{\Pi}_c\right) @\mathbf{x} \\
&\qquad +\; i\delta_{bc} \times \int d^3\mathbf{x} \left(\hat{\Phi}_d\hat{\Pi}_a \;-\; \hat{\Phi}_a\hat{\Pi}_d\right) @\mathbf{x} \;-\; i\delta_{ac} \times \int d^3\mathbf{x} \left(\hat{\Phi}_d\hat{\Pi}_b \;-\; \hat{\Phi}_b\hat{\Pi}_d\right) @\mathbf{x} \\
&= \; -i\delta_{bd} \times \hat{Q}_{ca} \;+\; i\delta_{ad} \times \hat{Q}_{cb} \;+\; i\delta_{bc} \times \hat{Q}_{da} \;-\; i\delta_{ac} \times \hat{Q}_{db} \\
&= \; -i\delta_{bc} \times \hat{Q}_{ad} \;+\; i\delta_{ac} \times \hat{Q}_{bd} \;+\; i\delta_{bd} \times \hat{Q}_{ac} \;-\; i\delta_{ad} \times \hat{Q}_{bc} \,.
\end{aligned}
\tag{S.54}
$$

Quod erat demonstrandum.

Note: since the charges are time independent, the fields in the above formulae may be evaluated at any time t, as long as it's the same time for all the operators.

Problem **3**(a):

In class, we have expanded a single free scalar fields $\hat{\Phi}(\mathbf{x})$ and its canonical conjugate $\hat{\Pi}(x)$ into creation and annihilation operators $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}$, see my notes for details. In the present N-field case, we may proceed exactly like in class, except that the creation and annihilation operators are labeled by the species index $a = 1, \ldots, N$ in addition to the momentum mode

$\mathbf{p}$, thus

$$\begin{aligned}\hat{\Phi}_a(x) &= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left(e^{-ikx}\hat{a}_{\mathbf{k},a} + e^{+ikx}\hat{a}^\dagger_{\mathbf{k},a}\right)^{k^0=+\omega_{\mathbf{k}}},\\ \hat{\Pi}_a(x) &= \partial^0\hat{\Phi}_a(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left(-i\omega_{\mathbf{k}}\,e^{-ikx}\hat{a}_{\mathbf{k},a} + i\omega_{\mathbf{k}}\,e^{+ikx}\hat{a}^\dagger_{\mathbf{k},a}\right)^{k^0=+\omega_{\mathbf{k}}}.\end{aligned} \tag{S.55}$$

Given this expansion of the quantum fields, we may expand integrals of fields bilinears into sums of $\hat{a}\hat{a}$, $\hat{a}\hat{a}^\dagger$, $\hat{a}^\dagger\hat{a}$, and $\hat{a}^\dagger\hat{a}^\dagger$ operators. In particular,

