Problem **1**(a):

In $N\times N$ matrix form, the local $SU(N)$ symmetry acts on the adjoint matter field $\Phi(x)$ and the gauge field $\mathcal{A}_\mu(x)$ according to

$$\Phi'(x)\;=\;U(x)\Phi(x)U^\dagger(x),\qquad \mathcal{A}'_\mu(x)\;=\;U(x)\mathcal{A}_\mu(x)U^\dagger(x)\;+\;i\partial_\mu U(x)\,U^\dagger(x). \tag{S.1}$$

Consequently, the covariant derivatives (4) become

$$D_\mu\Phi(x)\;\to\;D'_\mu\Phi'(x)\;=\;\partial_\mu\Phi(x)\;+\;i\left[\mathcal{A}'_\mu(x),\Phi'(x)\right] \tag{S.2}$$

where the first term on the RHS expands to

$$\begin{aligned}
\partial_\mu\Phi' &= \partial_\mu\bigl(U\Phi U^\dagger\bigr)\\
&= (\partial_\mu U)\Phi U^\dagger \;+\; U(\partial_\mu\Phi)U^\dagger \;+\; U\Phi\bigl(\partial_\mu U^\dagger\bigr)\\
&= UU^\dagger(\partial_\mu U)\Phi U^\dagger \;+\; U(\partial_\mu\Phi)U^\dagger \;-\; U\Phi U^\dagger(\partial_\mu U)U^\dagger\\
&= U\Bigl((U^\dagger\partial_\mu U)\Phi \;+\; \partial_\mu\Phi \;-\; \Phi(U^\dagger\partial_\mu U)\Bigr)U^\dagger\\
&= U\Bigl(\partial_\mu\Phi \;+\; \bigl[U^\dagger\partial_\mu U,\Phi\bigr]\Bigr)U^\dagger
\end{aligned} \tag{S.3}$$

while the second term expands to

$$\begin{aligned}
\bigl[\mathcal{A}'_\mu,\Phi'\bigr] &= \bigl[U\mathcal{A}_\mu U^\dagger, U\Phi U^\dagger\bigr] \;+\; \bigl[i(\partial_\mu U)U^\dagger, U\Phi U^\dagger\bigr]\\
&= U\Bigl(\bigl[\mathcal{A}_\mu,\Phi\bigr] \;+\; i\bigl[U^\dagger\partial_\mu U,\Phi\bigr]\Bigr)U^\dagger
\end{aligned} \tag{S.4}$$

Combining the two expansions, we arrive at

$$D'_\mu\Phi' \;=\; U\Bigl(\partial_\mu\Phi + \cancel{\bigl[U^\dagger\partial_\mu U,\Phi\bigr]} \;+\; i\bigl[\mathcal{A}_\mu,\Phi\bigr] \;-\; \cancel{\bigl[U^\dagger\partial_\mu U,\Phi\bigr]}\Bigr)U^\dagger \;=\; U\bigl(D_\mu\Phi\bigr)U^\dagger. \tag{S.5}$$

Thus, the $D_\mu\Phi(x)$ matrix transforms exactly like the $\Phi(x)$ matrix itself, which makes the D_μ derivative (4) covariant. *Quod erat demonstrandum.*

Problem $\mathbf{1}$(b):

Let's start with the second line of eq. (6). In the matrix form, the adjoint multiplet Φ is a matrix, the fundamental multiplet Ψ is a column vector, and their matrix product $\Phi\Psi$ is also a column vector. The covariant derivatives acts on these matrices and vectors as

$$D_\mu\Phi \;=\; \partial_\mu\Phi \;+\; i\big[\mathcal{A}_\mu,\Phi\big], \qquad D_\mu\Psi \;=\; \partial_\mu\Psi \;+\; i\mathcal{A}_\mu\Psi, \tag{S.6}$$

while

$$\begin{aligned} D_\mu(\Phi\Psi) &\stackrel{\text{def}}{=} \partial_\mu(\Phi\Psi) \;+\; i\mathcal{A}_\mu(\Phi\Psi) \\ &= (\partial_\mu\Phi)\Psi \;+\; \Phi(\partial_\mu\Psi) \;+\; i\big[\mathcal{A}_\mu,\Phi\big]\Psi \;+\; i\Phi\mathcal{A}_\mu\Psi \\ &= (D_\mu\Phi)\Psi \;+\; \Phi(D_\mu\Psi). \end{aligned} \tag{S.7}$$

Likewise, on the third line of eq. (6), Ξ is a matrix, $\Psi^\dagger$ is a row vector, and their matrix product $\Psi^\dagger\Xi$ is also a row vector. Therefore

$$D_\mu\Psi^\dagger \;=\; \partial_\mu\Psi^\dagger \;-\; i\Psi^\dagger\mathcal{A}_\mu\,, \qquad D_\mu\Xi \;=\; \partial_\mu\Xi \;+\; i\big[\mathcal{A}_\mu,\Xi\big], \tag{S.8}$$

while

$$\begin{aligned} D_\mu(\Psi^\dagger\Xi) &\stackrel{\text{def}}{=} \partial_\mu(\Psi^\dagger\Xi) \;-\; i(\Psi^\dagger\Xi)\mathcal{A}_\mu \\ &= (\partial_\mu\Psi^\dagger)\Phi \;+\; \Psi^\dagger(\partial_\mu\Xi) \;-\; i\Psi^\dagger\mathcal{A}_\mu\Xi \;-\; i\Psi^\dagger\big[\Xi,\mathcal{A}_\mu\big] \\ &= (D_\mu\Psi^\dagger)\Xi \;+\; \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{S.9}$$

Finally, on the first line of eq. (6), Φ and Ξ are both $N\times N$ matrices and their product $\Phi\Xi$ is also a matrix. Consequently,

$$\begin{aligned} D_\mu(\Phi\Xi) &\stackrel{\text{def}}{=} \partial_\mu(\Phi\Xi) \;+\; i\big[\mathcal{A}_\mu,\Phi\Xi\big] \\ &= (\partial_\mu\Phi)\Xi \;+\; \Phi(\partial_\mu\Xi) \;+\; i\big[\mathcal{A}_\mu,\Phi\big]\Xi \;+\; i\Phi\big[\mathcal{A}_\mu,\Xi\big] \\ &= (D_\mu\Phi)\Xi \;+\; \Phi(D_\mu\Xi). \end{aligned} \tag{S.10}$$

Quod erat demonstrandum.

Problem $\mathbf{1}$(c):

For the adjoint multiplet of fields $\Phi(x)$,

$$\begin{aligned} D_\mu D_\nu \Phi &= \partial_\mu (D_\nu \Phi) + i\big[\mathcal{A}_\mu, D_\nu \Phi\big] \\ &= \partial_\mu \Big(\partial_\nu \Phi + i\big[\mathcal{A}_\nu, \Phi\big]\Big) + i\Big[\mathcal{A}_\mu, \Big(\partial_\nu \Phi + i\big[\mathcal{A}_\nu, \Phi\big]\Big)\Big] \\ &= \color{blue}{\partial_\mu \partial_\nu \Phi} + \color{red}{i\big[\partial_\mu \mathcal{A}_\nu, \Phi\big]} + \color{blue}{i\big[\mathcal{A}_\nu, \partial_\mu \Phi\big]} + \color{blue}{i\big[\mathcal{A}_\mu, \partial_\nu \Phi\big]} - \color{red}{\big[\mathcal{A}_\mu, \big[\mathcal{A}_\nu, \Phi\big]\big]} \end{aligned} \tag{S.11}$$

where the blue marks terms that are symmetric WRT $\mu \leftrightarrow \nu$ while the asymmetric terms are marked in red. Only the red terms contribute to the difference

$$\begin{aligned} \big[D_\mu, D_\nu\big]\Phi &= D_\mu D_\nu \Phi - D_\nu D_\mu \Phi \\ &= i\big[\partial_\mu \mathcal{A}_\nu, \Phi\big] - \big[\mathcal{A}_\mu, \big[\mathcal{A}_\nu, \Phi\big]\big] - i\big[\partial_\nu \mathcal{A}_\mu, \Phi\big] + \big[\mathcal{A}_\nu, \big[\mathcal{A}_\mu, \Phi\big]\big] \\ &= i\big[\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \Phi\big] - \big[\big[\mathcal{A}_\mu, \mathcal{A}_\nu\big], \Phi\big] \end{aligned} \tag{S.12}$$

where the second term on the bottom line follows from the Jacobi identity for the matrix commutator:

$$\begin{aligned} -\big[\mathcal{A}_\mu, \big[\mathcal{A}_\nu, \Phi\big]\big] + \big[\mathcal{A}_\nu, \big[\mathcal{A}_\mu, \Phi\big]\big] &= +\big[\mathcal{A}_\mu, \big[\Phi, \mathcal{A}_\nu\big]\big] + \big[\mathcal{A}_\nu, \big[\mathcal{A}_\mu, \Phi\big]\big] \\ &= -\big[\Phi, \big[\mathcal{A}_\nu, \mathcal{A}_\mu\big]\big] = -\big[\big[\mathcal{A}_\mu, \mathcal{A}_\nu\big], \Phi\big]\,. \end{aligned} \tag{S.13}$$

