Problem **1**(a):

The quantum state $|p,\lambda\rangle$ has definite momentum p^μ, thus $\hat{P}_\mu\,|p,\lambda\rangle = p_\mu\,|p,\lambda\rangle$ and likewise

$$\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\,|p,\lambda\rangle \;=\; \tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}p^\gamma\hat{J}^{\alpha\beta}\,|p,\lambda\rangle\,. \tag{S.1}$$

To work out the RHS of this formula, let's spell out the operator

$$\hat{Q}_\mu \;\stackrel{\rm def}{=}\; \tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}p^\gamma\hat{J}^{\alpha\beta} \tag{S.2}$$

in components:

$$\hat{Q}_0 \;=\; \tfrac{1}{2}\epsilon_{0ijk}p^k\hat{J}^{ij} \;=\; \tfrac{1}{2}\epsilon^{ijk}p^k\times\epsilon^{ij\ell}\hat{J}^\ell \;=\; \mathbf{p}\cdot\hat{\mathbf{J}}, \tag{S.3}$$

$$\begin{aligned}\hat{Q}_i \;&=\; \tfrac{1}{2}\epsilon_{ijk0}p^0\hat{J}^{jk} \;+\; \tfrac{1}{2}\epsilon_{ij0k}p^k\hat{J}^{j0} \;+\; \tfrac{1}{2}\epsilon_{i0jk}p^k\hat{J}^{0j}\\ &=\; -\tfrac{1}{2}\epsilon^{ijk}E\times\epsilon^{ij\ell}\hat{J}^\ell \;+\; \tfrac{1}{2}\epsilon^{ijk}p^k\times\hat{K}^j \;-\; \tfrac{1}{2}\epsilon^{ijk}p^k(-\hat{K}^j)\\ &=\; -E\hat{J}^i \;+\; (\hat{\mathbf{K}}\times\mathbf{p})^i.\end{aligned} \tag{S.4}$$

For a massless particle with a lightlike momentum $p^\mu = (E, E\mathbf{v})$, $|\mathbf{v}| = 1$, the components of $\hat{Q}^\mu$ operators become

$$\begin{aligned}\hat{\mathbf{Q}} \;&=\; +E\hat{\mathbf{J}} \;+\; E\mathbf{v}\times\hat{\mathbf{K}} \;=\; E\hat{\mathbf{I}}\\ &\qquad\langle\!\langle \text{ for } \hat{\mathbf{I}} \text{ as in eq. (1) } \rangle\!\rangle,\\ \hat{Q}^0 \;&=\; E\mathbf{v}\cdot\hat{\mathbf{J}} \;=\; E\hat{\lambda}.\end{aligned} \tag{S.5}$$

When this operator acts on the particle state $|p,\lambda\rangle$ which has definite helicity and is also annihilated by the two transverse generators $\hat{\mathbf{I}}_\perp$ as in eq. (3), we get

$$\hat{Q}^0\,|p,\lambda\rangle \;=\; E\lambda\,|p,\lambda\rangle \;=\; \lambda p^0\,|p,\lambda\rangle \tag{S.6}$$

and also

$$\hat{\mathbf{I}}\,|p,\lambda\rangle \;=\; \lambda\mathbf{v}\,|p,\lambda\rangle \tag{S.7}$$

and hence

$$\hat{\mathbf{Q}}\,|p,\lambda\rangle \;=\; E\lambda\mathbf{v}\,|p,\lambda\rangle \;=\; \lambda\mathbf{p}\,|p,\lambda\rangle\,. \tag{S.8}$$

Altogether, in 4d terms we get

$$\hat{Q}^\mu\,|p,\lambda\rangle \;=\; \lambda p^\mu\,|p,\lambda\rangle \tag{S.9}$$

and hence (after lowering the index μ on both sides)

$$\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\,|p,\lambda\rangle \;=\; \lambda\hat{P}_\mu\,|p,\lambda\rangle\,. \tag{4}$$

Quod erat demonstrandum.

Problem **1**(b):

Consider a *continuous Lorentz transform* $x^\mu \to x'^\mu = L^\mu_{\ \nu}x^\nu$ acting on the $|p,\lambda\rangle$ state of a massless particle. The operators on both sides of both sides of eq. (4) transform as Lorentz vectors,

$$\hat{\mathcal{D}}(L)\hat{P}_\nu\hat{\mathcal{D}}^\dagger(L) \;=\; L_\nu^{\ \mu}\hat{P}_\mu\,, \quad \hat{\mathcal{D}}(L)\left(\tfrac{1}{2}\epsilon_{\nu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)\hat{\mathcal{D}}^\dagger(L) \;=\; L_\nu^{\ \mu}\left(\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right). \tag{S.10}$$

Consequently, the transformed state

$$\hat{\mathcal{D}}(L)\,|p,\lambda\rangle \;=\; |Lp,??\rangle \tag{S.11}$$

satisfies the same eq. (4) as the original state $|p,\lambda\rangle$. Indeed,

$$\begin{aligned} L_\nu^{\ \mu}\left(\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)|Lp,??\rangle \;&=\; \hat{\mathcal{D}}(L)\left(\tfrac{1}{2}\epsilon_{\nu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)\hat{\mathcal{D}}^\dagger(L)\times\hat{\mathcal{D}}\,|p,\lambda\rangle \\ &=\; \hat{\mathcal{D}}(L)\times\left(\tfrac{1}{2}\epsilon_{\nu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)|p,\lambda\rangle \\ \langle\!\langle\text{ by eq. (4) }\rangle\!\rangle\quad &=\; \hat{\mathcal{D}}(L)\times\lambda\hat{P}_\nu\,|p,\lambda\rangle \\ &=\; \lambda\times\hat{\mathcal{D}}(L)\hat{P}_\nu\hat{\mathcal{D}}^\dagger(L)\times\hat{\mathcal{D}}\,|p,\lambda\rangle \\ &=\; \lambda\times L_\nu^{\ \mu}\hat{P}_\mu\,|Lp,??\rangle\,, \end{aligned} \tag{S.12}$$

and hence

$$\left(\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\hat{J}^{\alpha\beta}\hat{P}^\gamma\right)|Lp,??\rangle \;=\; \lambda\hat{P}_\mu\,|Lp,??\rangle\,. \tag{S.13}$$

Note that this equation for the transformed state $|Lp,??\rangle$ has exactly the same helicity eigenvalue λ as the original eq. (4), which means that the transformed state has the same helicity as the original state. And since the momentum and the helicity completely determine the quantum state of a particle up to an overall phase, it follows that

$$\hat{\mathcal{D}}(L)\,|p,\lambda\rangle \;=\; |Lp,\,\text{same}\,\lambda\rangle \times e^{i\,\text{phase}}. \tag{11}$$

Thus, *for massless particles, the continuous Lorentz transforms preserve helicity!*

Problem **2**(a):