$$\begin{aligned}&\int d^3\mathbf{x}\,\hat{\Phi}_a(\mathbf{x},t)\hat{\Pi}_b(\text{same }\mathbf{x},\text{same }t) =\\ &= \int d^3\mathbf{x}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\int\frac{d^3\mathbf{p}}{2(2\pi)^3}\left(e^{-ikx}\hat{a}_{\mathbf{k},a} + e^{+ikx}\hat{a}^\dagger_{\mathbf{k},a}\right)\left(-ie^{-ipy}\hat{a}_{\mathbf{p},b} + ie^{+ipy}\hat{a}^\dagger_{\mathbf{p},b}\right)\\ &= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\int\frac{d^3\mathbf{p}}{2(2\pi)^3}\begin{pmatrix} -i\hat{a}_{\mathbf{k},a}\hat{a}_{\mathbf{p},b}\times\int d^3\mathbf{x}\,e^{i(\mathbf{k}+\mathbf{p})\mathbf{x}-i(\omega_{\mathbf{k}}+\omega_{\mathbf{p}})t}\\ +i\hat{a}_{\mathbf{k},a}\hat{a}^\dagger_{\mathbf{p},b}\times\int d^3\mathbf{x}\,e^{i(\mathbf{k}-\mathbf{p})\mathbf{x}-i(\omega_{\mathbf{k}}-\omega_{\mathbf{p}})t}\\ -i\hat{a}^\dagger_{\mathbf{k},a}\hat{a}_{\mathbf{p},b}\times\int d^3\mathbf{x}\,e^{i(-\mathbf{k}+\mathbf{p})\mathbf{x}-i(-\omega_{\mathbf{k}}+\omega_{\mathbf{p}})t}\\ +i\hat{a}^\dagger_{\mathbf{k},a}\hat{a}^\dagger_{\mathbf{p},b}\times\int d^3\mathbf{x}\,e^{i(-\mathbf{k}-\mathbf{p})\mathbf{x}-i(-\omega_{\mathbf{k}}-\omega_{\mathbf{p}})t}\end{pmatrix}\\ &= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\int\frac{d^3\mathbf{p}}{2(2\pi)^3}\begin{pmatrix} -i\hat{a}_{\mathbf{k},a}\hat{a}_{\mathbf{p},b}\times(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{p})e^{-2i\omega_{\mathbf{k}}t}\\ +i\hat{a}_{\mathbf{k},a}\hat{a}^\dagger_{\mathbf{p},b}\times(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})\\ -i\hat{a}^\dagger_{\mathbf{k},a}\hat{a}_{\mathbf{p},b}\times(2\pi)^3\delta^{(3)}(-\mathbf{k}+\mathbf{p})\\ +i\hat{a}^\dagger_{\mathbf{k},a}\hat{a}^\dagger_{\mathbf{p},b}\times(2\pi)^3\delta^{(3)}(-\mathbf{k}-\mathbf{p})e^{+2i\omega_{\mathbf{k}}t}\end{pmatrix}\\ &= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\frac{1}{2}\left(i\hat{a}_{\mathbf{k},a}\hat{a}^\dagger_{\mathbf{k},b} - i\hat{a}^\dagger_{\mathbf{k},a}\hat{a}_{\mathbf{k},b}\right)\\ &\quad+\int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\frac{1}{2}\left(-i\hat{a}_{\mathbf{k},a}\hat{a}_{-\mathbf{k},b}\times e^{-2it\omega_{\mathbf{k}}} + i\hat{a}^\dagger_{\mathbf{k},a}\hat{a}^\dagger_{-\mathbf{k},b}\times e^{+2it\omega_{\mathbf{k}}}\right),\end{aligned} \tag{S.56}$$

and likewise

$$\int d^3\mathbf{x}\,\hat\Phi_b(\mathbf{x},t)\hat\Pi_a(\text{same }\mathbf{x},\text{same }t) \;=$$
$$= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(i\hat a_{\mathbf{k},b}\hat a^\dagger_{\mathbf{k},a} \,-\, i\hat a^\dagger_{\mathbf{k},b}\hat a_{\mathbf{k},a}\right)$$
$$+ \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(-i\hat a_{\mathbf{k},b}\hat a_{-\mathbf{k},a}\times e^{-2it\omega_{\mathbf{k}}} \,+\, i\hat a^\dagger_{\mathbf{k},b}\hat a^\dagger_{-\mathbf{k},a}\times e^{+2it\omega_{\mathbf{k}}}\right). \tag{S.57}$$