Altogether, we have

$$\big[D_\mu, D_\nu\big]\Phi = i\big[\big(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]\big), \Phi\big] \equiv i\big[\mathcal{F}_{\mu\nu}, \Phi\big] = ig\big[F_{\mu\nu}, \Phi\big] \tag{S.14}$$

where the second equality follows from the definition of the non-abelian $\mathcal{F}_{\mu\nu}$ and the third equality from $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$. Finally, in components

$$ig\big[F_{\mu\nu}, \Phi\big] = igF^b_{\mu\nu}\Phi^c \times \left[\frac{\lambda^b}{2}, \frac{\lambda^c}{2}\right] = igF^b_{\mu\nu}\Phi^c \times if^{bca}\,\frac{\lambda^c}{2} \tag{S.15}$$

and hence $[D_\mu, D_\nu]\Phi^a = -gf^{abc}F^b_{\mu\nu}\Phi^c$.

Problem **1**(d):

In matrix notations, the non-abelian gauge symmetries act on vector potentials $\mathcal{A}_\mu(x)$ according to

$$\mathcal{A}'_\mu(x) \;=\; U(x)\mathcal{A}_\mu(x)U^\dagger(x) \;+\; i\partial_\mu U(x)\,U^\dagger(x). \tag{S.16}$$

Taking

$$\mathcal{F}_{\mu\nu}(x) \;\stackrel{\rm def}{=}\; \partial_\mu\mathcal{A}_\nu(x) \;-\; \partial_\nu\mathcal{A}_\mu(x) \;+\; i\,[\mathcal{A}_\mu(x),\mathcal{A}_\nu(x)] \tag{S.17}$$

as the definition of the tension fields $\mathcal{F}_{\mu\nu}(x)$, we then have

$$\mathcal{F}'_{\mu\nu}(x) \;=\; \partial_\mu\mathcal{A}'_\nu(x) \;-\; \partial_\nu\mathcal{A}'_\mu(x) \;+\; i\left[\mathcal{A}'_\mu(x),\mathcal{A}'_\nu(x)\right], \tag{S.18}$$

whatever that evaluates to. Specifically, the first term here evaluates to

$$\begin{aligned}\partial_\mu\mathcal{A}'_\nu \;&=\; \partial_\mu\left(U\mathcal{A}_\nu U^\dagger \;+\; i(\partial_\nu U)U^\dagger\right)\\ &=\; U(\partial_\mu\mathcal{A}_\nu)U^\dagger \;+\; \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger\right] \;+\; i(\partial_\mu\partial_\nu U)U^\dagger \;-\; i(\partial_\nu U)U^\dagger\times(\partial_\mu U)U^\dagger\end{aligned} \tag{S.19}$$

where the second equality follows from

$$\partial_\mu\left(U\mathcal{A}_\nu U^\dagger\right) \;=\; U(\partial_\mu\mathcal{A}_\nu)U^\dagger \;+\; \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger\right] \tag{S.20}$$

— *cf.* similar formula (S.3) — and

$$\partial_\mu\left((\partial_\nu U)U^\dagger\right) \;=\; (\partial_\mu\partial_\nu U)U^\dagger \;+\; (\partial_\nu U)(\partial_\mu U^\dagger) \;=\; (\partial_\mu\partial_\nu U)U^\dagger \;-\; (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger. \tag{S.21}$$

Likewise

$$\partial_\nu\mathcal{A}'_\mu \;=\; U(\partial_\nu\mathcal{A}_\mu)U^\dagger + \left[(\partial_\nu U)U^\dagger, U\mathcal{A}_\mu U^\dagger\right] + i(\partial_\nu\partial_\mu U)U^\dagger \;-\; i(\partial_\mu U)U^\dagger\times(\partial_\nu U)U^\dagger \tag{S.22}$$

and hence

$$\begin{aligned}\partial_\mu\mathcal{A}'_\nu \;-\; \partial_\nu\mathcal{A}'_\mu \;=\;& U\big(\partial_\mu\mathcal{A}_\nu \;-\; \partial_\nu\mathcal{A}_\mu\big)U^\dagger \;+\; \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger\right] \;-\; \left[(\partial_\nu U)U^\dagger, U\mathcal{A}_\mu U^\dagger\right]\\ &+\; 0 \;+\; i\left[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger\right].\end{aligned} \tag{S.23}$$

At the same time, the commutator part of the tension field transforms into

$$\begin{aligned} i\left[\mathcal{A}'_\mu,\mathcal{A}'_\nu\right] &= i\Big[\Big(U\mathcal{A}_\mu U^\dagger\,+\,i(\partial_\mu U)U^\dagger\Big)\,,\Big(U\mathcal{A}_\nu U^\dagger\,+\,i(\partial_\nu U)U^\dagger\Big)\Big] \\ &= i\Big[U\mathcal{A}_\mu U^\dagger, U\mathcal{A}_\nu U^\dagger\Big]\;-\;\Big[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger\Big] \\ &\quad -\;\Big[U\mathcal{A}_\mu U^\dagger,(\partial_\nu U)U^\dagger\Big]\;-\;i\Big[(\partial_\mu U)U^\dagger,(\partial_\nu U)U^\dagger\Big]\,, \end{aligned} \tag{S.24}$$

Combining eqs. (S.23) and (S.24) leads to massive cancellation of 6 out of 8 terms on the combined right hand side. Only the first terms on right hand sides of (S.23) and (S.24) survive the cancellation, thus

$$\begin{aligned} \partial_\mu\mathcal{A}'_\nu\,-\,\partial_\nu\mathcal{A}'_\mu\,+\,i\left[\mathcal{A}'_\mu,\mathcal{A}'_\nu\right] &= U\big(\partial_\mu\mathcal{A}_\nu\,-\,\partial_\nu\mathcal{A}_\mu\big)U^\dagger\;+\;i\Big[U\mathcal{A}_\mu U^\dagger, U\mathcal{A}_\nu U^\dagger\Big] \\ &= U\Big(\partial_\mu\mathcal{A}_\nu\,-\,\partial_\nu\mathcal{A}_\mu\,+\,[\mathcal{A}_\mu,\mathcal{A}_\nu]\Big)U^\dagger, \end{aligned} \tag{S.25}$$

or in other words,

$$\mathcal{F}'_{\mu\nu}(x)\;=\;U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x). \tag{S.26}$$

Quod erat demonstrandum.

Problem **1**(e):

There are two ways to prove the non-abelian differential identity (8): using part (b) and (c) and Jacobi identity for the commutators, or the hard calculation based directly on eq. (S.17). Let me start with the easier proof.