The Lorentz generators $\hat{J}^i$ and $\hat{K}^i$ obey commutation relations

$$\left[\hat{J}^i,\hat{J}^j\right] \;=\; i\epsilon^{ijk}\hat{J}^k, \qquad \left[\hat{J}^i,\hat{K}^j\right] \;=\; i\epsilon^{ijk}\hat{K}^k, \qquad \left[\hat{K}^i,\hat{K}^j\right] \;=\; -i\epsilon^{ijk}\hat{J}^k. \tag{S.14}$$

Consequently, for two components of $\hat{\mathbf{J}}_+$ or two components of $\hat{\mathbf{J}}_-$ we have

$$\begin{aligned}\left[\hat{J}^i_\pm,\hat{J}^j_\pm\right] &= \tfrac{1}{4}\left[\hat{J}^i,\hat{J}^j\right] \;\pm\; \tfrac{i}{4}\left[\hat{K}^i,\hat{J}^j\right] \;\pm\; \tfrac{i}{4}\left[\hat{J}^i,\hat{K}^j\right] \;-\; \tfrac{1}{4}\left[\hat{K}^i,\hat{K}^j\right]\\ &= \tfrac{1}{4}\,i\epsilon^{ijk}\hat{J}^k \;\pm\; \tfrac{i}{4}\,i\epsilon^{ijk}\hat{K}^k \;\pm\; \tfrac{i}{4}\,i\epsilon^{ijk}\hat{K}^k \;-\; \tfrac{1}{4}\,(-i)\epsilon^{ijk}\hat{J}^k\\ &= \tfrac{i}{2}\epsilon^{ijk}\hat{J}^k \;\mp\; \tfrac{1}{2}\epsilon^{ijk}\hat{K}^k \;=\; i\epsilon^{ijk}\times\left(\tfrac{1}{2}\hat{J}^k \;\pm\; \tfrac{i}{2}\hat{K}^k\right)\\ &= i\epsilon^{ijk}\hat{J}^k_\pm\,,\end{aligned} \tag{S.15}$$

while for one component of $\hat{\mathbf{J}}_+$ and one component of $\hat{\mathbf{J}}_-$ (or the other way around) we get

$$\begin{aligned}\left[\hat{J}^i_\pm,\hat{J}^j_\mp\right] &= \tfrac{1}{4}\left[\hat{J}^i,\hat{J}^j\right] \;\pm\; \tfrac{i}{4}\left[\hat{K}^i,\hat{J}^j\right] \;\mp\; \tfrac{i}{4}\left[\hat{J}^i,\hat{K}^j\right] \;+\; \tfrac{1}{4}\left[\hat{K}^i,\hat{K}^j\right]\\ &= \tfrac{1}{4}\,i\epsilon^{ijk}\hat{J}^k \;\pm\; \tfrac{i}{4}\,i\epsilon^{ijk}\hat{K}^k \;\mp\; \tfrac{i}{4}\,i\epsilon^{ijk}\hat{K}^k \;+\; \tfrac{1}{4}\,(-i)\epsilon^{ijk}\hat{J}^k\\ &= \tfrac{i}{4}\epsilon^{ijk}\bigl(\hat{J}^k \pm i\hat{K}^k \mp i\hat{K}^k - \hat{J}^k\bigr)\\ &= 0.\end{aligned} \tag{S.16}$$

Problem $\mathbf{2}$(b):

Let's start by calculating the squares of the matrices (9) representing the boost $B(r,\mathbf{n})$,

$$\big[M_{\mathbf{2}}(B)\big]^2 \;=\; \exp(-r\mathbf{n}\cdot\boldsymbol{\sigma}) \quad\text{and}\quad \big[M_{\overline{\mathbf{2}}}(B)\big]^2 \;=\; \exp(+r\mathbf{n}\cdot\boldsymbol{\sigma}). \tag{S.17}$$

For any unit vector $\mathbf{n}$,

$$(\mathbf{n}\cdot\boldsymbol{\sigma})^2 \;=\; \mathbf{n}^2 \;=\; 1, \tag{S.18}$$

hence for any integer $k\geq 0$

$$(\mathbf{n}\cdot\boldsymbol{\sigma})^k \;=\; \begin{cases} 1 & \text{for even } k,\\ (\mathbf{n}\cdot\boldsymbol{\sigma}) & \text{for odd } k. \end{cases} \tag{S.19}$$

Consequently, expanding the exponentials in eq. (S.17) into powers of the rapidity r yields

$$\begin{aligned}
\exp(\mp r\mathbf{n}\cdot\boldsymbol{\sigma}) \;&=\; \sum_{k=0}^{\infty}\frac{(\mp r)^k}{k!}\times(\mathbf{n}\cdot\boldsymbol{\sigma})^k\\
&=\; \sum_{\text{even}\,k}\frac{(\mp r)^k}{k!}\times 1 \;+\; \sum_{\text{odd}\,k}\frac{(\mp r)^k}{k!}\times(\mathbf{n}\cdot\boldsymbol{\sigma})\\
&=\; \cosh(r)\times 1 \;\mp\; \sinh(r)\times(\mathbf{n}\cdot\boldsymbol{\sigma})\\
&\qquad\langle\!\langle\,\text{translating from the rapidity to the } \beta \text{ and } \gamma \text{ parameters (10)}\,\rangle\!\rangle\\
&=\; \gamma\times 1 \;\mp\; \beta\gamma\times(\mathbf{n}\cdot\boldsymbol{\sigma})\\
&=\; \gamma\times\big(1\mp\beta\mathbf{n}\cdot\boldsymbol{\sigma}\big).
\end{aligned} \tag{S.20}$$

Thus,

$$\big[M_{\mathbf{2}}(B)\big]^2 \;=\; \gamma\big(1\,-\,\beta\mathbf{n}\cdot\boldsymbol{\sigma}\big), \qquad \big[M_{\overline{\mathbf{2}}}(B)\big]^2 \;=\; \gamma\big(1\,+\,\beta\mathbf{n}\cdot\boldsymbol{\sigma}\big), \tag{S.21}$$

and therefore

$$M_{\mathbf{2}}(B) \;=\; \sqrt{\gamma}\times\sqrt{1-\beta\mathbf{n}\cdot\boldsymbol{\sigma}}, \qquad M_{\overline{\mathbf{2}}}(B) \;=\; \sqrt{\gamma}\times\sqrt{1+\beta\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{11}$$

Quod erat demonstrandum.

