Problem **1**(a):

In the Noether's formula (1) for the stress-energy tensor, the ϕ_a stand for independent fields or field components, however they might be labeled. In the electromagnetic case, the independent fields are components of the 4–vector $A_\lambda(x)$, hence

$$\begin{aligned} T^{\mu\nu}_{\text{Noether}}(\text{EM}) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\,\partial^\nu A_\lambda \;-\; g^{\mu\nu}\,\mathcal{L} \\ &= -F^{\mu\lambda}\,\partial^\nu A_\lambda \;+\; \tfrac{1}{4}g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \end{aligned} \tag{S.1}$$

While the second term here is clearly both gauge invariant and symmetric in $\mu \leftrightarrow \nu$, the first term is neither.

Problem **1**(b):

Let $\mathcal{K}^{\lambda\mu,\nu} = -F^{\lambda\mu}A^\nu$ as in eq. (5). Then

$$\partial_\lambda\mathcal{K}^{\lambda\mu,\nu} \;=\; -\big(\partial_\lambda F^{\lambda\mu}\big)A^\nu \;-\; F^{\lambda\mu}\big(\partial_\lambda A^\nu\big) \;=\; -J^\mu A^\nu \;+\; F^{\mu\lambda}\big(\partial_\lambda A^\nu\big), \tag{S.2}$$

where the first term on the RHS vanishes for the free EM fields (*i.e.*, when $J^\mu = 0$). Consequently, for the free EM fields

$$\begin{aligned} T^{\mu\nu}_{\text{phys}} &= T^{\mu\nu}_{\text{Noether}} \;+\; \partial_\lambda\mathcal{K}^{\lambda\mu,\nu} \\ &= -F^{\mu\lambda}\big(\partial^\nu A_\lambda\big) \;+\; \tfrac{1}{4}g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda} \;+\; F^{\mu\lambda}\big(\partial_\lambda A^\nu\big) \\ &= -F^{\mu\lambda}\Big(\partial^\nu A_\lambda \;-\; \partial_\lambda A^\nu\Big) \;+\; \tfrac{1}{4}g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda} \\ &= -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}, \end{aligned} \tag{S.3}$$

exactly as in eq. (4).

Problem $\mathbf{1}$(c):

Let's start with the Lagrangian (3). In component form,

$$F^{i0} = -F^{0i} = E^i, \qquad F^{ij} = -\epsilon^{ijk}B^k. \tag{S.4}$$

Therefore, $F^{i0}F_{i0} = F^{0i}F_{0i} = -E^iE^i$ where the minus sign comes from raising one space index. Likewise, $F^{ij}F_{ij} = +\epsilon^{ijk}B^k\epsilon^{ij\ell}B^\ell = +2B^kB^k$ where the plus sign comes from raising two space indices at once. Thus,

$$\mathcal{L} = -\tfrac{1}{4}\Big(F^{\mu\nu}F_{\mu\nu} = F^{i0}F_{i0} + F^{0i}F_{0i} + F^{ij}F_{ij}\Big) = \tfrac{1}{2}\big(\mathbf{E}^2 - \mathbf{B}^2\big). \tag{S.5}$$

Now let's work out the stress-energy tensor (4) in components. For the energy density component T^{00}, we have

$$\mathcal{U} \equiv T^{00} = -F^{0i}F^{0}_{\ i} - \mathcal{L} = +\mathbf{E}^2 - \tfrac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = \tfrac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \tag{S.6}$$

in agreement with the standard electromagnetic formulæ (in the *rationalized* $c = 1$ units).

Likewise, the energy flux and the momentum density are

$$S^i \equiv T^{i0} = T^{0i} = -F^{0j}F^{i}_{\ j} = -(-E^j)(+\epsilon^{ijk}B^k) = +\epsilon^{ijk}E^jB^k = (\mathbf{E}\times\mathbf{B})^i, \tag{S.7}$$

in agreement with the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{B}$ (again, in the *rationalized* $c = 1$ units).

Finally, the (3–dimensional) stress tensor is

$$\begin{aligned} T^{ij}_{\mathrm{EM}} &= -F^{i\lambda}F^{j}_{\ \lambda} - g^{ij}\mathcal{L} = -F^{i0}F^{j}_{\ 0} - F^{ik}F^{j}_{\ k} + \delta^{ij}\mathcal{L} \\ &= -E^iE^j + \epsilon^{ik\ell}B^\ell\,\epsilon^{jkm}B^m + \tfrac{1}{2}\delta^{ij}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \\ &= -E^iE^j + \left(\delta^{ij}B^\ell B^\ell - B^iB^j\right) + \tfrac{1}{2}\delta^{ij}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \\ &= -E^iE^j - B^iB^j + \tfrac{1}{2}\delta^{ij}\left(\mathbf{E}^2 + \mathbf{B}^2\right). \end{aligned} \tag{S.8}$$

Again, this is the well-known Maxwell stress tensor for the EM fields (in the rationalized $c = 1$ units), except for the overall sign convention: In our convention, positive EM stress is compression rather than tension, thus EM pressure

$$P = +\tfrac{1}{3}T^{ii} = +\tfrac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right), \tag{S.9}$$

while most EM books define positive stress to be tension rather than compression, hence

$$T^{ij}_{\rm EM} = +E^iE^j + B^iB^j - \tfrac{1}{2}\delta^{ij}\left(\mathbf{E}^2+\mathbf{B}^2\right) \tag{S.10}$$

$$\text{but } P = -\tfrac{1}{3}T^{ii} = +\tfrac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right). \tag{S.11}$$

Problem **1**(d):

In a sense, eq. (7) follows from part (b), but it is just as easy to derive it directly from the Maxwell equations. Starting with eq. (4), we immediately have

$$\partial_\mu T^{\mu\nu}_{\rm EM} = -\big(\partial_\mu F^{\mu\lambda}\big)F^\nu_{\ \lambda} - F^{\mu\lambda}\big(\partial_\mu F^\nu_{\ \lambda}\big) + \tfrac{1}{2}F_{\kappa\lambda}\big(\partial^\nu F^{\kappa\lambda}\big), \tag{S.12}$$

where the last two terms cancel each other. Indeed, we may rewrite the second term here as

$$\begin{aligned}
\text{second term} &= -F^{\mu\lambda}\times\partial_\mu F^\nu_{\ \lambda} = -F_{\mu\lambda}\times\partial^\mu F^{\nu\lambda} \\
&\quad\langle\!\langle\, \text{exchanging summation indices } \mu\leftrightarrow\lambda \,\rangle\!\rangle \\
&= -F_{\lambda\mu}\times\partial^\lambda F^{\nu\mu} = +F_{\mu\lambda}\times\partial^\lambda F^{\nu\mu} \\
&= \textit{average_of}\Big(+F_{\mu\lambda}\times\partial^\lambda F^{\nu\mu},\, -F_{\mu\lambda}\times\partial^\mu F^{\nu\lambda}\Big) \\
&= \tfrac{1}{2}F_{\mu\lambda}\Big(\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu}\Big),
\end{aligned} \tag{S.13}$$

hence combining the second and the third term on the RHS of eq. (S.12) gives us

$$\begin{aligned}
\text{second term} + \text{third term} &= \tfrac{1}{2}F_{\mu\lambda}\Big(\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu}\Big) + \tfrac{1}{2}F_{\kappa\lambda}\big(\partial^\nu F^{\kappa\lambda}\big) \\
&\quad\langle\!\langle\, \text{renaming the summation index } \kappa\to\mu \text{ in the third term} \,\rangle\!\rangle \\
&= \tfrac{1}{2}F_{\mu\lambda}\Big(\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda}\Big) \\
&= 0
\end{aligned} \tag{S.14}$$

thanks to the homogeneous Maxwell equation

$$\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} = 0. \tag{S.15}$$

Thus, on the RHS of eq. (S.12) the second and the the third term cancel each other and we are left with the first term only:

$$\partial_\mu T^{\mu\nu}_{\text{EM}} = -\big(\partial_\mu F^{\mu\lambda}\big)F^\nu_{\ \lambda} = -\,J^\lambda\,F^\nu_{\ \lambda} \tag{S.16}$$

where the second equality comes from the inhomogeneous Maxwell equation $\partial_\mu F^{\mu\lambda} = J^\lambda$. This proves eq. (7). Finally, eq. (8) follows from eq. (7) and the net stress-energy conservation (6). *Quod erat demonstrandum.*