In part (b) we have proved the Leibniz rule for the covariant derivatives of a matrix product of two adjoint fields $\Phi(x)$ and $\Xi(x)$. Clearly, the same Leibniz rule also applies to the commutator $[\Phi,\Xi]$:

$$D_\mu[\Phi,\Xi]\;=\;\big[D_\mu\Phi,\Xi\big]\;+\;\big[\Phi,D_\mu\Xi\big]. \tag{S.27}$$

In particular, for $\Phi=\mathcal{F}_{\mu\nu}$ and arbitrary Ξ, we have

$$D_\lambda([\mathcal{F}_{\mu\nu},\Xi])\;=\;[D_\lambda\mathcal{F}_{\mu\nu},\Xi]\;+\;[\mathcal{F}_{\mu\nu},D_\lambda\Xi]. \tag{S.28}$$

On the other hand, in part (c) we saw that for any adjoint field $\Xi(x)$ we have $[\mathcal{F}_{\mu\nu},\Xi] = -i[D_\mu,D_\nu]\Xi$. Likewise, for the $D_\lambda\Xi(x)$ we also have $[\mathcal{F}_{\mu\nu},D_\lambda\Xi] = -i[D_\mu,D_\nu]D_\lambda\Xi$. Consequently, eq. (S.28) becomes

$$-iD_\lambda[D_\mu,D_\nu]\Xi \;=\; [D_\lambda\mathcal{F}_{\mu\nu},\Xi] \;-\; i[D_\mu,D_\nu]D_\lambda\Xi \tag{S.29}$$

and hence

$$i[D_\lambda\mathcal{F}_{\mu\nu},\Xi] \;=\; [D_\lambda,[D_\mu,D_\nu]]\Xi. \tag{S.30}$$

Now, let's sum 3 such formulae, one for each cyclic permutations of the indices λ,μ,ν. On the left hand side, this gives us

$$i\left[\big(D_\lambda\mathcal{F}_{\mu\nu}+D_\mu\mathcal{F}_{\nu\lambda}+D_\nu\mathcal{F}_{\lambda\mu}\big),\Xi\right] \;=\; \cdots$$

while on the right hand side we obtain

$$\cdots \;=\; \Big([D_\lambda,[D_\mu,D_\nu]] \,+\, [D_\mu,[D_\nu,D_\lambda]] \,+\, [D_\nu,[D_\lambda,D_\mu]]\Big)\Xi \;=\; 0 \tag{S.31}$$

due to Jacobi identity for the double commutators of the three covariant derivatives D_λ, D_μ, and D_ν. Consequently

$$\left[\big(D_\lambda\mathcal{F}_{\mu\nu}+D_\mu\mathcal{F}_{\nu\lambda}+D_\nu\mathcal{F}_{\lambda\mu}\big),\Xi\right] \;=\; 0,$$

and this must be true for any adjoint field $\Xi(x)$. Moreover, for any x,λ,μ,ν, the $N\times N$ matrix

$$D_\lambda\mathcal{F}_{\mu\nu} \,+\, D_\mu\mathcal{F}_{\nu\lambda} \,+\, D_\nu\mathcal{F}_{\lambda\mu}$$

is traceless, and the only way it may commute with all traceless hermitian matrices $\Xi(x)$ is by being zero, thus

$$D_\lambda\mathcal{F}_{\mu\nu} \,+\, D_\mu\mathcal{F}_{\nu\lambda} \,+\, D_\nu\mathcal{F}_{\lambda\mu} \;=\; 0. \tag{S.32}$$

This is my first proof of the non-abelian differential identity (8).

The second proof of the differential identity (8) follows directly from the definition (S.17) of the non-abelian tension fields and the covariant derivatives (4). Let's spell out $D_\lambda \mathcal{F}_{\mu\nu}$ in detail:

$$\begin{aligned} D_\lambda \mathcal{F}_{\mu\nu} &= \partial_\lambda \mathcal{F}_{\mu\nu} + i[\mathcal{A}_\lambda, \mathcal{F}_{\mu\nu}] \\ &= \partial_\lambda\big(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]\big) + i\left[\mathcal{A}_\lambda, \big(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]\big)\right] \\ &= \partial_\lambda \partial_\mu \mathcal{A}_\nu - \partial_\lambda \partial_\nu \mathcal{A}_\mu + i[\partial_\lambda \mathcal{A}_\mu, \mathcal{A}_\nu] + i[\mathcal{A}_\mu, \partial_\lambda \mathcal{A}_\nu] \\ &\quad + i[\mathcal{A}_\lambda, \partial_\mu \mathcal{A}_\nu] - i[\mathcal{A}_\lambda, \partial_\nu \mathcal{A}_\mu] - [\mathcal{A}_\lambda, [\mathcal{A}_\mu, \mathcal{A}_\nu]] \\ &= \Big(\partial_\lambda \partial_\mu \mathcal{A}_\nu - \partial_\lambda \partial_\nu \mathcal{A}_\mu\Big) + i\Big([\partial_\lambda \mathcal{A}_\mu, \mathcal{A}_\nu] - [\partial_\mu \mathcal{A}_\nu, \mathcal{A}_\lambda]\Big) \\ &\quad + i\Big([\mathcal{A}_\mu, \partial_\lambda \mathcal{A}_\nu] - [\mathcal{A}_\lambda, \partial_\nu \mathcal{A}_\mu]\Big) - \Big([\mathcal{A}_\lambda, [\mathcal{A}_\mu, \mathcal{A}_\nu]]\Big). \end{aligned} \tag{S.33}$$

On the bottom two lines here I have grouped terms in () so that after summing over cyclic permutations of the indices λ, μ, ν, we get a zero sum separately for each group. Indeed,

$$\begin{aligned} \Big(\partial_\lambda \partial_\mu \mathcal{A}_\nu - \partial_\lambda \partial_\nu \mathcal{A}_\mu\Big) + \text{cyclic} &= \Big(\partial_\lambda \partial_\mu \mathcal{A}_\nu - \partial_\nu \partial_\lambda \mathcal{A}_\mu\Big) + \text{cyclic} \\ &= 0 \quad \langle\!\langle \text{ by inspection } \rangle\!\rangle, \\ \Big([\partial_\lambda \mathcal{A}_\mu, \mathcal{A}_\nu] - [\partial_\mu \mathcal{A}_\nu, \mathcal{A}_\lambda]\Big) + \text{cyclic} &= 0 \quad \langle\!\langle \text{ by inspection } \rangle\!\rangle, \\ \Big([\mathcal{A}_\mu, \partial_\lambda \mathcal{A}_\nu] - [\mathcal{A}_\lambda, \partial_\nu \mathcal{A}_\mu]\Big) + \text{cyclic} &= 0 \quad \langle\!\langle \text{ by inspection } \rangle\!\rangle, \text{ and} \\ [\mathcal{A}_\lambda, [\mathcal{A}_\mu, \mathcal{A}_\nu]] + \text{cyclic} &= 0 \quad \langle\!\langle \text{ by Jacobi identity } \rangle\!\rangle. \end{aligned} \tag{S.34}$$

Therefore,

$$D_\lambda \mathcal{F}_{\mu\nu} + \text{cyclic} \equiv D_\lambda \mathcal{F}_{\mu\nu} + D_\mu \mathcal{F}_{\nu\lambda} + D_\nu \mathcal{F}_{\lambda\mu} = 0. \tag{8}$$

Problem **1**(f):

The Euler–Lagrange field equations follow from requiring zero first variation of the action $S = \int \mathcal{L}$ under infinitesimal variation of the independent fields $\mathcal{A}_\mu(x)$. Let's start by calculating the variation of the tension fields $\mathcal{F}_{\mu\nu}$:

$$\begin{aligned} \delta \mathcal{F}_{\mu\nu} &\equiv \delta\Big(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]\Big) \\ &= \partial_\mu \delta \mathcal{A}_\nu - \partial_\nu \delta \mathcal{A}_\mu + i\big[\delta \mathcal{A}_\mu, \mathcal{A}_\nu\big] + i\big[\mathcal{A}_\mu, \delta \mathcal{A}_\nu\big] \\ &= \Big(\partial_\mu \delta \mathcal{A}_\nu + i\big[\mathcal{A}_\mu, \delta \mathcal{A}_\nu\big]\Big) - \Big(\partial_\nu \delta \mathcal{A}_\mu + i\big[\mathcal{A}_\nu, \delta \mathcal{A}_\mu\big]\Big) \\ &= D_\mu \delta \mathcal{A}_\nu - D_\nu \delta \mathcal{A}_\mu \end{aligned} \tag{S.35}$$

where we treat the matrix-valued variations $\delta\mathcal{A}_\nu(x)$ as adjoint fields so their covariant derivative work according to eq. (4), $D_\mu\delta\mathcal{A}_\nu \equiv \partial_\mu\delta\mathcal{A}_\nu + i[\mathcal{A}_\mu,\delta\mathcal{A}_\nu]$ and likewise for the $D_\nu\delta\mathcal{A}_\mu$.