Problem $\mathbf{2}$(c):

Among the 3 Pauli matrices $\boldsymbol{\sigma}$, the σ_1 and the σ_3 matrices are real while the σ_2 is imaginary. At the same time, the σ_1 and the σ_3 anticommute with the σ_2 while the σ_2 commutes with itself. Consequently,

$$\sigma_2(\sigma_{1,3})^*\sigma_2 \;=\; +\sigma_2\sigma_{1,3}\sigma_2 \;=\; -\sigma_{1,3}\sigma_2\sigma_2 \;=\; -\sigma_{1,3}\,, \tag{S.22}$$

while

$$\sigma_2(\sigma_2)^*\sigma_2 \;=\; -\sigma_2\sigma_2\sigma_2 \;=\; -\sigma_2\,, \tag{S.23}$$

thus for all 3 Pauli matrices

$$\sigma_2\boldsymbol{\sigma}^*\sigma_2 \;=\; -\boldsymbol{\sigma}. \tag{S.24}$$

Now let's apply this identity to the $\mathbf{2}$ and the $\overline{\mathbf{2}}$ representations of the same Lorentz symmetry L. Any *continuous* Lorentz symmetry must be generated by some linear combinations of the angular momenta and boost generators, thus

$$\hat{L} \;=\; \exp\bigl(-i\mathbf{a}\cdot\hat{\mathbf{J}} \,-\, i\mathbf{b}\cdot\hat{\mathbf{K}}\bigr) \tag{S.25}$$

for some real 3-vectors $\mathbf{a}$ and $\mathbf{b}$. In the $\mathbf{2}$ representation, the $\hat{J}^i$ generators act as $\frac{1}{2}\sigma^i$ matrices while the $\hat{K}^i$ generators act as $-\frac{i}{2}\sigma^i$, thus

$$\mathbf{a}\cdot\hat{\mathbf{J}} \;+\; \mathbf{b}\cdot\hat{\mathbf{K}} \quad \text{acts as } \tfrac{1}{2}(\mathbf{a}-i\mathbf{b})\cdot\boldsymbol{\sigma} \tag{S.26}$$

and hence

$$M \;\stackrel{\rm def}{=}\; M_{\mathbf{2}}(L) \;=\; \exp\bigl(-\tfrac{i}{2}(\mathbf{a}-i\mathbf{b})\cdot\boldsymbol{\sigma}\bigr). \tag{S.27}$$

Likewise, in the $\overline{\mathbf{2}}$ representation, the $\hat{J}^i$ generators also act as $\frac{1}{2}\sigma^i$ matrices but the $\hat{K}^i$

generators act as $+\frac{i}{2}\sigma^i$, thus

$$\mathbf{a}\cdot\hat{\mathbf{J}}\;+\;\mathbf{b}\cdot\hat{\mathbf{K}}\quad\text{acts as }\tfrac{1}{2}(\mathbf{a}+i\mathbf{b})\cdot\boldsymbol{\sigma} \tag{S.28}$$

and hence

$$\overline{M}\;\stackrel{\text{def}}{=}\;M_{\overline{\mathbf{2}}}(L)\;=\;\exp\bigl(-\tfrac{i}{2}(\mathbf{a}+i\mathbf{b})\cdot\boldsymbol{\sigma}\bigr). \tag{S.29}$$

In terms of the complex 3-vector $\mathbf{c}=\frac{1}{2}\mathbf{a}-\frac{i}{2}\mathbf{b}$,

$$M\;=\;\exp(-i\mathbf{c}\cdot\boldsymbol{\sigma})\quad\text{and}\quad\overline{M}\;=\;\exp(-i\mathbf{c}^*\cdot\boldsymbol{\sigma}), \tag{S.30}$$

and thanks to eq. (S.24), and two such matrices are related to each other as

$$\overline{M}\;=\;\sigma_2M^*\sigma_2\quad\text{and}\quad M\;=\;\sigma_2\overline{M}^*\sigma_2. \tag{S.31}$$

To prove this relation, note that eq. (S.24) implies

$$\sigma_2(-i\mathbf{c}\cdot\boldsymbol{\sigma})^*\sigma_2\;=\;+i\mathbf{c}^*\cdot(\sigma_2\boldsymbol{\sigma}^*\sigma_2)\;=\;-i\mathbf{c}^*\cdot\boldsymbol{\sigma}, \tag{S.32}$$

hence

$$\sigma_2\bigl[(-i\mathbf{c}\cdot\boldsymbol{\sigma})^2\bigr]^*\sigma_2\;=\;\sigma_2(-i\mathbf{c}\cdot\boldsymbol{\sigma})^*\sigma_2\times\sigma_2(-i\mathbf{c}\cdot\boldsymbol{\sigma})^2\sigma_2\;=\;(-i\mathbf{c}^*\cdot\boldsymbol{\sigma})\times(-i\mathbf{c}^*\cdot\boldsymbol{\sigma})\;=\;(-i\mathbf{c}^*\cdot\boldsymbol{\sigma})^2, \tag{S.33}$$

and likewise for any integer power n,

$$\sigma_2\bigl[(-i\mathbf{c}\cdot\boldsymbol{\sigma})^n\bigr]^*\sigma_2\;=\;(-i\mathbf{c}^*\cdot\boldsymbol{\sigma})^n. \tag{S.34}$$

Consequently, for a power series like the exponential we have

$$\begin{aligned}\sigma_2M^*\sigma_2\;&=\;\sigma_2\left(\sum_{n=0}^{\infty}\frac{1}{n!}(-i\mathbf{c}\cdot\boldsymbol{\sigma})^n\right)^*\sigma_2\;=\;\sum_{n=0}^{\infty}\frac{1}{n!}\,\sigma_2\bigl((-i\mathbf{c}\cdot\boldsymbol{\sigma})^n\bigr)^*\sigma_2\\&=\;\sum_{n=0}^{\infty}\frac{1}{n!}(-i\mathbf{c}^*\cdot\boldsymbol{\sigma})^n\;=\;\overline{M},\end{aligned} \tag{S.35}$$

and likewise $\sigma_2\overline{M}^*\sigma_2=M$. *Quod erat demonstrandum.*

Problem **2**(d):

Let's start with reality. The matrix $V=V^\mu\overline{\sigma}_\mu$ is hermitian if and only if the 4-vector V^μ is real. For any matrix $M\in SL(2,\mathbf{C})$, the transform

$$V\to V' = MVM^\dagger \tag{S.36}$$

preserves hermiticity: if V is hermitian, then so is V'; indeed

$$\bigl(V'\bigr)^\dagger = \bigl(MVM^\dagger\bigr)^\dagger = \bigl(M^\dagger\bigr)^\dagger V^\dagger M^\dagger = MVM^\dagger = V'. \tag{S.37}$$

In terms of the 4-vectors, this means that if the V^μ is real than the $V'^\mu=L^\mu_{\ \nu}V^\nu$ is also real. In other words, the 4×4 matrix $L^\mu_{\ \nu}(M)$ is real.