Problem **1**(e):

For $\nu = 0$, on the LHS of eq. (7) we have

$$\partial_\mu T^{\mu 0} = \partial_0 T^{00} + \partial_i T^{i0} = \frac{\partial\mathcal{U}}{\partial t} + \nabla\cdot\mathbf{S}, \tag{S.17}$$

while on the RHS of eq. (7) we have

$$-J_\lambda F^{0\lambda} = +J^i F^{0i} = -\mathbf{J}\cdot\mathbf{E}. \tag{S.18}$$

Thus, eq. (7) for $\nu = 0$ is the *Poynting theorem*

$$\frac{\partial\mathcal{U}}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}, \tag{S.19}$$

which is the local form of the work-energy theorem for the EM fields: The rate of the EM energy's non-conservation is equal to the power expended by the EM forces on the electric current $\mathbf{J}$.

Now consider eq. (7) for $\nu = i = 1, 2, 3$. On the LHS, we have

$$\partial_\mu T^{\mu i} = \partial_0 T^{0i} + \partial_j T^{ji} = \frac{\partial S^i}{\partial t} + \nabla_j T^{ji}, \tag{S.20}$$

which is the local non-conservation of the i^{th} component of the EM momentum. Indeed, the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{B}$ gives not only the flux of the EM energy but also the density of the EM momentum, while the stress-tensor T^{ij} (in our sign convention) gives the flux of the EM momentum. Physically, this non-conservation is due to mechanical forces between the EM fields and the currents, so we should have

$$\frac{\partial S^i}{\partial t} + \nabla^j T^{ji} = -f^i, \tag{S.21}$$

where $\mathbf{f}$ is the density of the net EM forces on the charges and the currents,

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}. \tag{S.22}$$

And indeed, on the RHS of eq. (7) for $\nu = i$ we have

$$-J_\lambda F^{i\lambda} = -J^0 F^{i0} + J^j F^{ij} = -\rho E^i - \epsilon^{ijk} J^j B^k = -\big(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\big)^i = -f^i, \tag{S.23}$$

in perfect agreement with eq. (S.21). In other words, the $\nu = 1,2,3$ components of eq. (7) give the local form of the momentum-impulse theorem for the EM field.

Problem **2**(a):

As discussed in class — see my notes on gauge symmetries (pages 5–6), — for the EM field coupled to any kinds of charged scalars with a Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_a (D_\mu\Phi_a^*)(D^\mu\Phi_a) - V(\text{scalars}), \tag{S.24}$$

the Euler–Lagrange equations for the charged fields are

$$D^\mu D_\mu \Phi_a + \frac{\partial V}{\partial \Phi_a^*} = 0, \qquad D^\mu D_\mu \Phi_a^* + \frac{\partial V}{\partial \Phi_a} = 0, \tag{S.25}$$

while the electric current is

$$J^\mu = \sum_a \Big(-iq_a \Phi_a D^\mu \Phi_a^* + iq_a \Phi_a^* D^\mu \Phi_a\Big). \tag{S.26}$$

In particular, for the theory at hand there is only one charged scalar field Φ (and its conjugate Φ^*), the Lagrangian density is as in eq. (9), so the Euler–Lagrange equations for the charged fields are

$$D_\mu D^\mu \Phi \;+\; m^2 \Phi \;=\; 0 \;=\; D_\mu D^\mu \Phi^* \;+\; m^2 \Phi^*, \tag{S.27}$$

while the electric current is simply

$$J^\mu \;=\; -iq \Phi D^\mu \Phi^* \;+\; iq \Phi^* D^\mu \Phi. \tag{S.28}$$

This current is manifestly gauge invariant, and is conserved, $\partial_\mu J^\mu = 0$ when the scalar fields obey their equation of motion (S.27). Indeed, by the Leibniz rule for the covariant derivatives

$$\begin{aligned} \partial_\mu \big(\Phi D^\mu \Phi^*\big) \;&=\; (D_\mu \Phi)(D^\mu \Phi^*) \;+\; \Phi (D_\mu D^\mu \Phi^*) \\ \langle\!\langle\, \text{by eq. (S.27)} \,\rangle\!\rangle \qquad &=\; (D_\mu \Phi)(D^\mu \Phi^*) \;+\; \Phi(-m^2 \Phi^*), \end{aligned} \tag{S.29}$$

$$\begin{aligned} \partial_\mu \big(\Phi^* D^\mu \Phi\big) \;&=\; (D_\mu \Phi^*)(D^\mu \Phi) \;+\; \Phi^* (D_\mu D^\mu \Phi) \\ \langle\!\langle\, \text{by eq. (S.27)} \,\rangle\!\rangle \qquad &=\; (D_\mu \Phi^*)(D^\mu \Phi) \;+\; \Phi^*(-m^2 \Phi), \end{aligned} \tag{S.30}$$

hence

$$\partial_\mu J^\mu \;=\; -\cancel{iq (D_\mu \Phi)(D^\mu \Phi^*)} \;+\; \cancel{iqm^2 \Phi \Phi^*} \;+\; \cancel{iq (D_\mu \Phi^*)(D^\mu \Phi)} \;-\; \cancel{iqm^2 \Phi^* \Phi} \;=\; 0. \tag{S.31}$$

Problem **2**(b):

According to the Noether theorem,

$$\begin{aligned} T^{\mu\nu}_{\text{Noether}} \;&=\; \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\, \partial^\nu A_\lambda \;+\; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\, \partial^\nu \Phi \;+\; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)}\, \partial^\nu \Phi^* \;-\; g^{\mu\nu}\, \mathcal{L} \\ &=\; T^{\mu\nu}_{\text{Noether}}(\text{EM}) \;+\; T^{\mu\nu}_{\text{Noether}}(\text{matter}) \end{aligned} \tag{S.32}$$

where

$$T^{\mu\nu}_{\text{Noether}}(\text{EM}) \;=\; -F^{\mu\lambda}\, \partial^\nu A_\lambda \;+\; \tfrac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda} \tag{S.33}$$

similar to the free EM fields, and

$$T^{\mu\nu}_{\text{Noether}}(\text{matter}) \;=\; D^\mu \Phi^*\, \partial^\nu \Phi \;+\; D^\mu \Phi\, \partial^\nu \Phi^* \;-\; g^{\mu\nu} \big(D^\lambda \Phi^*\, D_\lambda \Phi \;-\; m^2 \Phi^* \Phi \big). \tag{S.34}$$

Both terms on the second line of eq. (S.32) lack $\mu \leftrightarrow \nu$ symmetry and gauge invariance and thus need $\partial_\lambda \mathcal{K}^{\lambda\mu,\nu}$ corrections for some $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$. We would like to show that the same $\mathcal{K}^{\lambda\mu,\nu} = -F^{\lambda\mu}A^\nu$ we used to improve the free electromagnetic stress-energy tensor will now symmetrize both the $T^{\mu\nu}_{\rm EM}$ and $T^{\mu\nu}_{\rm mat}$ at the same time!