It is easy to see that the bottommost lines of eqs. (S.57) and (S.56) are exactly the same, — indeed

$$\begin{aligned}
[\text{from eq. (S.57)}] &= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(-i\hat a_{\mathbf{k},b}\hat a_{-\mathbf{k},a}\times e^{-2it\omega_{\mathbf{k}}} \,+\, i\hat a^\dagger_{\mathbf{k},b}\hat a^\dagger_{-\mathbf{k},a}\times e^{+2it\omega_{\mathbf{k}}}\right) \;= \\
&\quad\langle\!\langle \text{ changing integration variable from } \mathbf{k} \text{ to } -\mathbf{k} \,\rangle\!\rangle \\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(-i\hat a_{-\mathbf{k},b}\hat a_{\mathbf{k},a}\times e^{-2it\omega_{\mathbf{k}}} \,+\, i\hat a^\dagger_{-\mathbf{k},b}\hat a^\dagger_{\mathbf{k},a}\times e^{+2it\omega_{\mathbf{k}}}\right) \\
&\quad\langle\!\langle \text{ commuting } \hat a_{-\mathbf{k},b} \text{ with } \hat a_{\mathbf{k},a} \text{ and } \hat a^\dagger_{-\mathbf{k},b} \text{ with } \hat a^\dagger_{\mathbf{k},a} \,\rangle\!\rangle \\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(-i\hat a_{\mathbf{k},a}\hat a_{-\mathbf{k},b}\times e^{-2it\omega_{\mathbf{k}}} \,+\, i\hat a^\dagger_{\mathbf{k},a}\hat a^\dagger_{-\mathbf{k},b}\times e^{+2it\omega_{\mathbf{k}}}\right) \\
&= [\text{from eq. (S.56)}],
\end{aligned} \tag{S.58}$$

— so they cancel out from the difference

$$\hat Q_{ab} \;= \int d^3\mathbf{x}\,\left(\hat\Phi_a(\mathbf{x},t)\hat\Pi_b(\mathbf{x},t) \,-\, \hat\Phi_b(\mathbf{x},t)\hat\Pi_a(\mathbf{x},t)\right). \tag{S.59}$$

Consequently, the expansion of the charges (S.59) into the creation and the annihilation operators comes solely from the second-from-the-bottom lines of eqs. (S.56) and (S.57), thus

$$\begin{aligned}
\hat Q_{ab} &= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(i\hat a_{\mathbf{k},a}\hat a^\dagger_{\mathbf{k},b} \,-\, i\hat a^\dagger_{\mathbf{k},a}\hat a_{\mathbf{k},b}\right) \,-\, \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(i\hat a_{\mathbf{k},b}\hat a^\dagger_{\mathbf{k},a} \,-\, i\hat a^\dagger_{\mathbf{k},b}\hat a_{\mathbf{k},a}\right) \\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(i\hat a_{\mathbf{k},a}\hat a^\dagger_{\mathbf{k},b} \,+\, i\hat a^\dagger_{\mathbf{k},b}\hat a_{\mathbf{k},a} \,-\, i\hat a^\dagger_{\mathbf{k},a}\hat a_{\mathbf{k},b} \,-\, \hat a_{\mathbf{k},b}\hat a^\dagger_{\mathbf{k},a}\right) \\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,\frac12\left(2i\hat a^\dagger_{\mathbf{k},b}\hat a_{\mathbf{k},a} \,+\, i[\hat a_{\mathbf{k},a},\hat a^\dagger_{\mathbf{k},b}] \,-\, 2i\hat a^\dagger_{\mathbf{k},b}\hat a_{\mathbf{k},a} \,-\, i[\hat a_{\mathbf{k},b},\hat a^\dagger_{\mathbf{k},a}]\right).
\end{aligned} \tag{S.60}$$

On the last line here, the two commutators vanish for $a\neq b$, while for $a=b$ they cancel each other. Either way, we may drop them from eq. (S.60), which leaves us with

$$\hat{Q}_{ab} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \Big(i\hat{a}^\dagger_{\mathbf{k},b}\hat{a}_{\mathbf{k},a} - i\hat{a}^\dagger_{\mathbf{k},b}\hat{a}_{\mathbf{k},a} \Big). \tag{21}$$

Quod erat demonstrandum.