Next, let's prove that $L^\mu_{\ \nu}(M)\in O(3,1)$ — it preserves the Lorentz metric $g^{\alpha\beta}$, or equivalently, for any V^μ, $g^{\alpha\beta}V'_\alpha V'_\beta=g^{\alpha\beta}V_\alpha V_\beta$. In terms of the 2×2 matrix $V=V^\mu\overline{\sigma}_\mu$, the Lorentz square of the 4-vector becomes the determinant:

$$g^{\alpha\beta}V_\alpha V_\beta = \det\bigl(V=V_\mu\sigma^\mu\bigr). \tag{S.38}$$

Indeed, from the explicit form of the 4 matrices

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\+i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}+1&0\\0&-1\end{pmatrix} \tag{S.39}$$

we have

$$V = V^\mu\overline{\sigma}_\mu = \begin{pmatrix}V_0+V_3 & V_1-iV_2\\ V_1+iV_2 & V_0-V_3\end{pmatrix}$$

and hence

$$\det(V) = (V_0+V_3)(V_0-V_3)-(V_1-iV_2)(V_1+iV_2) = V_0^2-V_3^2-V_1^2-V_2^2 = g^{\alpha\beta}V_\alpha V_\beta\,. \tag{S.40}$$

The determinant of a matrix product is the product of the individual matrices' determinants. Hence, for the transform (S.36),

$$\det(V') = \det(M)\times\det(V)\times\det(M^\dagger) = \det(V)\times\left|\det(M)\right|^2. \tag{S.41}$$

The M matrices of interest to us belong to the $SL(2,\mathbf{C})$ group — they are complex matrices with unit determinants. There are no other restrictions, but $\det(M) = 1$ is enough to assure $\det(V') = \det(V)$, *cf.* eq. (S.41). Thanks to the relation (S.38), this means

$$g^{\alpha\beta}V'_\alpha V'_\beta \;=\; \det(V') \;=\; \det(V) \;=\; g^{\alpha\beta}V_\alpha V_\beta \tag{S.42}$$

— which proves that the matrix $L^\mu_{\ \nu}(M)$ is indeed Lorentzian.

Problem **2**(e):

To prove that the Lorentz transform $L^\mu_{\ \nu}(M)$ is orthochronous, we need to show that for any V_μ in the forward light cone — that is, $V^2 > 0$ and $V_0 > 0$ — the V'_μ is also in the forward light cone. In matrix terms, $V^2 > 0$ means $\det(V) > 0$ while $V_0 > 0$ means $\mathrm{tr}(V) > 0$; together, these two conditions mean that the 2×2 hermitian matrix V is positive-definite. The transform (S.36) preserves positive definiteness: if for any complex 2-vector $\xi \neq 0$ we have $\xi^\dagger V \xi > 0$, then

$$\xi^\dagger V' \xi \;=\; \xi^\dagger M V M^\dagger \xi \;=\; (M^\dagger\xi)^\dagger V (M^\dagger \xi) \;>\; 0. \tag{S.43}$$

(Note that $M^\dagger\xi \neq 0$ for any $\xi \neq 0$ because $\det(M) \neq 0$.) Thus, for any $M \in SL(2,\mathbf{C})$ the Lorentz transform $V^\mu \to V'^\mu$ preserves the forward light cone — in other words, the $L^\mu_{\ \nu}(M)$ is orthochronous, $L^\mu_{\ \nu}(M) \in O^+(3,1)$.

The simplest proof that the Lorentz transform (16) is proper — $\det(L) = +1$ — for any $SL(2,\mathbf{C})$ matrix M is topological: The $SL(2,\mathbf{C})$ group manifold — which spans all matrices of the form

$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{complex } a,b,c,d, \quad ad \,-\, bc \;=\; 1 \tag{S.44}$$

is connected, so all such matrices are continuously connected to the $\mathbf{1}_{2\times 2}$ matrix. It is easy to see that for $M = 1$, the Lorentz transform $L(1)$ is trivial, $L^\mu_{\ \nu} = \delta^\mu_\nu$, hence all Lorentz transforms of the form (16) are continuously connected to the trivial transform. Consequently, they all must be continuous Lorentz transforms and therefore proper and orthochronous.

An alternative proof is more involved. It involves decomposition of M into a product of a unitary U and a Hermitian H, 2 Lemmas showing that $L(U)$ is a pure 3D rotation (without reflections) while $L(H)$ is a pure Lorentz boost, and the group law from part (f) which tells us that

$$L(M=UH) \;=\; L(U)\times L(H) \;=\; \text{rotation}\times\text{boost}, \tag{S.45}$$

which makes it a continuous Lorentz transform. But this is a rather long proof, so I leave it as optional exercise for the interested students.

Problem **2**(f):

Let $L_1 = L(M_1)$, $L_2 = L(M_2)$ and $L_{12} = L(M_2M_1)$ be Lorentz transforms constructed according to eq. (13–16) for some $SL(2,\mathbf{C})$ matrices M_1 and M_2 and their product M_2M_1. We want to prove that L_{12} obtains from a product of consecutive Lorentz transforms, first L_2 and then L_1, so let's consider how all these transforms act on some 4–vector V^μ. On one hand,

$$\bigl(L_{12}V\bigr)^\nu\,\overline{\sigma}_\nu \;=\; (M_2M_1)\times(V^\nu\overline{\sigma}_\nu)\times(M_2M_1)^\dagger \;=\; M_2M_1\times(V^\nu\overline{\sigma}_\nu)\times M_1^\dagger M_2^\dagger. \tag{S.46}$$

On the other hand,

$$\begin{aligned}\bigl(L_2L_1V\bigr)^\nu\,\overline{\sigma}_\nu \;&=\; M_2\times\bigl((L_1V)^\nu\overline{\sigma}_\nu\bigr)\times M_2^\dagger \;=\; M_2\times\bigl(M_1\times(V^\nu\overline{\sigma}_\nu)\times M_1^\dagger\bigr)\times M_2^\dagger\\ \text{also} \;&=\; M_2M_1\times(V^\nu\overline{\sigma}^\nu)\times M_1^\dagger M_2^\dagger.\end{aligned}$$

Thus we see that

$$\bigl(L_{12}V\bigr)^\nu\,\overline{\sigma}_\nu \;=\; \bigl(L_2L_1V\bigr)^\nu\,\overline{\sigma}_\nu \tag{S.47}$$

and therefore

$$\bigl(L_{12}V\bigr)^\nu \;=\; \bigl(L_2L_1V\bigr)^\nu. \tag{S.48}$$

Moreover, this holds true for any 4–vector V^μ, hence the Lorentz transforms $L_{12} = L(M_2M_1)$ and $L_2L_1 = L(M_2)L(M_1)$ must be equal to each other, *quod erat demonstrandum.*

Problem **2**(g):

For any Lie algebra equivalent to an angular momentum or its analytic continuation, the product of two doublets comprises a triplet and a singlet, $\mathbf{2}\otimes\mathbf{2}=\mathbf{3}\oplus\mathbf{1}$, or in (j) notations, $(\frac{1}{2})\otimes(\frac{1}{2}) = (1)\oplus(0)$. Furthermore, the triplet $\mathbf{3} = (1)$ is symmetric with respect to permutations of the two doublets while the singlet $\mathbf{1}=(0)$ is antisymmetric.