In light of eq. (S.35), the trace in the Yang–Mills Lagrangian (9) varies by

$$\begin{aligned}\delta\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\Big) &= 2\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}\delta\mathcal{F}_{\mu\nu}\Big) = 2\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}\big(D_\mu\delta\mathcal{A}_\nu - D_\nu\delta\mathcal{A}_\mu\big)\Big)\\ &= 4\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}D_\mu\delta\mathcal{A}_\nu\Big) \qquad \langle\!\langle\,\text{since } \mathcal{F}^{\mu\nu} = -\mathcal{F}^{\nu\mu}\,\rangle\!\rangle\\ &= -4\,\mathrm{tr}\Big(\big(D_\mu\mathcal{F}^{\mu\nu}\big)\delta\mathcal{A}_\nu\Big) + 4\partial_\mu\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu\Big)\end{aligned} \tag{S.36}$$

where the last equality follows from the Leibniz rule for the two adjoint fields $\mathcal{F}^{\mu\nu}$ and $\delta\mathcal{A}_\nu$:

$$\begin{aligned}\mathrm{tr}\big((D_\mu\mathcal{F}^{\mu\nu})\delta\mathcal{A}_\nu\big) + \mathrm{tr}\big(\mathcal{F}^{\mu\nu}(D_\mu\delta\mathcal{A}_\nu)\big) &=\\ &= \mathrm{tr}\big(D_\mu(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu)\big)\\ &= \mathrm{tr}\big(\partial_\mu(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu)\big) + i\,\mathrm{tr}\big(\big[\mathcal{A}_\mu,(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu)\big]\big)\\ &= \partial_\mu\,\mathrm{tr}(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu) + 0 \quad \langle\!\langle\,\text{since trace of a commutator is zero}\,\rangle\!\rangle.\end{aligned} \tag{S.37}$$

Thus,

$$\delta\mathcal{L}_{\mathrm{YM}} = \frac{2}{g^2}\,\mathrm{tr}\big((D_\mu\mathcal{F}^{\mu\nu})\delta\mathcal{A}_\nu\big) - \text{a total divergence} \tag{S.38}$$

so the net Yang-Mills action varies by

$$\delta S = \frac{2}{g^2}\int\! d^4x\ \mathrm{tr}\big(D_\mu\mathcal{F}^{\mu\nu}(x)\delta\mathcal{A}_\nu(x)\big) = \frac{1}{g^2}\int\! d^4x\,\sum_a D_\mu\mathcal{F}^{a\mu\nu}(x)\times\delta\mathcal{A}^a_\nu(x). \tag{S.39}$$

To make this variation vanish for any infinitesimal $\delta\mathcal{A}^a_\nu(x)$ we need $D_\mu\mathcal{F}^{a\mu\nu}(x)\equiv 0$, and this becomes the Euler–Lagrange equation for the Yang–Mills theory,

$$D_\mu\mathcal{F}^{a\mu\nu} = 0. \tag{S.40}$$

Problem **2**(a):

In problem **1**(f) we saw that under infinitesimal variations of the gauge fields, the YM Lagrangian varies by

$$\delta\mathcal{L}_{\mathrm{YM}} \;=\; \frac{1}{g^2}\sum_a D_\mu\mathcal{F}^{a\mu\nu}\times\delta\mathcal{A}^a_\nu \;+\; \partial_\mu(\cdots) \;=\; \sum_a D_\mu F^{a\mu\nu}\times\delta A^a_\nu \;+\; \partial_\mu(\cdots). \tag{S.41}$$

Now let's add the matter Lagrangian $\mathcal{L}_{\mathrm{mat}}(\phi, D\phi)$ for some matter fields in a non-trivial multiplet (or multiplets) of the gauge symmetry. When we vary the gauge fields $A^a_\nu(x)$ while keeping the matter fields $\phi(x)$ fixed, the covariant derivatives $D\phi$ vary due to $igA^a_\nu t^a\phi$ terms in $D_\nu\phi$, which leads to non-trivial variation

$$\delta\mathcal{L}_{\mathrm{mat}} \;=\; \sum_a \frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial A^a_\nu}\times\delta A^a_\nu \;\equiv\; -\sum_a J^{a\nu}\times\delta A^a_\nu\,. \tag{S.42}$$

Altogether, the net action of the theory varies by

$$\delta S \;=\; \int\! d^4x \sum_a \Big(D_\mu F^{a\mu\nu}(x) \;-\; J^{a\nu}(x)\Big)\times\delta A^a_\nu(x). \tag{S.43}$$

Requiring this variation to vanish for any $\delta A^a_\nu(x)$ leads to the field equations

$$D_\mu F^{a\mu\nu} \;=\; J^{a\nu}, \tag{S.44}$$

or in matrix notations $D_\mu F^{\mu\nu} = J^\nu$. This is the non-abelian version of the Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$.

In the abelian EM theory, the equations $\partial_\mu F^{\mu\nu} = J^\nu$ require the electric current to be conserved, $\partial_\nu J^\nu = \partial_\nu\partial_\mu F^{\mu\nu} = 0$ since $F^{\mu\nu} = -F^{\nu\mu}$ and the derivatives commute with each other. The non-abelian tension fields $F^{\mu\nu}$ are also antisymmetric in $\mu\leftrightarrow\nu$, but the covariant derivatives do not commute, $D_\mu D_\nu \neq D_\nu D_\mu$. Therefore,

$$D_\nu J^\nu \;=\; D_\nu D_\mu F^{\mu\nu} \;=\; \tfrac{1}{2}[D_\mu, D_\nu]\mathcal{F}^{\mu\nu} \;=\; \tfrac{i}{2}[\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}] \tag{S.45}$$

where the last equality works exactly as in problem **3**(c) — the $\mathcal{F}^{a\mu\nu}$ fields form an adjoint multiplet of fields, and for any such multiplet packed into an hermitian $N\times N$ matrix Φ,

$[D_\mu, D_\nu]\Phi = i[\mathcal{F}_{\mu\nu}, \Phi]$. However, unlike a generic matrix Φ which may commute or not commute with the $\mathcal{F}_{\mu\nu}$, for any μ and ν the $\mathcal{F}^{\mu\nu}$ matrix always commutes with *itself.* Thus,

$$[\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}] \;=\; 0 \quad \text{even before summing over } \mu \text{ and } \nu. \tag{S.46}$$

Of course, after the summing over μ and ν we still have a zero, thus $D_\nu D_\mu \mathcal{F}^{\mu\nu}(x) \equiv 0$.

Thus, consistency of the field equations (S.44) for the gauge fields requires the non-abelian currents $J^{a\mu}$ to be *covariantly conserved:*

$$D_\nu J^\nu \;=\; D_\mu D_\nu F^{\mu\nu} \;=\; 0, \tag{S.47}$$

or in components

$$\partial_\nu J^{a\nu} \;-\; f^{abc}\mathcal{A}^b_\nu J^{c\nu} \;=\; 0. \tag{S.48}$$

Note: because of the covariantizing term here, we do not have conserved net charges; alas,

$$\frac{d}{dt}\int d^3\mathbf{x}\, J^{a0}(\mathbf{x},t) \;\neq\; 0. \tag{S.49}$$

Problem **2**(b):

The currents J^a_μ come from the covariant derivatives in the Lagrangian for the scalar fields

$$\mathcal{L}_{\rm mat} \;=\; D_\mu\Psi^\dagger D^\mu\Psi \;-\; V(\Psi^\dagger\Psi). \tag{S.50}$$

Expanding the covariant derivatives $D_\mu\Psi^\dagger$ and $D^\mu\Psi$ in components of Ψ^i, Ψ^*_i, and A^a_μ, we obtain

$$D_\mu\Psi^*_i \;=\; \partial_\mu\Psi^*_i \;-\; \frac{ig}{2}\,A^a_\mu\Psi^*_i\big(\lambda^a\big)^i{}_j, \qquad D^\mu\Psi^i \;=\; \partial^\mu\Psi^i \;+\; \frac{ig}{2}\,A^{a\mu}\big(\lambda^a\big)^i{}_j\Psi^j, \tag{S.51}$$

and hence

$$\begin{aligned} J^a_\mu \;&=\; -\frac{\partial\mathcal{L}_{\rm mat}}{\partial A^a_\mu} \;=\; -D^\nu\Psi^*_i \times \frac{\partial D_\nu\Psi^i}{\partial A^a_\mu} \;-\; \frac{\partial D_\nu\Psi^*_i}{\partial A^a_\nu} \times D^\mu\Psi^i \\ &=\; \frac{g}{2}\Big(-iD^\nu\Psi^*_i \times \delta^\mu_\nu\big(\lambda^a\big)^i{}_j\Psi^j \;+\; i\delta^\mu_\nu\Psi^*_j\big(\lambda^a\big)^j{}_i \times D^\nu\Psi^i\Big) \\ &=\; \frac{g}{2}\Big(-i(D^\mu\Psi^\dagger)\lambda^a\Psi \;+\; i\Psi^\dagger\lambda^a(D^\mu\Psi)\Big) \\ &=\; -g\,\mathrm{Im}\Big(\Psi^\dagger\lambda^a D^\mu\Psi\Big), \end{aligned} \tag{S.52}$$

exactly as in eq. (16).