Indeed, we saw in problem 1(b) that

$$\partial_\lambda \mathcal{K}^{\lambda\mu,\nu} \;=\; -J^\mu A^\nu \;+\; F^{\mu\lambda}\bigl(\partial_\lambda A^\nu\bigr) \tag{S.2}$$

where the second term symmetrizes the EM stress-energy tensor,

$$T^{\mu\nu}_{\rm Noether}({\rm EM}) \;+\; (\text{second term in eq. (S.2)}) \;=\; T^{\mu\nu}_{\rm phys}({\rm EM}) \;=\; T^{\nu\mu}_{\rm phys}({\rm EM}). \tag{S.3}$$

At the same time, the first term in eq. (S.2) — namely the $-J^\mu A^\nu$ term — happens to symmetrize the matter stress-energy tensor. Indeed, consider the difference

$$\begin{aligned}
\Delta T^{\mu\nu}_{\rm matter} \;&\equiv\; T^{\mu\nu}_{\rm phys}(\text{matter}) \;-\; T^{\mu\nu}_{\rm Noether}(\text{matter}) \\
&=\; D^\mu\Phi^*(D^\nu\Phi - \partial^\nu\Phi) \;+\; D^\mu\Phi(D^\nu\Phi^* - \partial^\nu\Phi^*) \\
&=\; D^\mu\Phi^*(iqA^\nu\Phi) \;+\; D^\mu\Phi(-iqA^\nu\Phi^*) \\
&=\; -A^\nu\times\Bigl(iq\Phi^*D^\mu\Phi \;-\; iq\Phi D^\mu\Phi^*\Bigr) \\
&=\; -A^\nu J^\mu \;=\; \text{first term in eq. (S.2)},
\end{aligned} \tag{S.35}$$

which immediately tells us that

$$T^{\mu\nu}_{\rm Noether}(\text{matter}) \;+\; (\text{first term in eq. (S.2)}) \;=\; T^{\mu\nu}_{\rm phys}(\text{matter}) \;=\; T^{\nu\mu}_{\rm phys}(\text{matter}). \tag{S.36}$$

Altogether, the divergence of the $\mathcal{K}^{\lambda\mu,\nu}$ as in eq. (2) symmetrizes the net stress-energy tensor (12), *quod erat demonstrandum.*

Problem **2**(c):

Since the fields $\Phi(x)$ and $\Phi^*(x)$ have opposite electric charges, their product is neutral and therefore

$$\partial_\mu(\Phi^*\Phi) = D_\mu(\Phi^*\Phi) \;=\; (D_\mu\Phi^*)\Phi + \Phi^*(D_\mu\Phi). \tag{S.37}$$

In a similar manner

$$\begin{aligned}
\partial_\mu\left((D^\mu\Phi^*)\,(D^\nu\Phi)\right) \;&=\; (D_\mu D^\mu\Phi^*)\,(D^\nu\Phi) \;+\; (D^\mu\Phi^*)\,(D_\mu D^\nu\Phi)\\
&=\; \left(D^2\Phi^*\right)\left(D^\nu\Phi\right) \;+\; \left(D_\mu\Phi^*\right)\left(D^\mu D^\nu\Phi \;=\; D^\nu D^\mu\Phi \;+\; [D^\mu,D^\nu]\Phi\right)\\
&\qquad \langle\!\langle\, \text{using equation of motion } D^2\Phi^* = -m^2\Phi^* \text{ for the first term} \,\rangle\!\rangle\\
&\qquad\quad \langle\!\langle\, \text{and } [D^\mu,D^\nu]\Phi = iqF^{\mu\nu}\Phi \text{ for the second term} \,\rangle\!\rangle\\
&=\; \left(-m^2\Phi^*\right)\left(D^\nu\Phi\right) \;+\; \left(D_\mu\Phi^*\right)\left(D^\nu D^\mu\Phi\right) \;+\; \left(D_\mu\Phi^*\right)\left(iqF^{\mu\nu}\Phi\right),
\end{aligned} \tag{S.38}$$

and likewise

$$\begin{aligned}
\partial_\mu\left((D^\mu\Phi)\,(D^\nu\Phi^*)\right) \;&=\; (D_\mu D^\mu\Phi)\,(D^\nu\Phi^*) \;+\; (D^\mu\Phi)\,(D_\mu D^\nu\Phi^*)\\
&=\; \left(-m^2\Phi\right)\left(D^\nu\Phi^*\right) \;+\; \left(D_\mu\Phi\right)\left(D^\nu D^\mu\Phi^*\right) \;+\; \left(D_\mu\Phi\right)\left(-iqF^{\mu\nu}\Phi^*\right).
\end{aligned} \tag{S.39}$$

Finally,

$$\begin{aligned}
\partial_\mu\left[-g^{\mu\nu}\left(D_\lambda\Phi^*\,D^\lambda\Phi \;-\; m^2\,\Phi^*\Phi\right)\right] \;&=\; -\partial^\nu\left(D_\lambda\Phi^*\,D^\lambda\Phi\right) \;+\; m^2\partial^\nu\left(\Phi^*\Phi\right)\\
&=\; -\left(D^\nu D^\mu\Phi^*\right)\left(D_\mu\Phi\right) \;-\; \left(D_\mu\Phi^*\right)\left(D^\nu D^\mu\Phi\right)\\
&\qquad +\; m^2\Phi\,\left(D^\nu\Phi^*\right) \;+\; m^2\Phi^*\,\left(D^\nu\Phi\right).
\end{aligned} \tag{S.40}$$

Together, the left hand sides of eqs. (S.38), (S.39) and (S.40) comprise $\partial_\mu T^{\mu\nu}_{\rm mat}$ — *cf.* eq. (13). On the other hand, totaling up the right hand sides of these three equations results in massive cancellation of all terms except those containing the gauge field strength tensor $F^{\mu\nu}$. Therefore,

$$\begin{aligned}
\partial_\mu T^{\mu\nu}_{\rm mat} \;&=\; (D_\mu\Phi^*)\,(iqF^{\mu\nu}\Phi) \;+\; (D_\mu\Phi)\,(-iqF^{\mu\nu}\Phi^*) \;+\; \text{massive cancellation}\\
&=\; F^{\mu\nu}\left(iq\Phi\,D_\mu\Phi^* \;-\; iq\Phi^*\,D_\mu\Phi\right)\\
&=\; F^{\mu\nu}\times\left(-J_\mu\right) \;=\; +F^{\nu\lambda}J_\lambda
\end{aligned} \tag{S.41}$$

in accordance with eq. (15).

Finally, combining this formula with eq. (7) immediately shows us that the net stress-energy tensor (13) is conserved,

$$\partial_\mu T^{\mu\nu}_{\rm tot} \;=\; \partial_\mu T^{\mu\nu}_{\rm tot} \;+\; \partial_\mu T^{\mu\nu}_{\rm EM} \;=\; 0. \tag{S.42}$$

Quod erat demonstrandum.