For two separate and independent types of angular momenta $\mathbf{J}_+$ and $\mathbf{J}_-$ we combine the j_+ quantum numbers independently from the j_- and the j_- quantum numbers independently from the j_+. For two bi-spinors, this gives us

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2}) \;=\; (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0). \tag{S.49}$$

Furthermore, the symmetric part of this product should be either symmetric with respect to both the j_+ and the j_- indices or antisymmetric with respect to both indices, thus

$$\big[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\big]_{\text{sym}} \;=\; (1,1)\oplus(0,0). \tag{S.50}$$

Likewise, the antisymmetric part is either symmetric with respect to the j_+ but antisymmetric with respect to the j_- or the other way around, thus

$$\big[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\big]_{\text{antisym}} \;=\; (1,0)\oplus(0,1). \tag{S.51}$$

From the $SO^+(3,1)$ point of view, the bi-spinor $(\frac{1}{2},\frac{1}{2})$ is the Lorentz vector. A general 2-index Lorentz tensor transforms like a product of two such vectors, so from the $SL(2,\mathbf{C})$ point of view it's a product of two bi-spinors, which decomposes to irreducible multiplets according to eq. (S.49).

The Lorentz symmetry respects splitting of a general 2-index tensor into a symmetric tensor $T^{\mu\nu}=+T^{\nu\mu}$ and an antisymmetric tensor $F^{\mu\nu}=-F^{\nu\mu}$. The symmetric tensor corresponds to a symmetrized square of a bi-spinor, which decomposes into irreducible multiplets according to eq. (S.50). The singlet $(0,0)$ component is the Lorentz-invariant trace T^μ_μ while the $(1,1)$ irreducible multiplet is the traceless part of the symmetric tensor.

Likewise, the antisymmetric Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$ decomposes according to eq. (S.51). Here, the irreducible components $(1,0)$ and $(0,1)$ are complex but conjugate to each other; individually, they describe antisymmetric tensors subject to complex duality conditions $\frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}F_{\mu\nu} = \pm iF^{\kappa\lambda}$, or in 3D terms, $\mathbf{E} = \pm i\mathbf{B}$.

Problem **3**(a):

Suppose the γ^μ matrices obey the anticommutation relations (16). Then:

$$\beta^2 = \gamma^0\gamma^0 = g^{00} = +1, \tag{S.52}$$

$$\begin{aligned} \{\beta,\alpha^i\} &= \{\gamma^0,\gamma^0\gamma^i\} = \gamma^0\gamma^0\gamma^i + \gamma^0\gamma^i\gamma^0 \\ &= \gamma^0\times\{\gamma^0,\gamma^i\} = \gamma^0\times 0 = 0, \end{aligned} \tag{S.53}$$

$$\alpha^i\alpha^j = \gamma^0\gamma^i\gamma^0\gamma^j = -\gamma^0\gamma^0\gamma^i\gamma^j = -\gamma^i\gamma^j, \tag{S.54}$$

$$\Downarrow$$

$$\{\alpha^i,\alpha^j\} = -\{\gamma^i,\gamma^j\} = -2g^{ij} = +2\delta^{ij}, \tag{S.55}$$

quod erat demonstrandum.

Conversely, suppose $\beta^2 = 1$, $\{\beta,\alpha^i\} = 0$, and $\{\alpha^i,\alpha^j\} = +2\delta^{ij}$, and let us *define* $\gamma^0 \stackrel{\text{def}}{=} \beta$ and $\gamma^i \stackrel{\text{def}}{=} \beta\alpha^i$. In this case:

$$\gamma^0\gamma^0 = \beta\beta = +1 = g^{00}, \tag{S.56}$$

$$\{\gamma^0,\gamma^i\} = \{\beta,\beta\alpha^i\} = \beta\times\{\beta,\alpha^i\} = \beta\times 0 = 0 = 2g^{0i}, \tag{S.57}$$

$$\gamma^i\gamma^j = \beta\alpha^i\beta\alpha^j = -\beta\beta\alpha^i\alpha^j = -\alpha^i\alpha^j, \tag{S.58}$$

$$\Downarrow$$

$$\{\gamma^i,\gamma^j\} = -\{\alpha^i,\alpha^j\} = -2\delta^{ij} = +2g^{ij}, \tag{S.59}$$

and therefore

$$\forall\mu,\nu: \quad \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\,. \tag{1}$$

Quod erat demonstrandum.

Problem $\mathbf{3}$(b):

$$\gamma^\alpha \gamma_\alpha \;=\; \tfrac{1}{2}\{\gamma^\alpha,\gamma^\beta\}g_{\alpha\beta} \;=\; g^{\alpha\beta}g_{\alpha\beta} \;=\; 4; \tag{S.60}$$

$$\begin{aligned}\gamma^\alpha\gamma^\nu\gamma_\alpha \;&=\; (\gamma^\alpha\gamma^\nu \;=\; 2g^{\nu\alpha} \;-\; \gamma^\nu\gamma^\alpha)\gamma_\alpha \\ &=\; 2\gamma^\nu \;-\; \gamma^\nu(\gamma^\alpha\gamma_\alpha \;=\; 4) \;=\; -2\gamma^\nu\,;\end{aligned} \tag{S.61}$$

$$\begin{aligned}\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha \;&=\; (\gamma^\alpha\gamma^\mu \;=\; 2g^{\mu\alpha} \;-\; \gamma^\mu\gamma^\alpha)\gamma^\nu\gamma_\alpha \\ &=\; 2\gamma^\nu\gamma^\mu \;-\; \gamma^\mu(\gamma^\alpha\gamma^\nu\gamma_\alpha \;=\; -2\gamma^\nu) \\ &=\; 2\gamma^\nu\gamma^\mu \;+\; 2\gamma^\mu\gamma^\nu \;=\; 4g^{\mu\nu}\,;\end{aligned} \tag{S.62}$$

$$\begin{aligned}\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha \;&=\; (\gamma^\alpha\gamma^\lambda \;=\; 2g^{\mu\alpha} \;-\; \gamma^\lambda\gamma^\alpha)\gamma^\mu\gamma^\nu\gamma_\alpha \\ &=\; 2\gamma^\mu\gamma^\nu\gamma^\lambda \;-\; \gamma^\lambda(\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha \;=\; 4g^{\mu\nu}) \\ &=\; (2\gamma^\mu\gamma^\nu \;-\; 4g^{\mu\nu} \;=\; -2\gamma^\nu\gamma^\mu)\gamma^\lambda \\ &=\; -2\gamma^\nu\gamma^\mu\gamma^\lambda.\end{aligned} \tag{S.63}$$

Problem $\mathbf{3}$(c):

First, a Lemma: for any Lorentz vector a^μ, the $\not{a} \stackrel{\text{def}}{=} \gamma^\mu a_\mu$ matrix squares to

$$\not{a}\,\not{a} \;=\; \gamma^\mu a_\mu \gamma^\nu a_\nu \;=\; a_\mu a_\nu \times \Big(\gamma^\mu\gamma^\nu \;=\; g^{\mu\nu} - 2iS^{\mu\nu}\Big) \;=\; a^2 \;-\; i[a_\mu,a_\nu]\times S^{\mu\nu} \tag{S.64}$$

(where the last equality comes from $S^{\mu\nu} = -S^{\nu\mu}$.) For vectors a^μ whose components commute with each other, this formula simplifies to $\not{a}^2 \;=\; a^2$, in particular for the ordinary derivatives ∂^μ, $\not{\partial}^2 = \partial^2$. However, the covariant derivatives D_μ do not commute with each other. Instead, $[D_\mu, D_\nu] = iqF_{\mu\nu}(x)$ where q is the electric charge of the field on which D_μ act; for the electron field $\Psi(x)$, $q = -e$ and hence $[D_\mu, D_\nu]\Psi(x) = -ieF_{\mu\nu}(x)\Psi(x)$. Hence, according to the lemma (S.64),

$$\not{D}^2\Psi \;=\; D^2\Psi \;-\; eF_{\mu\nu}S^{\mu\nu}\Psi. \tag{S.65}$$

Now, suppose the electron field $\Psi(x)$ satisfies the covariant Dirac equation $(i\not{D} - m)\Psi = 0$.