Now let's combine these component currents in the matrix $J_\mu = J_\mu^a \times \frac{1}{2}\lambda^a$. Each term on the RHS of eq. (16) has form $\Psi_1^\dagger \lambda^a \Psi_2$ where either $\Psi_1^\dagger = \Psi^\dagger$ while $\Psi_2 = D_\mu \Psi$ or $\Psi_1 = D_\mu \Psi^\dagger$ while $\Psi_2 = \Psi$. For each of these combinations, the identity (17) leads to

$$\sum_a \left(\lambda^a\right)^i_{\ j} \times \left(\Psi_1^\dagger \lambda^a \Psi_2\right) \;=\; 2\Psi_2^i \Psi_{1j}^* \;-\; \frac{2}{N}\left(\Psi_1^\dagger \Psi_2\right) \times \delta^i_{\ j} \,. \tag{S.53}$$

Thus, altogether the matrix-valued $SU(N)$ symmetry current J_μ has matrix elements

$$\left(J_\mu\right)^i_{\ j} \;=\; \frac{ig}{2}\Big((D_\mu\Psi)^i \Psi^*_{\ j} \;-\; \Psi^i (D_\mu \Psi^*)_j\Big) \;-\; \frac{ig}{2N}\,\delta^i_{\ j} \times \Big(\Psi^\dagger D_\mu \Psi \;-\; (D_\mu \Psi^\dagger)\Psi\Big), \tag{S.54}$$

or in matrix form

$$\begin{aligned} J_\mu \;\stackrel{\rm def}{=}\; \sum_a J_\mu^a \times \tfrac{1}{2}\lambda^a \;=\;& \frac{ig}{2}\Big((D_\mu\Psi)\otimes\Psi^\dagger \;-\; \Psi\otimes D_\mu\Psi^\dagger\Big) \\ &-\; \frac{ig}{2N}\Big(\Psi^\dagger D_\mu\Psi \;-\; (D_\mu\Psi^\dagger)\Psi\Big) \times \mathbf{1}_{N\times N} \,. \end{aligned} \tag{18}$$

Quod erat demonstrandum.

Problem **2**(c):

Under a local $SU(N)$ symmetry $U(x)$, the scalar fields $\Psi(x)$ and $\Psi^\dagger(x)$ and their *covariant* derivatives $D_\mu\Psi(x)$ and $D_\mu\Psi^\dagger(x)$ transform according to

$$\begin{aligned} \Psi(x) \;\mapsto\; U(x)\Psi(x), &\quad \Psi^\dagger(x) \;\mapsto\; \Psi^\dagger(x) U^\dagger(x), \\ D_\mu\Psi(x) \;\mapsto\; U(x) D_\mu\Psi(x), &\quad D_\mu\Psi^\dagger(x) \;\mapsto\; (D_\mu\Psi^\dagger(x)) U^\dagger(x). \end{aligned} \tag{S.55}$$

Consequently,

$$D_\mu\Psi^i(x) \times \Psi_j^*(x) \;\mapsto\; U^i_{\ k}(x) D_\mu \Psi^k \times \Psi_\ell^* (U^\dagger(x))^\ell_{\ j} \,, \tag{S.56}$$

or in the matrix form

$$D_\mu\Psi(x)\otimes\Psi^\dagger(x) \;\mapsto\; U(x)\Big(D_\mu\Psi(x)\otimes\Psi^\dagger(x)\Big) \times U^\dagger(x). \tag{S.57}$$

Likewise,

$$\Psi(x)\otimes D_\mu\Psi^\dagger(x) \;\mapsto\; U(x)\Big(\Psi(x)\otimes D_\mu\Psi^\dagger(x)\Big) \times U^\dagger(x). \tag{S.58}$$

The first term on the RHS of eq. (18) for the matrix-valued current combines these two expressions, so the whole first term transforms under the local symmetries as

$$\text{(first term)} \;\mapsto\; U(x) \times \text{(first term)} \times U^\dagger(x). \tag{S.59}$$

As to the second term on the RHS of eq. (18), it's gauge invariant because

$$\Psi^\dagger(x) D_\mu \Psi(x) \;\mapsto\; \Psi^\dagger(x) U^\dagger(x) U(x) D_\mu \Psi(x) \;=\; \Psi^\dagger(x) D_\mu \Psi(x) \tag{S.60}$$

and likewise

$$D_\mu \Psi^\dagger(x) \Psi(x) \;\mapsto\; D_\mu \Psi^\dagger(x) \Psi(x). \tag{S.61}$$

However, for a unitary $U(x)$ matrix we may treat the unit matrix as transforming according to

$$\mathbf{1} \;\mapsto\; \mathbf{1} \;=\; U(x) \times \mathbf{1} \times U^\dagger(x), \tag{S.62}$$

so the gauge invariant second term on the RHS of eq. (18) can also be viewed as transforming according to

$$\text{(second term)} \;\mapsto\; U(x) \times \text{(second term)} \times U^\dagger(x). \tag{S.63}$$

Altogether, combining eqs. (S.59) and (S.63), we find that the matrix-valued currents $J_\mu(x)$ transform under the gauge symmetries as

$$J_\mu(x) \;\mapsto\; U(x) \times J_\mu(x) \times U^\dagger(x), \tag{S.64}$$

exactly as in eq. (12). Thus, the component currents $J^a_\mu(x)$ transform into each other as members of the adjoint multiplet of the $SU(N)$ group.

Problem **2**(d):

First, let's derive the Leibniz rule for the adjoint multiplets of the form $Q^a = \Psi^\dagger \lambda^a \Psi'$:

$$D_\mu\bigl(\Psi^\dagger\lambda^a\Psi'\bigr) \;=\; D_\mu\Psi^\dagger\lambda^a\Psi' \;+\; \Psi^\dagger\lambda^a D_\mu\Psi'. \tag{S.65}$$

Proof:

$$\begin{aligned} D_\mu\bigl(\Psi^\dagger\lambda^a\Psi'\bigr) \;&\equiv\; \partial_\mu\bigl(\Psi^\dagger\lambda^a\Psi'\bigr) \;-\; gf^{abc}A^b_\mu\bigl(\Psi^\dagger\lambda^c\Psi'\bigr) \\ &=\; (\partial_\mu\Psi^\dagger)\lambda^a\Psi' \;+\; \Psi^\dagger\lambda^a(\partial_\mu\Psi') \;-\; gA^b_\mu\,\Psi^\dagger\bigl(f^{abc}\lambda^c \;=\; -\tfrac{i}{2}[\lambda^a,\lambda^b]\bigr)\Psi' \\ &=\; \Bigl(\partial_\mu\Psi^\dagger \;-\; \frac{ig}{2}\,A^b_\mu\Psi^\dagger\lambda^b\Bigr)\lambda^a\Psi' \;+\; \Psi^\dagger\lambda^a\Bigl(\partial_\mu\Psi' \;+\; \frac{ig}{2}\,A^b_\mu\lambda^b\Psi'\Bigr) \\ &=\; D_\mu\Psi^\dagger\lambda^a\Psi' \;+\; \Psi^\dagger\lambda^a D_\mu\Psi'. \end{aligned} \tag{S.66}$$

Thanks to this lemma, the non-abelian currents (S.52) satisfy

$$\begin{aligned} D_\mu J^{a\mu} \;&=\; -\frac{ig}{2}D_\mu\Bigl(\Psi^\dagger\lambda^a D^\mu\Psi \;-\; D^\mu\Psi^\dagger\lambda^a\Psi\Bigr) \\ &=\; -\frac{ig}{2}\Bigl(\cancel{D_\mu\Psi^\dagger\lambda^a D^\mu\Psi} \;+\; \Psi^\dagger\lambda^a D_\mu D^\mu\Psi \;-\; D_\mu D^\mu\Psi^\dagger\lambda^a\Psi \;-\; \cancel{D^\mu\Psi^\dagger\lambda^a D_\mu\Psi}\Bigr) \\ &=\; g\,\mathrm{Im}\Bigl(\Psi^\dagger\lambda^a D_\mu D^\mu\Psi\Bigr). \end{aligned} \tag{S.67}$$