Problem **3**(a):

Classically,

$$\frac{d}{dt}\mathbf{L}_{\rm mech} \;=\; \mathbf{v}\times\vec{\pi} \;+\; \mathbf{x}\times\mathbf{F} \;=\; 0 \;+\; \mathbf{x}\times\mathbf{F} \tag{S.43}$$

where $\mathbf{F}$ is the net force on the charged particle. In presence of the EM fields (16), this force is

$$\mathbf{F} \;=\; q\mathbf{E} \;+\; \frac{q}{c}\mathbf{v}\times\mathbf{B} \;=\; \frac{qQ}{r^2}\,\mathbf{n} \;+\; \frac{qM}{cr^2}\,\mathbf{v}\times\mathbf{n}, \tag{S.44}$$

hence

$$\frac{d}{dt}\mathbf{L}_{\rm mech} \;=\; (\mathbf{x}=r\mathbf{n})\times\mathbf{F} \;=\; 0 \;+\; \frac{qM}{cr}\,\mathbf{n}\times(\mathbf{v}\times\mathbf{n}) \;=\; \frac{qM}{cr}\big(\mathbf{v} \;-\; (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\big). \tag{S.45}$$

At the same time,

$$\frac{d}{dt}\mathbf{J}_{\rm EM} \;=\; -\frac{qM}{c}\,\frac{d\mathbf{n}}{dt} \;=\; -\frac{qM}{c}\,\frac{\mathbf{v} \;-\; (\mathbf{v}\cdot\mathbf{n})\mathbf{n}}{r}\,. \tag{S.46}$$

By inspection of the last two formulae, the separate angular momenta $\mathbf{L}_{\rm mech}$ and $\mathbf{J}_{\rm EM}$ are not conserved, but the net angular momentum (17) is conserved,

$$\frac{d}{dt}\mathbf{J}_{\rm net} \;=\; \frac{d}{dt}\mathbf{L}_{\rm mech} \;+\; \frac{d}{dt}\mathbf{J}_{\rm EM} \;=\; 0. \tag{S.47}$$

Quod erat demonstrandum.

Problem **3**(b):

Let's start by verifying eq. (22). Since the 3 coordinate operators $\hat{x}_i$ commute with each other, we have

$$[\hat{x}_i, \hat{J}_j^{\mathrm{EM}}] \;=\; 0 \tag{S.48}$$

and therefore

$$\begin{aligned}
[\hat{x}_i, \hat{J}_j] \;&=\; [\hat{x}_i, \hat{L}_j] \\
&=\; [\hat{x}_i, \epsilon_{jk\ell}\hat{x}_k\hat{\pi}_\ell] \;=\; \epsilon_{jk\ell}\hat{x}_k[\hat{x}_i, \hat{\pi}_\ell] \\
&=\; \epsilon_{jk\ell}\hat{x}_k \times i\hbar\delta_{i\ell} \;=\; i\hbar\epsilon_{jki}\hat{x}_k \\
&=\; i\hbar\epsilon_{ijk}\hat{x}_k\,.
\end{aligned} \tag{S.49}$$

Verifying eq. (23) takes more work. First,

$$\begin{aligned}
[\hat{\pi}_i, \hat{L}_j] \;&=\; [\hat{\pi}_i, \epsilon_{jk\ell}\hat{x}_k\hat{\pi}_\ell] \\
&=\; \epsilon_{jk\ell}[\hat{\pi}_i, \hat{x}_k] \times \hat{\pi}_\ell \;+\; \epsilon_{jk\ell}\hat{x}_k \times [\hat{\pi}_i, \hat{\pi}_\ell] \\
&=\; \epsilon_{jk\ell} \times -i\hbar\delta_{ik} \times \hat{\pi}_\ell \;+\; \epsilon_{jk\ell}\hat{x}_k \times \frac{iqM\hbar}{c}\,\epsilon_{i\ell m}\frac{\hat{x}_m}{\hat{r}^3} \\
&=\; -i\hbar\epsilon_{ji\ell}\hat{\pi}_\ell \;+\; \frac{iqM\hbar}{c}\,(\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{mk})\,\frac{\hat{x}_k\hat{x}_m}{\hat{r}^3} \\
&=\; +i\hbar\epsilon_{ij\ell}\hat{\pi}_\ell \;+\; \frac{iqM\hbar}{c}\,\frac{\hat{n}_i\hat{n}_j - \delta_{ij}}{\hat{r}}
\end{aligned} \tag{S.50}$$

where $\hat{n}_i \stackrel{\mathrm{def}}{=} \hat{x}_i/\hat{r}$. On the bottom line of this formula, the first term is precisely what we want in eq. (23), but the second term is something we do not want. Fortunately, this second term is canceled by the commutator of $\hat{\pi}_i$ with the other part of the net angular momentum,

$$\begin{aligned}
[\hat{\pi}_i, \hat{J}_j^{\mathrm{EM}}] \;&=\; -\frac{qM}{c}\,[\hat{\pi}_i, \hat{n}_j] \\
&=\; -\frac{qM}{c} \times -i\hbar\widehat{\frac{\partial n_j}{\partial x_i}} \\
&=\; +\frac{i\hbar qM}{c} \times \frac{\delta_{ij} - \hat{n}_i\hat{n}_j}{\hat{r}}\,.
\end{aligned} \tag{S.51}$$

Thus altogether,

$$[\hat{\pi}_i, \hat{J}_j^{\mathrm{net}}] \;=\; [\hat{\pi}_i, \hat{L}_j] \;+\; [\hat{\pi}_i, \hat{J}_j^{\mathrm{EM}}] \;=\; +i\hbar\epsilon_{ij\ell}\hat{\pi}_\ell \;+\; 0, \tag{S.52}$$

precisely as in eq. (23).

Finally, eq. (24) follows from eqs. (22) and (23). Indeed, eq. (22) implies that $\hat{r}^2 = \hat{x}_i\hat{x}_i$ and hence $\hat{r}$ commute with the $\hat{J}_j$, and therefore

$$[\hat{J}_i^{\rm EM},\hat{J}_j] \;=\; -\frac{qM}{c}\left[\frac{\hat{x}_i}{\hat{r}},\hat{J}_j\right] \;=\; -\frac{qM}{c}\,\frac{1}{\hat{r}}\,[\hat{x}_i,\hat{J}_j] \;=\; -\frac{qM}{c}\,\frac{1}{\hat{r}}\times i\hbar\epsilon_{ijk}\hat{x}_k \;=\; i\hbar\epsilon_{ijk}\hat{J}_k^{\rm EM}. \quad \text{(S.53)}$$

At the same time, eqs. (22) and (23) together lead to

$$\begin{aligned}
[\hat{L}_i,\hat{J}_j] \;&=\; [\epsilon_{ik\ell}\hat{x}_k\hat{\pi}_\ell,\hat{J}_j] \\
&=\; \epsilon_{ik\ell}\hat{x}_k\times\Big([\hat{\pi}_\ell,\hat{J}_j] \;=\; i\hbar\epsilon_{\ell jm}\hat{\pi}_m\Big) \;+\; \epsilon_{ik\ell}\Big([\hat{x}_k,\hat{J}_j] \;=\; i\hbar\epsilon_{kjn}\hat{x}_n\Big)\times\hat{\pi}_\ell \qquad \text{(S.54)}\\
&=\; i\hbar\hat{x}_n\hat{\pi}_m\times\Big(\delta_{kn}\epsilon_{ik\ell}\epsilon_{\ell jm} \;+\; \delta_{\ell m}\epsilon_{ik\ell}\epsilon_{kjn}\Big)
\end{aligned}$$

where

$$\begin{aligned}
\delta_{kn}\epsilon_{ik\ell}\epsilon_{\ell jm} \;+\; \delta_{\ell m}\epsilon_{ik\ell}\epsilon_{kjn} \;&=\; \big(\delta_{ij}\delta_{nm}-\delta_{im}\delta_{nj}\big) \;+\; \big(\delta_{mj}\delta_{ni}-\delta_{ji}\delta_{mn}\big) \\
&=\; \delta_{mj}\delta_{ni} \;-\; \delta_{im}\delta_{nj} \qquad \text{(S.55)}\\
&=\; \epsilon_{ijk}\epsilon_{knm}\,,
\end{aligned}$$

hence

$$\begin{aligned}
[\hat{L}_i,\hat{J}_j] \;&=\; i\hbar\hat{x}_n\hat{\pi}_m\times\epsilon_{ijk}\epsilon_{knm} \\
&=\; i\hbar\epsilon_{ijk}\times\epsilon_{knm}\hat{x}_n\hat{\pi}_m \qquad \text{(S.56)}\\
&=\; i\hbar\epsilon_{ijk}\times\hat{L}_k\,.
\end{aligned}$$