<u>Problem **3**(b):</u>

Reversing eqs. (23), we have

$$\begin{aligned} \hat{a}_{\mathbf{k},1} &= \frac{\hat{a}_\mathbf{k} + \hat{b}_\mathbf{k}}{\sqrt{2}}, \quad & \hat{a}_{\mathbf{k},2} &= \frac{-i\hat{a}_\mathbf{k} + i\hat{b}_\mathbf{k}}{\sqrt{2}}, \\ \hat{a}^\dagger_{\mathbf{k},1} &= \frac{\hat{a}^\dagger_\mathbf{k} + \hat{b}^\dagger_\mathbf{k}}{\sqrt{2}}, \quad & \hat{a}^\dagger_{\mathbf{k},2} &= \frac{i\hat{a}^\dagger_\mathbf{k} - i\hat{b}^\dagger_\mathbf{k}}{\sqrt{2}}. \end{aligned} \tag{S.61}$$

Consequently, in the integrand of eq. (20) for the $\hat{Q}_{21}$ we get

$$\begin{aligned} -i\hat{a}^\dagger_{\mathbf{k},2}\hat{a}_{\mathbf{k},1} + i\hat{a}^\dagger_{\mathbf{k},1}\hat{a}_{\mathbf{k},2} &= \frac{-i}{2}(i\hat{a}^\dagger_\mathbf{k} - i\hat{b}^\dagger_\mathbf{k})(\hat{a}_\mathbf{k} + \hat{b}_\mathbf{k}) + \frac{i}{2}(\hat{a}^\dagger_\mathbf{k} + \hat{b}^\dagger_\mathbf{k})(-i\hat{a}_\mathbf{k} + i\hat{b}_\mathbf{k}) \\ &= \frac{1}{2}\Big(\hat{a}^\dagger_\mathbf{k}\hat{a}_\mathbf{k} + \cancel{\hat{a}^\dagger_\mathbf{k}\hat{b}_\mathbf{k}} - \cancel{\hat{b}^\dagger_\mathbf{k}\hat{a}_\mathbf{k}} - \hat{b}^\dagger_\mathbf{k}\hat{b}_\mathbf{k}\Big) \\ &\quad + \frac{1}{2}\Big(\hat{a}^\dagger_\mathbf{k}\hat{a}_\mathbf{k} + \cancel{\hat{b}^\dagger_\mathbf{k}\hat{a}_\mathbf{k}} - \cancel{\hat{a}^\dagger_\mathbf{k}\hat{b}_\mathbf{k}} - \hat{b}^\dagger_\mathbf{k}\hat{b}_\mathbf{k}\Big) \\ &= \hat{a}^\dagger_\mathbf{k}\hat{a}_\mathbf{k} - \hat{b}^\dagger_\mathbf{k}\hat{b}_\mathbf{k} \end{aligned} \tag{S.62}$$

and therefore

$$\hat{Q}_{21} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \Big(\hat{a}^\dagger_\mathbf{k}\hat{a}_\mathbf{k} - \hat{b}^\dagger_\mathbf{k}\hat{b}_\mathbf{k} \Big) = \hat{N}_{\text{particles}} - \hat{N}_{\text{antiparticles}}. \tag{24}$$

Quod erat demonstrandum.

Problem **3**(c):

First, let's verify eqs. (25). Plugging $a=2, b=1$ and ($c=1$ or $c=2$) into eq. (19), we get

$$[\hat{Q}_{21},\hat{\Phi}_1(x)] \;=\; -i\hat{\Phi}_2(x) \;+\; i0, \quad [\hat{Q}_{21},\hat{\Phi}_2(x)] \;=\; -i0 \;+\; i\hat{\Phi}_1(x), \tag{S.63}$$