Now let the scalar fields satisfy their covariant equations of motion

$$D_\mu\frac{\partial\mathcal{L}}{\partial(D_\mu\Psi_i)} \;=\; \frac{\partial\mathcal{L}}{\partial\Psi_i}\,, \qquad D_\mu\frac{\partial\mathcal{L}}{\partial(D_\mu\Psi^{*i})} \;=\; \frac{\partial\mathcal{L}}{\partial\Psi^{*i}}\,. \tag{S.68}$$

For the Lagrangian (3.13) these equations read

$$\begin{aligned} D_\mu D^\mu\Psi^{*i} \;&=\; -\frac{\partial V}{\partial\Psi_i} \;=\; -\Psi^{i*}\times\left(m^2 \;+\; \frac{\lambda}{2}\Psi^\dagger\Psi\right), \\ D_\mu D^\mu\Psi_i \;&=\; -\frac{\partial V}{\partial\Psi^{*i}} \;=\; -\left(m^2 \;+\; \frac{\lambda}{2}\Psi^\dagger\Psi\right)\times\Psi_i, \end{aligned} \tag{S.69}$$

so for the fields obeying these equations

$$\Psi^\dagger\lambda^a D_\mu D^\mu\Psi \;=\; -\left(m^2 \;+\; \frac{\lambda}{2}\Psi^\dagger\Psi\right)\times\Psi^\dagger\lambda^a\Psi \;=\; (\text{real})\times(\text{real}) \tag{S.70}$$

for any hermitian matrix λ^a, and therefore

$$D_\mu J^{a\mu} \;=\; g\,\mathrm{Im}\Bigl(\Psi^\dagger\lambda^a D_\mu D^\mu\Psi\Bigr) \;=\; 0. \tag{S.71}$$

Problem $\mathbf{3}$(a):

$$
\begin{aligned}
\left[\hat{J}^i,\hat{J}^j\right] &\equiv \tfrac{1}{4}\epsilon^{ik\ell}\epsilon^{jmn}\left[\hat{J}^{k\ell},\hat{J}^{mn}\right] \;=\; \langle\!\langle\text{ by eq. (19) }\rangle\!\rangle\\
&= \tfrac{1}{4}\epsilon^{ik\ell}\epsilon^{jmn}\left(-ig^{km}\hat{J}^{\ell n}+ig^{kn}\hat{J}^{\ell m}+ig^{\ell m}\hat{J}^{kn}-g^{\ell n}\hat{J}^{km}\right)\\
&\qquad \langle\!\langle\text{ by antisymmetry of the }\epsilon\text{'s }\rangle\!\rangle\\
&= \epsilon^{ik\ell}\epsilon^{jmn}\times -ig^{km}\hat{J}^{\ell n}\\
&= i\hat{J}^{\ell n}\times\left(-g^{km}\epsilon^{ik\ell}\epsilon^{jmn} = +\delta^{km}\epsilon^{ik\ell}\epsilon^{jmn} = \delta^{ij}\delta^{\ell n}-\delta^{in}\delta^{\ell j}\right)\\
&= 0 - i\hat{J}^{ji} = +i\hat{J}^{ij}\\
&\equiv +i\epsilon^{ijk}\hat{J}^k.
\end{aligned}
\tag{S.72}
$$

$$
\begin{aligned}
\left[\hat{J}^i,\hat{K}^j\right] &\equiv \tfrac{1}{2}\epsilon^{ik\ell}\left[\hat{J}^{k\ell},\hat{J}^{0j}\right] \;=\; \langle\!\langle\text{ by eq. (19) }\rangle\!\rangle\\
&= \tfrac{1}{2}\epsilon^{ik\ell}\left(-ig^{k0}\hat{J}^{\ell j}+ig^{kj}\hat{J}^{\ell 0}+ig^{\ell 0}\hat{J}^{kj}-ig^{\ell j}\hat{J}^{k0}\right)\\
&= \tfrac{1}{2}\epsilon^{ik\ell}\left(0-i\delta^{kj}\hat{J}^{\ell 0}+0+i\delta^{\ell j}\hat{J}^{k0}\right)\\
&\equiv \tfrac{1}{2}\epsilon^{ik\ell}\left(+i\delta^{kj}\hat{K}^{\ell}-i\delta^{\ell j}\hat{K}^{k}\right)\\
&= \tfrac{1}{2}\epsilon^{ij\ell}\hat{K}^{\ell}-\tfrac{1}{2}\epsilon^{ikj}\hat{K}^{j}\\
&= \epsilon^{ijk}\hat{K}^k,
\end{aligned}
\tag{S.73}
$$

$$
\begin{aligned}
\left[\hat{K}^i,\hat{K}^j\right] &\equiv \left[\hat{J}^{0i},\hat{J}^{0j}\right] \;=\; \langle\!\langle\text{ by eq. (19) }\rangle\!\rangle\\
&= -ig^{00}\hat{J}^{ij}+ig^{0j}\hat{J}^{i0}+ig^{i0}\hat{J}^{0j}-ig^{ij}\hat{J}^{00}\\
&= -i\hat{J}^{ij}+0+0+0,\\
&\equiv -i\epsilon^{ijk}\hat{J}^k.
\end{aligned}
\tag{S.74}
$$

Problem $\mathbf{3}$(b):

$$
\begin{aligned}
\left[\hat{V}^i,\hat{J}^j\right] &\equiv \tfrac{1}{2}\epsilon^{jk\ell}\left[\hat{V}^i,\hat{J}^{k\ell}\right] = \tfrac{1}{2}\epsilon^{jk\ell}\left(ig^{ik}\hat{V}^\ell-ig^{i\ell}\hat{V}^k\right)\\
&= \tfrac{1}{2}\epsilon^{jk\ell}\left(-i\delta^{ik}\hat{V}^\ell+\delta^{i\ell}\hat{V}^k\right) = -\tfrac{i}{2}\epsilon^{ji\ell}\hat{V}^\ell+\tfrac{i}{2}\epsilon^{jki}\hat{V}^k\\
&= i\epsilon^{ijk}\hat{V}^k,
\end{aligned}
\tag{S.75}
$$

$$
\begin{aligned}
\left[\hat{V}^0,\hat{J}^j\right] &= \tfrac{1}{2}\epsilon^{jk\ell}\left[\hat{V}^0,\hat{J}^{k\ell}\right] = \tfrac{1}{2}\epsilon^{jk\ell}\left(\cancel{ig^{0k}\hat{V}^\ell}-\cancel{ig^{0\ell}\hat{V}^k}\right)\\
&= 0,
\end{aligned}
\tag{S.76}
$$

$$\left[\hat{V}^i, \hat{K}^j\right] = \left[\hat{V}^i, \hat{J}^{0j}\right] = ig^{i0}\hat{V}^j - ig^{ij}\hat{V}^0$$
$$= +i\delta^{ij}\hat{V}^0, \tag{S.77}$$
$$\left[\hat{V}^0, \hat{K}^j\right] = \left[\hat{V}^0, \hat{J}^{0j}\right] = ig^{00}\hat{V}^j - ig^{0j}\hat{V}^0$$
$$= +i\hat{V}^j. \tag{S.78}$$

Note that the Hamiltonian of a relativistic theory is a member of a 4-vector multiplet $\hat{P}^\mu = (\hat{H}, \hat{\mathbf{P}})$ where $\hat{\mathbf{P}}$ is the net momentum operator. Applying the above equations to the $\hat{P}^\mu$ vectors, we obtain

$$\begin{aligned}
\left[\hat{P}^i, \hat{J}^j\right] &= i\epsilon^{ijk}\hat{P}^k, \\
\left[\hat{H}, \hat{J}^j\right] &= 0, \\
\left[\hat{P}^i, \hat{K}^j\right] &= +i\delta^{ij}\hat{H}, \\
\left[\hat{H}, \hat{K}^j\right] &= +i\hat{P}^j.
\end{aligned} \tag{S.79}$$

In particular, the Hamiltonian $\hat{H}$ commutes with the three angular momenta $\hat{J}^j$ but it does not commute with the three generators $\hat{K}^k$ of the Lorentz boosts.