Finally, combining eqs. (S.53) and (S.56), we arrive at

$$[\hat{J}_i,\hat{J}_j] \;=\; [\hat{L}_j,\hat{J}_j] \;+\; [\hat{J}_i^{\rm EM},\hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\hat{L}_k \;+\; i\hbar\epsilon_{ijk}\hat{J}_k^{\rm EM} \;=\; i\hbar\epsilon_{ijk}\hat{J}_k\,, \qquad \text{(S.57)}$$

precisely as in eq. (24). *Quod erat demonstrandum.*

Problem **3**(c):

Eqs. (22–24) imply that the $\hat{\mathbf{x}}$, $\hat{\vec{\pi}}$, and $\hat{\mathbf{J}}$ operators act as vectors under the space rotations generated by the angular momenta $\hat{J}_j$. Consequently, all the scalar combinations made from these operators act as scalars under such rotations and therefore commute with the $\hat{J}_j$. In particular, eq. (22) implies that $\hat{r}^2 = \hat{x}_i\hat{x}_i$ commutes with all the $\hat{J}_j$ and hence the

$\hat{r}$ and the $1/\hat{r}$ operators also commute with all the $\hat{J}_j$. In the same way, eq. (8) implies that the $\hat{\vec{\pi}}^2 = \hat{\pi}_i\hat{\pi}_i$ operator also commutes with all the $\hat{J}_j$. Thus, both terms in eq. (25) for the Hamiltonian commute with the angular momenta $\hat{J}_j$, so the whole Hamiltonian also commutes with them,

$$[\hat{H}, \hat{J}_j] \;=\; 0. \tag{S.58}$$

Therefore, in the Heisenberg picture of QM, the angular momentum operators $\hat{J}_j$ are time-independent. In other words, the $\hat{J}_j$ are conserved operators.

Problem **3**(d):

By definition

$$\hat{\mathbf{J}} \;=\; \hat{\mathbf{x}}\times\hat{\vec{\pi}} \;+\; \hat{\mathbf{J}}^{\rm EM} \;=\; \hat{\mathbf{x}}\times\hat{\mathbf{p}} \;-\; \frac{q}{c}\,\hat{\mathbf{x}}\times\mathbf{A}(\hat{\mathbf{x}}) \;-\; \frac{qM}{c}\,\hat{\mathbf{n}}\,. \tag{S.59}$$

In this formula

$$\mathbf{x}\times\mathbf{A} \;=\; r\mathbf{n}\times M\,\frac{\pm1-\cos\theta}{r\sin\theta}\,\mathbf{e}_\phi \;=\; M\,\frac{\pm1-\cos\theta}{\sin\theta}\,\big(\mathbf{n}\times\mathbf{e}_\phi \;=\; -\mathbf{e}_\theta\big) \tag{S.60}$$

where

$$\mathbf{e}_\theta \;=\; (+\cos\theta\cos\phi, +\cos\theta\sin\phi, -\sin\theta) \tag{S.61}$$

is the unit vector in the θ direction. Focusing on the z component $\hat{J}_z$ of the angular momentum, we have

$$[\mathbf{x}\times\mathbf{A}]_z \;=\; M\,\frac{\pm1-\cos\theta}{\sin\theta}\,(+\sin\theta) \;=\; M\,(\pm1-\cos\theta), \tag{S.62}$$

hence

$$\left[-\frac{q}{c}\,\mathbf{x}\times\mathbf{A}(\mathbf{x}) \;-\; \frac{qM}{c}\,\mathbf{n}\right]_z \;=\; -\frac{Mq}{c}\,(\pm1-\cos\theta) \;-\; \frac{qM}{c}\,\cos\theta \;=\; \mp\frac{Mq}{c} \tag{S.63}$$

and therefore

$$\hat{J}_z \;=\; [\hat{\mathbf{x}}\times\hat{\mathbf{p}}]_z \;\mp\frac{Mq}{c}\,. \tag{S.64}$$

Finally, in the polar coordinate basis, the $[\hat{\mathbf{x}}\times\hat{\mathbf{p}}]_z$ operator acts as $-i\hbar\partial/\partial\phi$, thus altogether

$$\hat{J}_z\psi(r,\theta,\phi) \;=\; -i\hbar\frac{\partial\psi}{\partial\phi} \;\mp\; \frac{Mq}{c}\times\psi, \tag{S.65}$$

precisely as in eq. (27).

Problem **3**(e):

According to eqs. (S.60) and (S.61),

$$[\mathbf{x}\times\mathbf{A}]_x \ \pm'\ i[\mathbf{x}\times\mathbf{A}]_y = \ -M\,\frac{\pm 1-\cos\theta}{\sin\theta}\,\cos\theta\exp(\pm' i\phi) \tag{S.66}$$

where $\pm$ denotes the gauge choice (Northern vs. Southern hemisphere) while $\pm'$ is a separate sign choice, same on both sides of this equation. Likewise,

$$n_x \ \pm'\ in_y \ = \ \sin\theta\exp(\pm' i\phi), \tag{S.67}$$

hence

$$\begin{aligned}\left[-\frac{q}{c}(\mathbf{x}\times\mathbf{A}) \ - \ \frac{qM}{c}\mathbf{n}\right]_x \ \pm'\ i\left[-\frac{q}{c}(\mathbf{x}\times\mathbf{A}) \ - \ \frac{qM}{c}\mathbf{n}\right]_y \ &= \\ &= \ \frac{qM}{c}\,\exp(\pm' i\phi)\times\left(\frac{(\pm 1-\cos\theta)\cos\theta}{\sin\theta} \ - \ \sin\theta\right) \\ &= \ \frac{qM}{c}\,\exp(\pm' i\phi)\times\frac{\pm\cos\theta-1}{\sin\theta}\,.\end{aligned} \tag{S.68}$$

Also, in the spherical coordinates

$$[\hat{\mathbf{x}}\times\hat{\mathbf{p}}]_x \ \pm'\ i[\hat{\mathbf{x}}\times\hat{\mathbf{p}}]_y \ = \ \hbar\exp(\pm' i\phi)\left(\pm'\frac{\partial}{\partial\theta} \ + \ i\coth\theta\,\frac{\partial}{\partial\phi}\right); \tag{S.69}$$

you can find this formula in any undergraduate QM textbook. Thus altogether, plugging the last two formulae into eq. (S.59) for the net angular momentum, we get

$$\hat{J}_x \ \pm'\ i\hat{J}_y \ = \ \hbar\exp(\pm' i\phi)\left(\pm'\frac{\partial}{\partial\theta} \ + \ i\coth\theta\,\frac{\partial}{\partial\phi} \ + \ \frac{qM}{\hbar c}\,\frac{\pm\cos\theta-1}{\sin\theta}\right), \tag{S.70}$$

in perfect agreement with eqs. (28). *Quod erat demonstrandum.*

Problem **3**(f):