and consequently

$$\begin{aligned}
[\hat{Q}_{21},\hat{\Phi}(x)] \;&=\; \frac{[\hat{Q}_{21},\hat{\Phi}_1(x)] \,+\, i[\hat{Q}_{21},\hat{\Phi}_2(x)]}{\sqrt{2}} \;=\; \frac{-i\hat{\Phi}_2(x) \,-\, \hat{\Phi}_1(x)}{\sqrt{2}} \;=\; -\hat{\Phi}(x),\\
[\hat{Q}_{21},\hat{\Phi}^\dagger(x)] \;&=\; \frac{[\hat{Q}_{21},\hat{\Phi}_1(x)] \,-\, i[\hat{Q}_{21},\hat{\Phi}_2(x)]}{\sqrt{2}} \;=\; \frac{-i\hat{\Phi}_2(x) \,+\, \hat{\Phi}_1(x)}{\sqrt{2}} \;=\; +\hat{\Phi}^\dagger(x),
\end{aligned} \tag{25}$$

Now let's use the Campbell identity:

$$\begin{aligned}
\exp(i\theta\hat{Q}_{21})\hat{\Phi}(x)\exp(-i\theta\hat{Q}_{21}) \;&=\; \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}[\hat{Q}_{21},[\cdots[\hat{Q}_{21},\hat{\Phi}(x)]\cdots]]_{n\,\mathrm{times}}\\
&=\; \hat{\Phi}(x) \;+\; i\theta[\hat{Q}_{12},\hat{\Phi}(x)] \;+\; \frac{(i\theta)^2}{2}[\hat{Q}_{21},[\hat{Q}_{21},\hat{\Phi}(x)]] \;+\; \cdots.
\end{aligned} \tag{25}$$

From eq. (24) for the $\hat{\Phi}$, it's obvious that the multiple commutators of $\hat{\Phi}(x)$ with the charge $\hat{Q}_{21}$ amount to $\pm\hat{\Phi}(x)$, specifically

$$[\hat{Q}_{21},[\cdots[\hat{Q}_{21},\hat{\Phi}(x)]\cdots]]_{n\,\mathrm{times}} \;=\; (-1)^n\hat{\Phi}(x), \tag{S.64}$$

hence by the Campbell identity

$$\exp(i\theta\hat{Q}_{21})\hat{\Phi}(x)\exp(-i\theta\hat{Q}_{21}) \;=\; \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}\,(-1)^n\hat{\Phi}(x) \;=\; \exp(-i\theta)\times\hat{\Phi}(x). \tag{27.a}$$

Likewise, all multiple commutators of the $\hat{\Phi}^\dagger(x)$ with the charge $\hat{Q}_{21}$ amount to $+\hat{\Phi}^\dagger(x)$,

$$[\hat{Q}_{21},[\cdots[\hat{Q}_{21},\hat{\Phi}^\dagger(x)]\cdots]]_{n\,\mathrm{times}} \;=\; +\hat{\Phi}^\dagger(x), \tag{S.65}$$

hence by the Campbell identity

$$\begin{aligned}\exp(i\theta\hat{Q}_{21})\hat{\Phi}^\dagger(x)\exp(-i\theta\hat{Q}_{21}) &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}[\hat{Q}_{21},[\cdots[\hat{Q}_{21},\hat{\Phi}^\dagger(x)]\cdots]]_{n\,\text{times}} \\ &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}\times\hat{\Phi}^\dagger(x) \;=\; \exp(+i\theta)\times\hat{\Phi}^\dagger(x).\end{aligned} \tag{27.b}$$

Quod erat demonstrandum.