Then for any differential operator $\mathcal{D}$, $\mathcal{D}\times(i\not{D}-m)\Psi=0$, and in particular

$$(-i\not{D}-m)\times(i\not{D}-m)\Psi \;=\; 0. \tag{S.66}$$

The LHS of this formula amounts to

$$(-i\not{D}-m)\times(i\not{D}-m)\Psi \;=\; \bigl(\not{D}^2\,+\,m^2\bigr)\Psi \;=\; \bigl(D^2\,-\,eF_{\mu\nu}S^{\mu\nu}\,+\,m^2\bigr)\Psi, \tag{S.67}$$

which immediately leads to eq. (18).

Problem **3**(d):

The anti-commutation relations (17) imply $\gamma^\mu\gamma^\nu=\pm\gamma^\nu\gamma^\mu$ where the sign is '+' for $\mu=\nu$ and '−' otherwise. Hence for any product Γ of the γ matrices, $\gamma^\mu\Gamma=(-1)^n\Gamma\gamma^\mu$, where n is the number of $\gamma^{\nu\neq\mu}$ factors of Γ. For the $\Gamma=\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, $n=3$ for any $\mu=0,1,2,3$, hence $\gamma^\mu\gamma^5=-\gamma^5\gamma^\mu$.

As to the spin matrices, $\gamma^5\gamma^\mu\gamma^\nu=-\gamma^\mu\gamma^5\gamma^\nu=+\gamma^\mu\gamma^\nu\gamma^5$ and therefore $\gamma^5S^{\mu\nu}=+S^{\mu\nu}\gamma^5$.

Problem **3**(e):

First, the hermiticity:

$$\begin{aligned}\bigl(\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3\bigr)^\dagger \;&=\; -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \;=\; -i(-\gamma^3)(-\gamma^2)(-\gamma^1)(+\gamma^0)\\ &=\; +i\gamma^3\gamma^2\gamma^1\gamma^0 \;=\; +i(\gamma^3\gamma^2\gamma^1)\times\gamma^0 \;=\; +i(-1)^3\gamma^0\times\gamma^3\gamma^2\gamma^1\\ &=\; +i(-1)^3\gamma^0\times(-1)^2\gamma^1\times\gamma^3\gamma^2 \;=\; +i(-1)^3\gamma^0\times(-1)^2\gamma^1\times(-1)\gamma^2\gamma^3\\ &=\; (-1)^6\times(+i\gamma^0\gamma^1\gamma^2\gamma^3) \;\equiv\; +1\times\gamma^5.\end{aligned} \tag{S.68}$$

Second, the square:

$$\begin{aligned}(\gamma^5)^2 \;&=\; \gamma^5(\gamma^5)^\dagger \;=\; (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^3\gamma^2\gamma^1\gamma^0) \;=\; -\gamma^0\gamma^1\gamma^2(\gamma^3\gamma^3)\gamma^2\gamma^1\gamma^0\\ &=\; +\gamma^0\gamma^1(\gamma^2\gamma^2)\gamma^1\gamma^0 = \;-\gamma^0(\gamma^1\gamma^1)\gamma^0 \;=\; +\gamma^0\gamma^0 \;=\; +1.\end{aligned} \tag{S.69}$$

Problem **3**(f):

Since the four Dirac matrices γ^0, γ^1, γ^2, γ^3 all anticommute with each other, we have

$$\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu \;=\; \gamma^{[0}\gamma^1\gamma^2\gamma^{3]} \;=\; 24\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -24i\gamma^5. \tag{S.70}$$

To prove the other identity, we note that a totally antisymmetric product $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}$ vanishes unless the Lorentz indices $\kappa, \lambda, \mu, \nu$ are all distinct — which makes them $0, 1, 2, 3$ in some order. For such indices, the anticommutativity of the Dirac matrices implies $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = -\epsilon^{\kappa\lambda\mu\nu} \times \gamma^0\gamma^1\gamma^2\gamma^3$ (note that $\epsilon^{0123} = -1$), and hence

$$\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} \;=\; -24\epsilon^{\kappa\lambda\mu\nu} \times \gamma^0\gamma^1\gamma^2\gamma^3 \;=\; +24i\epsilon^{\kappa\lambda\mu\nu} \times \gamma^5. \tag{S.71}$$

Problem **3**(g):

$$\begin{aligned} 6i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5 &= \tfrac{6}{24}\,\gamma_\kappa\,\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} \\ &= \tfrac{1}{4}\,\gamma_\kappa\,\Big(\gamma^\kappa\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} \;-\; \gamma^{[\lambda)}\gamma^\kappa\gamma^{(\mu}\gamma^{\nu]} \;+\; \gamma^{[\lambda}\gamma^{\mu)}\gamma^\kappa\gamma^{(\nu]} \;-\; \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}\gamma^\kappa\Big) \\ &= \tfrac{1}{4}\Big(4\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} \;+\; 2\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} \;+\; 4g^{[\lambda\mu}\gamma^{\nu]} \;+\; 2\gamma^{[\nu}\gamma^\mu\gamma^{\lambda]}\Big) \\ &= \tfrac{1}{4}(4+2+0-2) \times \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} \;=\; \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}. \end{aligned} \tag{S.72}$$

Problem **3**(h):

Proof by inspection: In the Weyl basis and in the 2×2 block form, the 16 matrices are

$$\begin{aligned} &\mathbf{1}_{4\times4} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\ &\tfrac{1}{2}\gamma^{[i}\gamma^{j]} = -i\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \quad \tfrac{1}{2}\gamma^{[0}\gamma^{i]} = \begin{pmatrix} -\sigma^i & 0 \\ 0 & +\sigma^i \end{pmatrix}, \\ &\gamma^5\gamma^0 = \begin{pmatrix} 0 & -\mathbf{1} \\ +\mathbf{1} & 0 \end{pmatrix}, \quad \gamma^5\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & +\mathbf{1} \end{pmatrix}, \end{aligned} \tag{S.73}$$

and their linear independence is self-evident. Since there are only 16 independent 4×4 matrices altogether, any such matrix Γ is a linear combination of the matrices (S.73). *Quod erat demonstrandum.*