Problem **3**(c):

In the ordinary quantum mechanics, it is often said that generators of continuous symmetries must commute with the Hamiltonian operator. However, this is true only for the symmetries that act in a time independent manner — for example, rotating the 3D space by the same angle at all times t. But when the transformation rules of a symmetry depend on time, the Hamiltonian must change to account for this time dependence.

In a Lorentz boost, the transform $\mathbf{x} \to \mathbf{x}'$ obviously depends on time, which changes the way the transformed quantum fields such as $\hat{\Phi}'(\mathbf{x}, t)$ depend on t. Consequently, the Hamiltonian $\hat{H}$ of the theory must change so that the new Heisenberg equations would match the new time dependence. In terms of the generators, this means that the boost generators $\hat{K}^i$ should **not** commute with the Hamiltonian.

Note that this non-commutativity is not caused by the Lorentz boosts affecting the time itself, $t' = L^0{}_\mu x^\mu \neq t$. Even in non-relativistic theories — where the time is absolute — the generators of symmetries which affect the other variables in a time-dependent matter do not commute with the Hamiltonian $\hat{H}$.

Indeed, consider a Galilean transform from one non-relativistic moving frame into another, $\mathbf{x}' = \mathbf{x}+\mathbf{v}t$ but $t' = t$. This is a good symmetry of non-relativistic particles interacting with each other but not subject to any external potential,

$$\hat{H} \;=\; \sum_a \frac{1}{2M_a}\hat{\mathbf{p}}_a^2 \;+\; \tfrac{1}{2}\sum_{a\neq b} V(\hat{\mathbf{x}}_a - \hat{\mathbf{x}}_b). \tag{S.80}$$

A unitary operator $\hat{\mathcal{G}}$ realizing a Galilean symmetry acts on coordinate and momentum operators as

$$\hat{\mathcal{G}}\hat{\mathbf{x}}_a\hat{\mathcal{G}}^\dagger \;=\; \hat{\mathbf{x}}_a \;+\; \mathbf{v}t, \quad \hat{\mathcal{G}}\hat{\mathbf{p}}_a\hat{\mathcal{G}}^\dagger \;=\; \hat{\mathbf{p}}_a \;+\; M_a\mathbf{v}, \tag{S.81}$$

and it also transforms the Hamiltonian into

$$\hat{\mathcal{G}}\hat{H}\hat{\mathcal{G}}^\dagger \;=\; \hat{H} \;+\; \mathbf{v}\cdot\hat{\mathbf{P}}_{\rm tot} \;+\; \tfrac{1}{2}M_{\rm tot}\mathbf{v}^2. \tag{S.82}$$

In terms of the Galilean boost generators $\hat{\mathbf{K}}_G$,

$$\hat{\mathcal{G}} \;=\; \exp\bigl(-i\mathbf{v}\cdot\hat{\mathbf{K}}_G\bigr), \tag{S.83}$$

so under an infinitesimal boost $\mathbf{v} = \vec{\epsilon}$, various operators transform according to

$$\widehat{\mathcal{O}} \;\to\; \widehat{\mathcal{O}} \;+\; \delta\widehat{\mathcal{O}}, \qquad \delta\widehat{\mathcal{O}} \;=\; -i\epsilon^i\left[\hat{K}^i_G, \widehat{\mathcal{O}}\right]. \tag{S.84}$$

Consequently, the commutation relations with the boost generators follow from the infinitesimal boosts, for example

$$\begin{aligned}
\delta\hat{\mathbf{x}}_a \;&=\; \vec{\epsilon}\,t &&\Longrightarrow \left[\hat{x}^i_a, \hat{K}^j_G\right] \;=\; i\delta^{ij}\times t,\\
\delta\hat{\mathbf{p}}_a \;&=\; \vec{\epsilon}\,M_a &&\Longrightarrow \left[\hat{p}^i_a, \hat{K}^j_G\right] \;=\; i\delta^{ij}\times M_a,\\
\delta\hat{H} \;&=\; \vec{\epsilon}\cdot\hat{\mathbf{P}}_{\rm tot} &&\Longrightarrow \left[\hat{H}, \hat{K}^j_G\right] \;=\; i\hat{P}^i_{\rm tot}\,.
\end{aligned} \tag{S.85}$$

In particular, the Hamiltonian does NOT commute with the Galilean boost generators.

Problem **4**(a):

Consider a linear combination $\frac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu}$ of Lorentz generators with some generic coefficients $N_{\mu\nu} = -N_{\nu\mu}$. The infinitesimal Lorents transform

$$L^{\mu}_{\ \nu} \ = \ \exp(i\epsilon N)^{\mu}_{\ \nu} \ = \ \delta^{\mu}_{\nu} \ + \ i\epsilon N^{\mu}_{\ \nu} \ + \ O(\epsilon^2) \tag{S.86}$$

preserves a given momentum p^μ, $L^{\mu}_{\ \nu}P^\nu = P^\mu$ if and only if

$$N^{\mu}_{\ \nu}p^\nu \ = \ 0. \tag{S.87}$$

In 3D terms, $N^{ij} = \epsilon^{ijk}a^k$ and $N^{0k} = -N^{k0} = b^k$ for some 3-vectors $\mathbf{a}$ and $\mathbf{b}$, the generator in question is

$$\tfrac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu} \ = \ \mathbf{a}\cdot\hat{\mathbf{J}} \ + \ \mathbf{b}\cdot\hat{\mathbf{K}}, \tag{S.88}$$

and

$$\begin{aligned} N^{0\nu}p_\nu \ &= \ -N^{0j}p^j \ = \ -\mathbf{b}\cdot\mathbf{p}, \\ N^{i\nu}p_\nu \ &= \ -N^{ij}p^j \ + \ N^{i0}p^0 \ = \ -\epsilon^{ijk}p^j a^k \ - \ b^i E, \end{aligned} \tag{S.89}$$

so the condition (S.87) becomes

$$\mathbf{a}\times\mathbf{p} \ - \ \mathbf{b}E \ = \ 0 \quad \text{and} \quad \mathbf{b}\cdot\mathbf{p} \ = \ 0. \tag{S.90}$$

Actually, the second condition here is redundant, so the general solution is

$$\text{any } \mathbf{a}, \quad \mathbf{b} \ = \ \mathbf{a}\times\frac{\mathbf{p}}{E} \tag{S.91}$$

and hence

$$\tfrac{1}{2}N_{\mu\nu}\hat{J}^{\mu\nu} \ = \ \mathbf{a}\cdot\hat{\mathbf{J}} \ + \ \frac{(\mathbf{a}\times\mathbf{p})\cdot\hat{\mathbf{K}}}{E} \ = \ \frac{\mathbf{a}}{E}\cdot\Big(E\hat{\mathbf{J}} \ + \ \mathbf{p}\times\hat{\mathbf{K}}\Big) \quad \text{for any } \mathbf{a}. \tag{S.92}$$

In other words, the Lorentz symmetries preserving the momentum p^μ have 3 generators, namely the components of the 3-vector

$$\hat{\mathbf{R}} \ = \ E\,\hat{\mathbf{J}} \ + \ \mathbf{p}\times\hat{\mathbf{K}}\,. \tag{22}$$

Quod erat demonstrandum.

Problem **4**(b):

Consider a massive particle moving at a slower-than-light speed β in z direction, so the energy is $E = \gamma m$ (where $\gamma = 1/\sqrt{1-\beta^2}$) while the 3-momentum is $\mathbf{p} = (0, 0, \beta\gamma m)$. Consequently, the three components of the $\hat{\mathbf{R}}$ vector for this energy-momentum are

$$\hat{R}^x = \gamma m\hat{J}^x - \beta\gamma m\hat{K}^y, \quad \hat{R}^y = \gamma m\hat{J}^y + \beta\gamma m\hat{K}^x, \quad \hat{R}^z = \gamma m\hat{J}^z. \tag{S.93}$$

These 3 operators generate the little group $G(p)$ of the particle's 4–momentum $p^\mu = (E, \mathbf{p})$. To see that this little group happens to be isomorphic to the 3D rotation group $SO(3)$, we need to find 3 linear combinations $\tilde{J}^{x,y,z}$ of the operators (S.93) which obey the angular-momentum commutation relations

$$\left[\tilde{J}^i, \tilde{J}^j\right] = i\epsilon^{ijk}\tilde{J}^k. \tag{S.94}$$