Problem **3**(d):
First of all, let's check that the charge operators $\hat{Q}_{ab}$ are Hermitian for any $a\neq b$: Since the classical fields $\Phi_a(x)$ and $\Pi_a(x)$ are real, their quantum counterparts $\hat{\Phi}_a(x)$ and $\hat{\Phi}_a(x)$ are Hermitian. Moreover, at equal times the fields $\hat{\Phi}_a(x)$ and $\hat{\Pi}_b(x)$ with $a\neq b$ commute with each other, and likewise $\hat{\Phi}_b(x)$ commutes with the $\hat{\Pi}_a(x)$. Consequently, in the integrand of eq. (18)

$$\Big(\hat{\Phi}_a(x)\hat{\Pi}_b(x)\Big)^\dagger \;=\; \hat{\Pi}_b^\dagger(x)\hat{\Phi}_a^\dagger(x) \;=\; \hat{\Pi}_b(x)\hat{\Phi}_a(x) = \;\hat{\Phi}_a(x)\hat{\Pi}_b(x) \quad \langle\!\langle\,\text{for } a\neq b\,\rangle\!\rangle, \tag{S.66}$$

and likewise

$$\Big(\hat{\Phi}_b(x)\hat{\Pi}_a(x)\Big)^\dagger \;=\; \hat{\Phi}_b(x)\hat{\Pi}_a(x) \quad \langle\!\langle\,\text{for } a\neq b\,\rangle\!\rangle. \tag{S.67}$$

Thus the whole integrand of eq. (18) is Hermitian, so the integral is Hermitian too. And this establishes the Hermiticity of all the charge operators $\hat{Q}_{ab}$.

Next, for any real coefficients A_{ab} the combination of the Hermitian charge operators

$$\hat{V} \;=\; \frac{1}{2}\sum_{a,b}A_{ab}\hat{Q}_{ab} \tag{S.68}$$

is itself a Hermitian operator. Consequently, the exponent $\hat{U}=\exp(-i\hat{V})$ is a unitary operator.

Problem $\mathbf{3}$(e):

By the Campbell identity (25), for $\hat{U} = \exp(-i\hat{V})$

$$\hat{U}\hat{\Phi}_a(x)\hat{U}^\dagger \;=\; e^{-i\hat{V}}\hat{\Phi}_a(x)e^{+i\hat{V}} \;=\; \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\,[\hat{V},[\cdots[\hat{V},\hat{\Phi}(x)]]]_{n\,\text{times}}\,, \tag{S.69}$$

so let's calculate the multiple commutators of the $\hat{V}$ operator (S.68) with the quantum field $\Phi_a(x)$. The first commutator follows from eq. (19):

$$\begin{aligned}
[\hat{V},\hat{\Phi}_a(x)] \;&=\; \frac{1}{2}\sum_{bc}A_{bc}[\hat{Q}_{bc},\hat{\Phi}_a(x)] \\
&\qquad\langle\!\langle\text{ note indices } a,b,c \text{ different from eq. (19) }\rangle\!\rangle \\
&=\; \frac{1}{2}\sum_{bc}A_{bc}\times\Big(-i\delta_{ca}\hat{\Phi}_b(x)\;+\;i\delta_{ba}\hat{\Phi}_c(x)\Big) \\
&=\; \frac{-i}{2}\sum_b A_{ba}\hat{\Phi}_b(x)\;+\;\frac{i}{2}\sum_c A_{ac}\hat{\Phi}_c(x) \\
&\qquad\langle\!\langle\text{ by antisymmetry } A_{ba}=-A_{ab}\,\rangle\!\rangle \\
&=\; +i\sum_c A_{ac}\hat{\Phi}_c\,.
\end{aligned} \tag{S.70}$$

The second commutator follows by iterating this formula:

$$\begin{aligned}
[\hat{V},[\hat{V},\hat{\Phi}_a(x)] \;&=\; \left[\hat{V},i\sum_c A_{ac}\hat{\Phi}_c(x)\right] \;=\; i\sum_c A_{ac}[\hat{V},\hat{\Phi}_c(x)] \\
&=\; i\sum_c A_{ac}\times i\sum_d A_{cd}\hat{\Phi}_d(x) \;=\; i^2\sum_d (A^2)_{ad}\hat{\Phi}_d(x),
\end{aligned} \tag{S.71}$$

where A^2 is the matrix square of A. From this formula, it's clear how further iterations for the higher multiple commutators work — they use the higher powers of the matrix A,

$$[\hat{V},[\cdots[\hat{V},\hat{\Phi}_a(x)]]]_{n\,\text{times}} \;=\; i^n\sum_b (A^n)_{ab}\hat{\Phi}_b(x). \tag{S.72}$$