Algebraic Proof: Without making any assumption about the matrix form of the γ^μ operators, let us consider the Clifford algebra (17). Using $\gamma^\mu\gamma^\nu = \pm\gamma^\nu\gamma^\mu$ where the sign is + for $\mu = \nu$ and − for $\mu \neq \nu$, we may re-order any product of the γ matrices as $\pm\gamma^0\cdots\gamma^0\,\gamma^1\cdots\gamma^1\,\gamma^2\cdots\gamma^2\,\gamma^3\cdots\gamma^3$. Moreover, since each γ^μ squares to +1 or −1, we may further simplify the product in question to to $\pm(\gamma^0 \text{ or } 1)\times(\gamma^1 \text{ or } 1)\times(\gamma^2 \text{ or } 1)\times(\gamma^3 \text{ or } 1)$. The net result is (up to a sign or $\pm i$ factor) one of the 16 matrices: 1, or a γ^μ, or a $\gamma^\mu\gamma^\nu$ for $\mu \neq \nu$ which equals to $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]}$, or a $\gamma^\lambda\gamma^\mu\gamma^\nu$ for 3 different λ, μ, ν which equals to $\frac{1}{6}\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = i\epsilon^{\lambda\mu\nu\rho}\gamma^5\gamma_\rho$ (*cf.* part (g)), or $\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^5$. Consequently, any operator Γ algebraically constructed of the γ's is a linear combination of these 16 matrices.

Incidentally, this proof explains why the Dirac matrices are 4×4 in $d = 4$ spacetime dimensions: the 16 linearly-independent products of Dirac matrices require matrix size to be $\sqrt{16} = 4$.

Technically, we may also use matrices of size $4n \times 4n$, but then we would have $\gamma^\mu = \gamma^\mu_{4\times 4}\otimes \mathbf{1}_{n\times n}$, and ditto for all their products. Physically, this means combining the Dirac spinor index with some other index $i = 1,\ldots,n$ which has nothing to do with Lorentz symmetry. Nobody wants such an index confusion, so physicists always stick to 4×4 Dirac matrices in 4 spacetime dimensions.

Problem **4**(a):

I have already written down the γ^5 and the $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} = -2iS^{\mu\nu}$ in the Weyl basis. (S.73), but here is the detailed calculation, in case you need it:

$$\begin{aligned}
\gamma^5 &= i\begin{pmatrix} 0 & \sigma^0 \\ \overline{\sigma}^0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ \overline{\sigma}^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ \overline{\sigma}^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ \overline{\sigma}^3 & 0 \end{pmatrix} \\
&= i\begin{pmatrix} \sigma^0\overline{\sigma}^1\sigma^2\overline{\sigma}^3 & 0 \\ 0 & \overline{\sigma}^0\sigma^1\overline{\sigma}^2\sigma^3 \end{pmatrix} = \begin{pmatrix} +i\sigma^1\sigma^2\sigma^3 & 0 \\ 0 & -i\sigma^1\sigma^2\sigma^3 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}.
\end{aligned} \tag{4}$$

Problem **4**(b):

In the Weyl basis for the γ matrices,

$$\gamma^\mu\gamma^\nu \;=\; \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \overline{\sigma}^\nu & 0 \end{pmatrix} \;=\; \begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu \end{pmatrix}, \tag{S.74}$$

and hence

$$S^{ij} \;=\; \frac{i}{4}\begin{pmatrix} -\sigma^{[i}\sigma^{j]} & 0 \\ 0 & -\sigma^{[i}\sigma^{j]} \end{pmatrix} \;=\; \frac{\epsilon^{ijk}}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{S.75}$$

while

$$S^{0k} \;=\; -S^{k0} \;=\; \tfrac{i}{2}\gamma^0\gamma^k \;=\; \frac{i}{2}\begin{pmatrix} -\sigma^k & 0 \\ 0 & +\sigma^k \end{pmatrix}. \tag{S.76}$$

By inspection, these spin matrices agree with the $\mathbf{2}+\overline{\mathbf{2}}$ reducible representation of the Lorentz generators,

$$S^{ij} \;=\; \epsilon^{ijk}\begin{pmatrix} \hat{J}^k_{\mathbf{2}} & 0 \\ 0 & \hat{J}^k_{\overline{\mathbf{2}}} \end{pmatrix} \quad \text{while} \quad S^{0i} \;=\; -S^{i0} \;=\; \begin{pmatrix} \hat{K}^i_{\mathbf{2}} & 0 \\ 0 & \hat{K}^i_{\overline{\mathbf{2}}} \end{pmatrix}, \tag{S.77}$$

or in one formula

$$S^{\mu\nu} \;=\; \begin{pmatrix} \hat{J}^{\mu\nu}_{\mathbf{2}} & 0 \\ 0 & \hat{J}^{\mu\nu}_{\overline{\mathbf{2}}} \end{pmatrix}. \tag{5}$$

Problem **4**(c):

In problem 2(c) we saw that for any continuous Lorentz transform L,

$$M_R(L) \;=\; \sigma_2 M_L^*(L)\sigma_2 \quad \text{and} \quad M_L(L) \;=\; \sigma_2 M_R^*\sigma_2\,. \tag{S.78}$$

Consequently, considering the transformation laws (23) of the Weyl spinors $\psi_L(x)$ and $\psi_R(x)$ and their complex conjugates, we see that

$$\begin{aligned} \psi'_L(x') \;&=\; M_L \times \psi_L(x), \\ \psi'^*_L(x') \;&=\; M_L^* \times \psi_L(x), \\ \sigma_2 \times \psi'^*_L(x') \;&=\; \sigma_2 \times M_L^* \times \psi_L^*(x) \;=\; \sigma_2 M_L^*\sigma_2 \times \sigma_2\psi_L^*(x) \;=\; M_R \times \sigma_2\psi_L^*(x), \end{aligned} \tag{S.79}$$

thus $\sigma_2 \times \psi_L^*(x)$ transforms under the continuous Lorentz transforms exactly like the $\psi_R(x)$.

Likewise,

$$\begin{aligned}
\psi'_R(x') &= M_R \times \psi_R(x), \\
\psi'^*_R(x') &= M^*_R \times \psi_R(x), \\
\sigma_2 \times \psi'^*_R(x') &= \sigma_2 \times M^*_R \times \psi^*_R(x) \;=\; \sigma_2 M^*_R \sigma_2 \times \sigma_2 \psi^*_R(x) \;=\; M_L \times \sigma_2 \psi^*_R(x),
\end{aligned} \tag{S.80}$$

thus $\sigma_2 \times \psi^*_R(x)$ transforms under the continuous Lorentz transforms exactly like the $\psi_L(x)$.