Because of the spherical symmetry of the quantum system in question, we expect all the eigenstates to have wavefunctions of the form

$$\psi(r,\theta,\phi) \;=\; f(r)\times g(\theta)\times h(\phi). \tag{S.71}$$

Moreover, in light of eq. (27), the states of definite m should have

$$h(\phi) \;=\; \exp(im'\phi) \quad\text{for}\quad m' \;=\; m \pm \frac{qM}{\hbar c}\,. \tag{S.72}$$

Or rather, in the Northern hemisphere gauge

$$h_N(\phi) \;=\; \exp(im_N\phi) \quad\text{for}\quad m_N \;=\; m + \frac{qM}{\hbar c}\,, \tag{S.73}$$

while in the Southern hemisphere gauge

$$h_S(\phi) \;=\; \exp(im_S\phi) \quad\text{for}\quad m_S \;=\; m - \frac{qM}{\hbar c}\,. \tag{S.74}$$

Both h_N and h_s must be single-valued functions of the angle ϕ, so both m_N and m_S must be integer. Consequently:

1. $qM/\hbar c$ must be integer or half-integer — this is the Dirac's charge quantization condition.
2. For integer $qM/\hbar c$, the eigenvalue m of the $\hat{J}_z$ must be integer, and hence j must also be integer. But for a half-integer $qM/\hbar c$, the eigenvalue m must be half-integer, and hence j must also be half-integer.

Now consider a multiplet of states $|j,m\rangle$ of definite j and all possible m ranging from $-j$ to $+j$ by 1. In this multiplet, the state with maximal $m=+j$ must be annihilated by the $\hat{J}_+$ operator,

$$(\hat{J}_x + i\hat{J}_y)\,|j,m=j\rangle \;=\; 0. \tag{S.75}$$

In polar coordinates, this operator acts as in the top eq. (26), so for a wave function of the form (S.71) with $h(\phi)$ as in eq. (S.72), we have

$$\begin{aligned}\hat{J}_+\psi(r,\theta,\phi) &= \hbar\exp(+i\phi)f(r)h(\phi)\times\left(\begin{aligned}&+\frac{dg}{d\theta}-\left(m'=m\pm\frac{qM}{\hbar c}\right)\coth\theta\times g(\theta)\\ &\qquad -\frac{qM}{\hbar c}\,\frac{1\mp\cos\theta}{\sin\theta}\times g(\theta)\end{aligned}\right)\\ &= \hbar\exp(+i\phi)f(r)h(\phi)\times\left(+\frac{dg}{d\theta}-\left(m\coth\theta+\frac{qM}{\hbar c}\,\frac{1}{\sin\theta}\right)\times g(\theta)\right).\end{aligned} \tag{S.76}$$

For the state with $m=+j$ the LHS here must vanish, so the $g(\theta)$ function must obey the differential equation

$$\frac{dg}{d\theta} = \left(m\coth\theta+\frac{qM}{\hbar c}\,\frac{1}{\sin\theta}\right)\times g. \tag{S.77}$$

Consequently,

$$\begin{aligned}d\log g(\theta) &= \frac{dg}{g} = \left(m\coth\theta+\frac{qM}{\hbar c}\,\frac{1}{\sin\theta}\right)d\theta\\ &= \left(m-\frac{qM}{\hbar c}\right)\times\frac{\cos\theta-1}{2\sin\theta}\,d\theta + \left(m+\frac{qM}{\hbar c}\right)\times\frac{\cos\theta+1}{2\sin\theta}\,d\theta\\ &= \left(m-\frac{qM}{\hbar c}\right)\times\left(\frac{-\sin(\theta/2)d\theta}{2\cos(\theta/2)} = d\log\cos(\theta/2)\right)\\ &\quad + \left(m+\frac{qM}{\hbar c}\right)\times\left(\frac{\cos(\theta/2)d\theta}{2\sin(\theta/2)} = d\log\sin(\theta/2)\right)\end{aligned} \tag{S.78}$$

and therefore

$$g(\theta) = \text{const}\times\big(\cos(\theta/2)\big)^{n_1}\times\big(\sin(\theta/2)\big)^{n_2}\quad\text{for}\quad n_{1,2} = (m=j)\mp\frac{qM}{\hbar c}. \tag{S.79}$$

To make this solution regular at both $\theta=0$ and $\theta=\pi$, both n_1 and n_2 must be non-negative integers. Consequently, we need

$$j = \left|\frac{qM}{\hbar c}\right| + \text{a non-negative integer}, \tag{S.80}$$

precisely as in eq. (29). *Quod erat demonstrandum.*

Problem **3**(g):

First,

$$\hat{\mathbf{J}}^2 \;=\; \left(\hat{\mathbf{L}} + \frac{qM}{c}\hat{\mathbf{n}}\right)^2 \;=\; \hat{\mathbf{L}}^2 \;+\; \left(\frac{qM}{c}\right)^2 \;+\; \frac{qM}{c}\big(\hat{\mathbf{n}}\cdot\hat{\mathbf{L}} + \hat{\mathbf{L}}\cdot\hat{\mathbf{n}}\big) \;=\; \hat{\mathbf{L}}^2 \;+\; \left(\frac{qM}{c}\right)^2 \tag{S.81}$$

because

$$\hat{\mathbf{n}}\cdot\hat{\mathbf{L}} \;=\; \hat{\mathbf{L}}\cdot\hat{\mathbf{n}} \;=\; 0. \tag{S.82}$$

Second,

$$\hat{\mathbf{L}}^2 \;=\; \big(\hat{\mathbf{x}}\times\hat{\vec{\pi}}\big)^2 \;=\; \hat{r}^2\big(\hat{\vec{\pi}}^2 \;-\; \hat{\pi}_r^2\big), \tag{S.83}$$

which obtains exactly as in QM of a particle in a central potential without the magnetic field. To be safe, I'll derive this formula for the present case in a moment. But once we have this formula, eq. (30) follows immediately from eqs. (S.81) and (S.83).

Now let's derive eq. (S.83). Classically, it follows from the basic vector algebra:

$$\mathbf{L}^2 \;=\; (\mathbf{x}\times\vec{\pi})^2 \;=\; \mathbf{x}^2\,\vec{\pi}^2 \;-\; (\mathbf{x}\cdot\vec{\pi})^2 \;=\; r^2\big(\vec{\pi}^2 \;-\; (\mathbf{n}\cdot\vec{\pi})^2\big) \;=\; r^2\big(\vec{\pi}^2 \;-\; \pi_r^2\big). \tag{S.84}$$

But in the quantum mechanics, we have to watch out for the commutators, thus

$$\begin{aligned}\hat{\mathbf{L}}^2 \;&=\; \big(\epsilon_{ijk}\hat{x}_j\hat{\pi}_k\big)\,\big(\epsilon_{i\ell m}\hat{x}_\ell\hat{\pi}_m\big)\\ &=\; \hat{x}_j\hat{\pi}_k\hat{x}_\ell\hat{p}_m \times \big(\epsilon_{ijk}\epsilon_{i\ell m} \;=\; \delta_{j\ell}\delta_{km} \;-\; \delta_{jm}\delta_{k\ell}\big)\\ &=\; \hat{x}_j\hat{\pi}_k\hat{x}_j\hat{\pi}_k \;-\; \hat{x}_j\hat{\pi}_k\hat{x}_k\hat{\pi}_j\,,\end{aligned} \tag{S.85}$$

where

$$\hat{x}_j\hat{\pi}_k\hat{x}_j\hat{\pi}_k \;=\; \hat{x}_j\hat{x}_j\hat{\pi}_k\hat{\pi}_k \;+\; \hat{x}_j\big([\hat{\pi}_k,\hat{x}_j] = -i\hbar\delta_{jk}\big)\hat{\pi}_k \;=\; \hat{r}^2\,\hat{\vec{\pi}}^2 \;-\; i\hbar\hat{x}_j\hat{\pi}_j\,, \tag{S.86}$$