As written, this formula works for all non-negative integers n, even for $n=0$ where

$$\begin{aligned}
A^0 \;=\; 1_{N\times N} \quad&\Longrightarrow\quad (A^0)_{ab}\;=\;\delta_{ab} \quad\Longrightarrow \\
&\Longrightarrow\quad [\hat{V},[\cdots[\hat{V},\hat{\Phi}_a(x)]]]_{0\,\text{times}} \;=\; i^0\sum_b \delta_{ab}\hat{\Phi}_b(x) \;=\; \hat{\Phi}_a(x).
\end{aligned} \tag{S.73}$$

Finally, let's plug all the commutators (S.72) into the Campbell identity (S.69):

$$\begin{aligned} \hat{U}\hat{\Phi}_a(x)\hat{U}^\dagger &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\, i^n \sum_b (A^n)_{ab}\hat{\Phi}_b(x) \\ &= \sum_b \left(\sum_{n=0}^{\infty} \frac{1}{n!}(A^n)_{ab}\right) \times \hat{\Phi}_b(x) \\ &= \sum_b \Big(\exp(A)\Big)_{ab} \hat{\Phi}_b(x) \end{aligned} \tag{S.74}$$

where $\exp(A)$ is the matrix exponential of A. Moreover, we have chosen the antisymmetric matrix A such that its matrix exponential is precisely the desired $SO(N)$ matrix R, thus eq. (S.74) amounts to the $SO(N)$ symmetry transform in the field space,

$$\hat{U}\hat{\Phi}_a(x)\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{\Phi}_a(x). \tag{29}$$

Quod erat demonstrandum.

Problem **3**(f):

All the $SO(N)$ charges $\hat{Q}_{ab}$ commute with the Hamiltonian and also with the net momentum operator $\hat{\mathbf{P}}_{\rm net}$. Consequently, the symmetry operators $\hat{U}$ also commutes with the Hamiltonian and with the net momentum operator. Consequently, when $\hat{U}$ acts on a quantum field $\hat{\Phi}_a(x)$ as in eq. (29), it cannot mix the creation and the annihilation operators comprising the $\hat{\Phi}_a(x)$ with each other unless they carry exactly the same energies and momenta. In particular, $\hat{U}$ cannot mix the creation operators with the annihilation operators since they have opposite signs of energies. Instead, all $\hat{U}$ can do is change the species index of the creation and the annihilation operators, thus

$$\begin{aligned} \hat{U}\hat{a}^\dagger_{\mathbf{p},a}\hat{U}^\dagger &= \text{linear combination of } \hat{a}^\dagger_{\mathbf{p},b} \text{ for the same } \mathbf{p}, \\ \hat{U}\hat{a}_{\mathbf{p},a}\hat{U}^\dagger &= \text{linear combination of } \hat{a}_{\mathbf{p},b} \text{ for the same } \mathbf{p}. \end{aligned} \tag{S.75}$$

In light of eqs. (28) for the quantum fields themselves, the coefficients of these linear combinations must be R_{ab}, thus

$$\hat{U}\hat{a}^\dagger_{\mathbf{p},a}\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{a}^\dagger_{\mathbf{p},b}\,, \quad \hat{U}\hat{a}_{\mathbf{p},a}\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{a}_{\mathbf{p},b}\,. \tag{30}$$

Problem **3**(g):

Eq. (31) follows from eq. (30) for the creation operators and from the way any n-particle state obtains from acting with the creation operators on the vacuum state,

$$|n:(\mathbf{p}_1,a_1),\ldots,(\mathbf{p}_n,a_n)\rangle \;=\; \hat a^\dagger_{\mathbf{p}_1,a_1}\cdots\hat a^\dagger_{\mathbf{p}_n,a_n}\,|0\rangle\,. \tag{S.76}$$