My choice of the (properly normalized) generators $\tilde{J}^{x,y,z}$ is spelled out in eqs. (23). Let's see that they indeed obey the commutation relations (S.94):

$$\begin{aligned}
\left[\hat{J}^z, \widetilde{J}^x\right] &= \gamma\left[\hat{J}^z, \hat{J}^x\right] - \beta\gamma\left[\hat{J}^z, \hat{K}^y\right] = \gamma \times i\hat{J}^y - \beta\gamma \times (-i\hat{K}^x) \\
&= i\big(\gamma\hat{j}^y + \beta\gamma\hat{K}^x\big) = i\widetilde{J}^y, \\
\left[\hat{J}^z, \widetilde{J}^y\right] &= \gamma\left[\hat{J}^z, \hat{J}^y\right] + \beta\gamma\left[\hat{J}^z, \hat{K}^x\right] = \gamma \times (-i\hat{J}^x) + \beta\gamma \times (+i\hat{K}^y) \\
&= -i\big(\gamma\hat{J}^x - i\beta\gamma\hat{K}^y\big) = -i\widetilde{J}^x, \\
\left[\widetilde{J}^x, \widetilde{J}^y\right] &= \gamma^2\left[\hat{J}^x, \hat{J}^y\right] - \beta\gamma^2\left[\hat{K}^y, \hat{J}^y\right] + \beta\gamma^2\left[\hat{J}^x, \hat{K}^x\right] - \beta^2\gamma^2\left[\hat{K}^y, \hat{K}^x\right] \\
&= \gamma^2 \times i\hat{J}^z - 0 + 0 - \beta^2\gamma^2 \times i\hat{J}^z \\
&= i\hat{J}^z \times \big(\gamma^2(1-\beta^2) = 1\big) = i\hat{J}^z.
\end{aligned} \tag{S.95}$$

Quod erat demonstrandum.

Problem **4**(c):

For a massless particle, we cannot rescale the little group generators as in eqs. (23) since the $1/m$ factor for the $\tilde{J}^{x,y}$ generators would be infinite. Instead, the best we can do is to use the $1/E$ factor for all three rescaled generators, hence eqs. (24) and (25). Consequently, instead of eqs. (S.95) for the commutators of the rescaled generators, we get

$$\begin{aligned}
\left[\hat{J}^z, \hat{I}^x\right] &= \left[\hat{J}^z, \hat{J}^x\right] - \left[\hat{J}^z, \hat{K}^y\right] = i\hat{J}^y - (-i\hat{K}^x) = i\hat{I}^y, \\
\left[\hat{J}^z, \hat{I}^y\right] &= \left[\hat{J}^z, \hat{J}^y\right] + \left[\hat{J}^z, \hat{K}^x\right] = (-i\hat{J}^x) - (+i\hat{K}^y) = -i\hat{I}^x, \\
\left[\hat{I}^x, \hat{I}^y\right] &= \left[\hat{J}^x, \hat{J}^y\right] - \left[\hat{K}^y, \hat{J}^y\right] + \left[\hat{J}^x, \hat{K}^x\right] - \left[\hat{K}^y, \hat{K}^x\right] \\
&= i\hat{J}^z - 0 + 0 - i\hat{J}^z = 0,
\end{aligned} \tag{S.96}$$

precisely as in eq. (26).

The commutation relations (26) are different from the angular-momentum commutation relations, and they cannot be brought to the form (S.94) by any finite rescaling of the generators. Consequently, the little group of a light-like momentum $p^\mu = (E, 0, 0, E)$ is NOT isomorphic to the $SO(3)$.

Instead, the commutation relations (26) are similar to the commutation relations between the z component of the angular momentum and the x and y components of the linear momentum,

$$\left[\hat{J}^z, \hat{P}^x\right] = +i\hat{P}^y, \quad \left[\hat{J}^z, \hat{P}^y\right] = -i\hat{P}^x, \quad \left[\hat{P}^x, \hat{P}^y\right] = 0. \tag{S.97}$$

The $\hat{J}^z$ operator generates rotations around the z axis, *i.e.*, within the xy plane, while the $\hat{P}^x$ and $\hat{P}^i$ operators generate translations in that plane. Together, they generate the group ISO(2) of isometries — rotations and translations — in 2 space dimensions.

Thus, we see that the little group $G(p)$ of a light-like momentum of a massless particle is isomorphic to the ISO(2).

Problem **4**(d):

Finally, consider a particle moving faster-than-light in the z direction, so its momentum $p^\mu = (p^0, 0, 0, p^3)$ has $p^3 > p^0$. Such a tachyon must have negative mass^2, so let us denote

$$M_i^2 = -m^2 = -p_\mu p^\mu > 0, \quad \gamma_i = \frac{p^0}{M_i}, \quad \beta = \frac{p^3}{p_0} \tag{S.98}$$

where the subscript i stands for 'imaginary' and $\beta > 1$ is the faster-than-light speed. In these notations, let us rescale the little group generators (22) according to

$$\begin{aligned}
\tilde{K}^x &= \frac{+1}{M_i}\,\hat{R}^y = \beta\gamma_i\hat{K}^x + \gamma_i\hat{J}^y, \\
\tilde{K}^y &= \frac{-1}{M_i}\,\hat{R}^x = \beta\gamma_i\hat{K}^y - \gamma_i\hat{J}^x, \\
\tilde{J}^z &= \frac{1}{\gamma_i M_i}\,\hat{R}^z = \hat{J}^z, \quad \text{the helicity.}
\end{aligned} \tag{S.99}$$

Consequently, the commutation relations of the rescaled operators become

$$\begin{aligned}
\left[\hat{J}^z, \tilde{K}^x\right] &= \beta\gamma_i\left[\hat{J}^z, \hat{K}^x\right] + \gamma_i\left[\hat{J}^z, \hat{J}^y\right] \\
&= \beta\gamma_i(+i\hat{K}^y) + \gamma^i(-i\hat{J}^x) \\
&= +i\tilde{K}^y,
\end{aligned} \tag{S.100}$$

$$\begin{aligned}
\left[\hat{J}^z, \tilde{K}^y\right] &= \beta\gamma_i\left[\hat{J}^z, \hat{K}^y\right] - \gamma_i\left[\hat{J}^z, \hat{J}^x\right] \\
&= \beta\gamma_i(-i\hat{K}^y) + \gamma^i(+i\hat{J}^x) \\
&= -i\tilde{K}^x,
\end{aligned} \tag{S.101}$$

$$\begin{aligned}
\left[\tilde{K}^x, \tilde{K}^y\right] &= \beta^2\gamma_i^2\left[\hat{K}^x, \hat{K}^y\right] + \beta\gamma_i^2\left[\hat{J}^y, \hat{K}^y\right] - \beta\gamma_i^2\left[\hat{K}^x, \hat{J}^x\right] - \gamma_i^2\left[\hat{J}^y, \hat{J}^x\right] \\
&= \beta^2\gamma_i^2 \times (-i\hat{J}^z) + \beta\gamma_i^2 \times 0 - \beta\gamma_i^2 \times 0 - \gamma_i^2 \times (-i\hat{J}^z) \\
&= -i(\beta^2 - 1)\gamma_i^2 \times \hat{J}^z \\
&= -i\hat{J}^z,
\end{aligned} \tag{S.102}$$

where the last equality follows from the kinematic relation

$$(\beta^2 - 1) \times \gamma_i^2 = \frac{p_3^2 - p_0^2}{p_0^2} \times \frac{p_0^2}{M_i^2} = \frac{p_3^2 - p_0^2}{M_i^2} = 1 \tag{S.103}$$

Altogether, the generators (S.99) obey the commutation relations

$$\left[\hat{J}^z, \tilde{K}^x\right] = +i\tilde{K}^y, \quad \left[\hat{J}^z, \tilde{K}^y\right] = -i\tilde{K}^x, \quad \left[\tilde{K}^x, \tilde{K}^y\right] = -i\hat{J}^z, \tag{S.104}$$

which are exactly similar to the $SO^+(2,1)$ commutation relations

$$\left[\hat{J}^z, \hat{K}^x\right] = +i\hat{K}^y, \quad \left[\hat{J}^z, \hat{K}^y\right] = -i\hat{K}^x, \quad \left[\hat{K}^x, \hat{K}^y\right] = -i\hat{J}^z. \tag{S.105}$$

Therefore, the little group $G(p)$ of a tachyonic momentum is isomorphic to the continuous Lorentz group $SO^+(2,1)$ in $2+1$ dimensions.