while

$$\begin{aligned}\hat{x}_j\hat{\pi}_k\hat{x}_k\hat{\pi}_j \;&=\; [\hat{x}_j,\hat{\pi}_k]\hat{x}_k\hat{\pi}_j \;+\; \hat{\pi}_k\hat{x}_k\hat{x}_j\hat{\pi}_j\\ &=\; [\hat{x}_j,\hat{\pi}_k]\hat{x}_k\hat{\pi}_j \;+\; [\hat{\pi}_k,\hat{x}_k]\hat{x}_j\hat{\pi}_j \;+\; (\hat{x}_k\hat{\pi}_k)(\hat{x}_j\hat{\pi}_j)\\ &=\; i\hbar\delta_{jk}\hat{x}_k\hat{\pi}_j \;-\; i\hbar\delta_{kk}\hat{x}_j\hat{\pi}_j + \;(\hat{x}_k\hat{\pi}_k)^2\\ &=\; i\hbar(\hat{\mathbf{x}}\cdot\hat{\vec{\pi}}) \;-\; 3i\hbar(\hat{\mathbf{x}}\cdot\hat{\vec{\pi}}) \;+\; (\hat{\mathbf{x}}\cdot\hat{\vec{\pi}})^2 \;=\; (\hat{\mathbf{x}}\cdot\hat{\vec{\pi}})^2 \;-\; 2i\hbar(\hat{\mathbf{x}}\cdot\hat{\vec{\pi}}).\end{aligned} \tag{S.87}$$

Altogether, this gives us

$$\hat{\mathbf{L}}^2 \;=\; \hat{r}^2\,\vec{\hat{\pi}}^2 \;-\; (\hat{\mathbf{x}}\cdot\vec{\hat{\pi}})^2 \;+\; i\hbar(\hat{\mathbf{x}}\cdot\vec{\hat{\pi}}). \tag{S.88}$$

Now let's compare the second and the third terms here to $\hat{r}^2\hat{\pi}_r^2$. First,

$$\hat{\pi}_r \;\stackrel{\text{def}}{=}\; \tfrac{1}{2}\big(\hat{n}_i\hat{\pi}_i+\hat{\pi}_i\hat{n}_i\big) \;=\; \hat{n}_i\hat{\pi}_i \;+\; \tfrac{1}{2}[\hat{\pi}_i,\hat{n}_i] \;=\; \hat{n}_i\hat{\pi}_i \;+\; \tfrac{1}{2}(-i\hbar)\left(\frac{\partial n_i}{\partial x_i}\;=\;\frac{2}{r}\right) \;=\; \hat{n}_i\hat{\pi}_i \;-\; \frac{i\hbar}{\hat{r}}\,. \tag{S.89}$$

Second,

$$\hat{r}^2\hat{\pi}_r^2 \;=\; \hat{r}\hat{\pi}_r\hat{r}\hat{\pi}_r \;+\; \hat{r}[\hat{r},\hat{\pi}_r]\hat{\pi}_r \tag{S.90}$$

where

$$[\hat{r},\hat{\pi}_r] \;=\; [\hat{r},\hat{n}_i\hat{\pi}_i] \;=\; \hat{n}_i[\hat{r},\hat{\pi}_i] =\; \hat{n}_i\left(i\hbar\widehat{\frac{\partial r}{\partial x_i}}\;=\;i\hbar\hat{n}_i\right) \;=\; i\hbar. \tag{S.91}$$

Consequently,

$$\hat{r}^2\hat{\pi}_r^2 \;=\; (\hat{r}\hat{\pi}_r)^2 \;+\; i\hbar(\hat{r}\hat{\pi}_r) \;=\; (\hat{x}_i\hat{\pi}_i-i\hbar)^2 \;+\; i\hbar(\hat{x}_i\hat{\pi}_i-i\hbar) \;=\; (\hat{x}_i\hat{\pi}_i)^2 \;-\; i\hbar(\hat{x}_i\hat{\pi}_i), \tag{S.92}$$

and comparing this formula to the RHS of eq. (S.88), we immediately see that

$$\hat{\mathbf{L}}^2 \;=\; \hat{r}^2\,\vec{\hat{\pi}}^2 \;-\; \hat{r}^2\hat{\pi}_r^2, \tag{S.93}$$

precisely as in eq. (S.83).

This completes our derivation of eq. (S.83) and hence eq. (30)

Problem **3**(h):

In light of eq. (30), the Hamiltonian of the charged particle orbiting a dyon can be written as

$$\hat{H} \;=\; \frac{\hat{\pi}_r^2}{2m} \;+\; \frac{\hat{\mathbf{J}}^2 - (qM/c)^2}{2m\hat{r}^2} \;-\; \frac{qQ}{\hat{r}}\,. \tag{S.94}$$

In the coordinate basis

$$\hat{\pi}_r \;=\; -i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right), \qquad \hat{\pi}_r^2 \;=\; -\hbar^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right), \tag{S.95}$$

so the radial wave function $f(r)$ (*cf.* eq. (S.71)) of a bound state $|n_r,j,m\rangle$ of energy $E<0$ and angular momentum j obeys the radial Schrödinger equation

$$\frac{\hbar^2}{2m}\left(-f''(r) \;-\; \frac{2}{r}\,f'(r) \;+\; \frac{j(j+1)-(qM/\hbar c)^2}{r^2}\,f(r)\right) \;-\; \frac{qQ}{r}\,f(r) \;=\; Ef(r). \tag{S.96}$$

This equation looks exactly like the radial Schrödinger equation for the hydrogen atom — except for having

$$\lambda(\lambda+1) \;\stackrel{\rm def}{=}\; j(j+1) \;-\; (qM/\hbar c)^2 \tag{18}$$

instead of $\ell(\ell+1)$ — and it can be solved in exactly the same way. You can find a solution — and there are many different way to solve eq. (S.96) — in any undergraduate QM textbook; but since λ is generally non-integral while many textbook solutions make use of ℓ being an integer, let me write down a solution of my own.

First, let me introduce a couple of parameters:

$$\kappa \;=\; \frac{1}{\hbar}\sqrt{-2mE} \tag{S.97}$$

for a bound state of negative energy $E<0$, and

$$\nu \;=\; \frac{qQm}{\hbar^2\kappa}\,. \tag{S.98}$$

In terms of these parameters (as well as λ), eq. (S.96) becomes

$$f'' \;+\; \frac{2}{r}\times f' \;-\; \frac{\lambda(\lambda+1)}{r^2}\times f \;+\; \frac{2\nu\kappa}{r}\times f \;=\; \kappa^2\times f. \tag{S.99}$$

Now let's take the asymptotic limits $r\to\infty$ and $r\to 0$. For $r\to\infty$, we may crudely approximate eq. (S.99) as

$$f'' \;\approx\; \kappa^2 f, \tag{S.100}$$

so the normalizable solution behaves as $f(r) \sim \exp(-\kappa r)$. In the opposite limit of $r \to 0$, we approximate eq. (S.99) as

$$f'' \;+\; \frac{2}{r} \times f' \;-\; \frac{\lambda(\lambda+1)}{r^2} \times f \;\approx\; 0, \tag{S.101}$$

with the normalizable solution being $f \sim r^\lambda$. In light of these asymptotic limits, we let

$$f(r) \;=\; r^\lambda \times \exp(-\kappa r) \times \Phi(r) \tag{S.102}$$

for some (hopefully) regular function $\Phi(r)$. Following eq. (S.102), we have

$$f'(r) \;=\; r^\lambda \exp(-\kappa r) \times \left(\Phi' \;+\; \frac{\lambda}{r}\Phi \;-\; \kappa\Phi\right), \tag{S.103}$$

$$f''(r) \;=\; r^\lambda \exp(-\kappa r) \times \left(\Phi'' \;+\; \left(\frac{2\lambda}{r} - 2\kappa\right)\Phi' \;+\; \left(\frac{\lambda(\lambda-1)}{r^2} - \frac{2\lambda\kappa}{r} + \kappa^2\right)\Phi\right), \tag{S.104}$$

and consequently eq. (S.99) becomes

$$\Phi'' \;+\; 2\left(\frac{\lambda+1}{r} - \kappa\right) \times \Phi' \;+\; 2\frac{(\nu-\lambda-1)\kappa}{r} \times \Phi. \tag{S.105}$$