Strictly speaking, if some of the n particles have exactly the same momenta and also are of the same species, we would need some extra normalization factors in this formula. Although eq. (31) holds true even when such normalization factors are taken into account, proving it would take some extra afforst to deal with those factors. So to keep our life simple, we note that in the infinite space exactly equal momenta of several particles are extremely unlikely to happen (probability $=0$), so we assume that all the momenta is distinct. Consequently, none of the n particles are in exactly the same quantum state, so the normalization factor in eq. (S.76) is simply 1.

In light of eq. (21) for the charges, they all annihilate the vacuum state $|0\rangle$, hence $\hat U\,|0\rangle=|0\rangle$. Consequently, for the one-particle states $\hat a^\dagger_{\mathbf{p},a}\,|0\rangle$ we have

$$\hat U\hat a^\dagger_{\mathbf{p},a}\,|0\rangle \;=\; \bigl(\hat U\hat a^\dagger_{\mathbf{p},a}\hat U^\dagger\bigr)\times\hat U\,|0\rangle \;=\; \sum_b R_{ab}\hat a^\dagger_{\mathbf{p},b}\times|0\rangle \quad\Longrightarrow\quad \hat U\,|1(\mathbf{p},a)\rangle \;=\; \sum_b R_{ab}\,|1(\mathbf{p},b)\rangle\,. \tag{S.77}$$

In other words, $\hat U$ rotates the species index of the particle by R but leaves its momentum unchanged.

Likewise, for any n–particle state

$$\begin{aligned}
\hat U\bigl|n:&(\mathbf{p}_1,a_1),(\mathbf{p}_2,a_2),\ldots,(\mathbf{p}_n,a_n)\bigr\rangle \;= \\
&=\; \hat U\hat a^\dagger_{\mathbf{p}_1,a_1}\hat a^\dagger_{\mathbf{p}_2,a_2}\cdots\hat a^\dagger_{\mathbf{p}_n,a_n}\,|0\rangle \\
&=\; \bigl(\hat U\hat a^\dagger_{\mathbf{p}_1,a_1}\hat U^\dagger\bigr)\times\bigl(\hat U\hat a^\dagger_{\mathbf{p}_2,a_2}\hat U^\dagger\bigr)\times\cdot\times\bigl(\hat U\hat a^\dagger_{\mathbf{p}_2,a_2}\hat U^\dagger\bigr)\times\hat U\,|0\rangle \\
&=\; \sum_{b_1}R_{a_1,b_1}\hat a^\dagger_{\mathbf{p}_1,b_1}\times\sum_{b_2}R_{a_2,b_2}\hat a^\dagger_{\mathbf{p}_2,b_2}\times\cdots\times\sum_{b_n}R_{a_n,b_n}\hat a^\dagger_{\mathbf{p}_n,b_n}\times|0\rangle \\
&=\; \sum_{b_1,b_2,\ldots,b_n}R_{a_1,b_1}R_{a_2,b_2}\cdots R_{a_n,b_n}\times\hat a^\dagger_{\mathbf{p}_1,b_1}\hat a^\dagger_{\mathbf{p}_2,b_2}\cdots\hat a^\dagger_{\mathbf{p}_n,b_n}\,|0\rangle \\
&=\; \sum_{b_1,b_2,\ldots,b_n}R_{a_1,b_1}R_{a_2,b_2}\cdots R_{a_n,b_n}\times|n:(\mathbf{p}_1,b_1),(\mathbf{p}_2,b_2),\ldots,(\mathbf{p}_n,b_n)\rangle\,.
\end{aligned} \tag{30}$$

Again, for each particle in the n–particle state its species index is rotated by R but the momentum stays unchanged. *Quod erat demonstrandum.*