To solve this equation, we rewrite it as

$$r \times \Bigl(\Phi'' \;-\; 2\kappa\Phi'\Bigr) \;+\; 2\Bigl((\lambda+1)\Phi' \;+\; (\nu-\lambda-1)\kappa\Phi\Bigr) \;=\; 0 \tag{S.106}$$

and then Laplace transform it to a first-order differential equation. Thus, we look for $\Phi(r)$ in the form of a contour integral in the complex plane,

$$\Phi(r) \;=\; \int\limits_\Gamma dt\, e^{tr} \times F(t) \tag{S.107}$$

for some analytic function $F(t)$ and some contour Γ. To allow integration by parts, Γ should be either a closed contour, or else both ends should extend to ∞ in directions along which the integrand dies off rapidly enough.

Given the Laplace transform (S.107) of the Φ function itself, we have

$$\frac{d\Phi}{dr} = \int\limits_{\gamma} dt\, e^{tr} \times t \times F(t), \tag{S.108}$$

$$\frac{d^2\Phi}{dr^2} = \int\limits_{\gamma} dt\, e^{tr} \times t^2 \times F(t), \tag{S.109}$$

while

$$\begin{aligned} r \times \Phi(r) &= \int\limits_{\Gamma} dt \left(re^{tr} = \frac{\partial e^{tr}}{\partial t} \right) \times F(t) \\ \langle\!\langle \text{ by parts } \rangle\!\rangle \quad &= -\int\limits_{\Gamma} dt\, e^{tr} \times \frac{dF}{dt}, \end{aligned} \tag{S.110}$$

and likewise

$$r \times \Big(\Phi'' - 2\kappa\Phi' \Big) = -\int\limits_{\Gamma} dt\, e^{tr} \times \frac{d}{dt}\Big(t^2 F(t) - 2\kappa t F(t) \Big). \tag{S.111}$$

Plugging all these formulae into eq. (S.106), we may recast it as an equation for the $F(t)$, namely

$$-\frac{d}{dt}\Big((t^2 - 2\kappa t) F(t) \Big) + 2\big((\lambda+1)t + (\nu - \lambda - 1)\kappa \big) F(t) = 0. \tag{S.112}$$

This is a fairly easy first-order differential equation. To solve, we rewrite it as

$$t(t-2\kappa) \times \frac{dF}{dt} = 2(\lambda t + \nu\kappa - \lambda\kappa) \times F(t), \tag{S.113}$$

hence

$$\frac{dF/dt}{F} = \frac{2\lambda t + 2\kappa\nu - 2\kappa\lambda}{t(t-2\kappa)} = \frac{\lambda - \nu}{t} + \frac{\lambda + \nu}{t - 2\kappa}, \tag{S.114}$$

$$\begin{aligned} d\log F &= \frac{dF}{F} = (\lambda - \nu)\frac{dt}{t} + (\lambda + \nu)\frac{dt}{t - 2\kappa} \\ &= (\lambda - \nu) d\log(t) + (\lambda + \nu) d\log(t - 2\kappa), \end{aligned} \tag{S.115}$$

$$F(t) \;=\; \text{const} \times t^{\lambda-\nu} \times (t-2\kappa)^{\lambda+\nu}, \tag{S.116}$$

and therefore

$$\psi_{\text{radial}} \;=\; f(r) \;= \text{const} \times r^{\lambda} e^{-\kappa r} \times \int_{\Gamma} dt\, e^{tr} \times t^{\lambda-\nu} \times (t-2\kappa)^{\lambda+\nu}. \tag{S.117}$$

It remain to determine the integration contour Γ in this formula. For generic λ and ν, the integrand in eq. (S.117) has two branch cuts, one from $t=0$ to $t=2\kappa$ and the other from $t=0$ to $t=\infty$; let's lay the combined branch cut along the real axis, from $t=-\infty$ to $t=+2\kappa$. Since there are no other singularities, the integration contour must therefore surround this cut, with both running to $-\infty$ on two sides of the cut. Consequently, the integral in eq. (S.117) becomes

$$2 \times \int_{-\infty}^{+2\kappa} dt\, e^{rt} \times \text{disc}\Big[t^{\lambda-\nu} \times (t-2\kappa)^{\lambda+\nu}\Big] \tag{S.118}$$

where 'disc' stands for the discontinuity of $[\cdots]$ across the real axis.

For large r, the exponential e^{rt} grows rapidly with t, so the integral (S.118) is dominated by its right end at $t=2\kappa$, so asymptotically

$$\text{for } r\to\infty: \quad \text{the integral} \;\sim\; e^{+2\kappa r} \times r^{\text{some power}} \tag{S.119}$$

and therefore

$$\psi_{\text{rad}}(r) \;\sim\; e^{+\kappa r} \times r^{\text{some power}}. \tag{S.120}$$

Such a radial wave function is un-normalizable, so there are no good solutions for generic λ and ν.

To get a good, normalizable solution of the radial wave equation, we need the integrand of eq. (S.117) to have a different geometry of singularities that would allow a different kind of an integration contour. Such geometry obtains when $\lambda - \nu$ is a negative integer, *i.e.*

$$\nu \;=\; \lambda \;+\; n_r, \quad n_r \;=\; 1,2,3,4,\ldots, \tag{S.121}$$

hence

$$\kappa \;=\; \frac{mqQ}{\nu\hbar^2} \;=\; \frac{mqQ}{(\lambda+n_r)\hbar^2} \tag{S.122}$$

and the bound state energy

$$E \;=\; -\frac{\hbar^2\kappa^2}{2m} \;=\; -\frac{m(qQ)^2}{2\hbar^2(\lambda+n_r)^2} \tag{S.123}$$

precisely as in eq. (17). Indeed, for $\lambda - \nu = -n_r$, the integrand

$$e^{tr} \times t^{\lambda-\nu} \times (t-2\kappa)^{\lambda+\nu} \tag{S.124}$$

has an isolated pole at $t = 0$ in addition to a branch cut from $t = -\infty$ to $t = +2\kappa$. Let's reroute the branch cut so it lies away from the pole at $t = 0$. Then in addition to the integration contour surrounding the branch cut, we have another option for the contour — a small circle around the pole around $t = 0$. For such a contour, the integral extracts the residue of this pole, hence

$$\psi_{\rm rad}(r) \;=\; {\rm const} \times r^\lambda e^{-\kappa r} \times \mathop{\rm Residue}_{@t=0}\left[\frac{e^{tr}\times(t-2\kappa)^{2\lambda+n_r}}{t^{n_r}}\right]. \tag{S.125}$$

As a function of r, the residue here is a polynomial of degree $n_r - 1$, so

$$\text{for } r\to\infty \quad \psi_{\rm rad} \;\sim\; e^{-\kappa r}\times r^{\text{some power}} \tag{S.126}$$

which makes for a normalizable radial wavefunction.