Problem **4**(d):

Given the γ^μ matrices (19) and the decomposition (23) of the Dirac spinor field $\Psi(x)$ into 2 Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$, we have

$$(i\gamma^\mu\partial_\mu \,-\, m)\Psi = \begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar\sigma^\mu\partial_\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \;=\; \begin{pmatrix} -m\psi_L \,+\, i\sigma^\mu\partial_\mu\psi_R \\ i\bar\sigma^\mu\partial_\mu\psi_L \,-\, m\psi_R \end{pmatrix} \tag{S.81}$$

while

$$\overline{\Psi} \;=\; \Psi^\dagger\gamma^0 \;=\; (\,\psi^\dagger_L \quad \psi^\dagger_R\,) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;=\; (\,\psi^\dagger_R \quad \psi^\dagger_L\,)\,. \tag{S.82}$$

Consequently, the Dirac Lagrangian becomes

$$\begin{aligned}
\mathcal{L} &= \overline{\Psi}\,(i\gamma^\mu\partial_\mu \,-\, m)\Psi \\
&= \psi^\dagger_R\Big(-m\psi_L \,+\, i\sigma^\mu\partial_\mu\psi_R\Big) \;+\; \psi^\dagger_L\Big(i\bar\sigma^\mu\partial_\mu\psi_L \,-\, m\psi_R\Big) \\
&= i\psi^\dagger_R\sigma^\mu\partial_\mu\psi_R \;+\; i\psi^\dagger_L\bar\sigma^\mu\partial_\mu\psi_L \;-\; m\,\psi^\dagger_L\psi_R \;-\; m\,\psi^\dagger_R\psi_L\,.
\end{aligned} \tag{S.83}$$

Problem **4**(e):

For the massless fermions — and only for the massless fermions, — the last two terms in the Lagrangian (S.83) go away, and we are left with

$$\mathcal{L}_{\text{massless}} \;=\; i\psi^\dagger_L\bar\sigma^\mu\partial_\mu\psi_L \;+\; i\psi^\dagger_R\sigma^\mu\partial_\mu\psi_R \tag{S.84}$$

where the two Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ are completely independent from each other. In particular, they obey independent Weyl equations

$$i\bar{\sigma}^\mu \partial_\mu \psi_L \;=\; 0 \quad \text{and} \quad i\sigma^\mu \partial_\mu \psi_R \;=\; 0. \tag{S.85}$$

On the other hand, for $m \neq 0$ the two last terms in the Lagrangian (S.83) connect the ψ_L and ψ_R to each other and we cannot have one without the other. In particular, their equations of motion become mixed,

$$i\bar{\sigma}^\mu \partial_\mu \psi_L \;=\; m\psi_R \quad \text{and} \quad i\sigma^\mu \partial_\mu \psi_R \;=\; m\psi_L\,. \tag{S.86}$$

Problem **5**(a):

As we saw in class, the Dirac equation $(i\not\partial - m)\Psi = 0$ implies the Klein–Gordon equation $(\partial^2 + m^2)\Psi = 0$, hence any plane-wave solution of the Dirac equation must have on-shell momentum p^μ with $p^2 = m^2$. But there is more to the Dirac equation than the Klein–Gordon equation, hence eqs. (25) for the spinor coefficients of the $e^{\mp ipx}$ plane waves. Indeed, for $\Psi_\alpha(x) = e^{-ipx} u_\alpha$ the Dirac equation becomes

$$(i\not\partial - m)e^{-ipx}u \;=\; e^{-ipx} \times (\not p - m)u, \tag{S.87}$$

so the constant spinor u_α must obey the matrix equation $(\not p - m)u = 0$. Likewise, for $\Psi_\alpha(x) = e^{+ipx} v_\alpha$ the Dirac equation becomes

$$(i\not\partial - m)e^{+ipx}v \;=\; e^{+ipx} \times (-\not p - m)v, \tag{S.88}$$

so the spinor v_α must obey the matrix equation $(\not p + m)v = 0$. Conversely, for any on-shell momentum p and any spinors u_α and v_α obeying the matrix equations (25) the plane waves $e^{-ipx}u_\alpha$ and $e^{+ipx}v_\alpha$ obey the Dirac equation.

Problem **5**(b):
For $\mathbf{p} = \mathbf{0}$, $p^0 = +m$ and $\not{p} - m = m(\gamma^0 - 1)$. Hence, the $u(\mathbf{p} = \mathbf{0}, s)$ spinors satisfy $(\gamma^0 - 1)u = 0$, or in the Weyl basis

$$\begin{pmatrix} -\mathbf{1}_{2\times 2} & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & -\mathbf{1}_{2\times 2} \end{pmatrix} u = 0 \quad \Longrightarrow \quad u = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \tag{S.89}$$

where ζ is an arbitrary two-component spinor. Its normalization follows from $u^\dagger u = 2\,\zeta^\dagger\zeta$, so if we want $u^\dagger u = 2E_{\mathbf{p}} = 2m$ (for $\mathbf{p} = 0$), then we need $\zeta^\dagger\zeta = m$. Equivalently, we want $\zeta = \sqrt{m}\,\xi$ — and hence u as in eq. (27) — for a conventionally normalized spinor ξ with $\xi^\dagger\xi = 0$.

Note that there are two independent choices of ξ, normalized to $\xi_s^\dagger\xi_{s'} = \delta_{s,s'}$, so they give rise to two independent $u_\alpha(\mathbf{0}, s)$ spinors normalized to $u^\dagger(\mathbf{0}, s)u(\mathbf{0}, s') = 2m\delta_{s,s'} = 2E_{\mathbf{p}}\delta_{s,s'}$. They correspond to the two spin states of the $\mathbf{p} = 0$ electron. In terms of the spin vector, $\mathbf{S} = \frac{1}{2}\xi_s^\dagger\boldsymbol{\sigma}\xi_s$.

Problem **5**(c):
The Dirac equation is Lorentz-covariant, so we may obtain solutions for all $p^\mu = (+E_{\mathbf{p}}, \mathbf{p})$ by simply Lorentz-boosting the solutions (27) from the rest frame where $p_0^\mu = (+m, \mathbf{0})$. Thus,

$$u(p,s) = M_D(B)\,u(p_0,s) = \begin{pmatrix} M_L & 0 \\ 0 & M_R \end{pmatrix} \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{m}\,M_L\,\xi_s \\ \sqrt{m}\,M_R\,\xi_s \end{pmatrix} \tag{S.90}$$

where M_D, M_L, and M_R are respectively Dirac-spinor, LH–Weyl-spinor, and RH–Weyl-spinor representations or the Lorentz boost B from $p^\mu = (m, \mathbf{0})$ to $p^\mu = (E, \mathbf{p})$. As we saw in problem 2(b), for a boost of velocity β in the direction $\mathbf{n}$,

$$M_L = \sqrt{\gamma} \times \sqrt{1 - \beta\mathbf{n}\cdot\boldsymbol{\sigma}}\,, \quad M_R = \sqrt{\gamma} \times \sqrt{1 + \beta\mathbf{n}\cdot\boldsymbol{\sigma}}\,. \tag{11}$$

For the boost in question

$$\gamma = \frac{E}{m}, \quad \gamma\beta\mathbf{n} = \frac{\mathbf{p}}{m}, \tag{S.91}$$

hence

$$\sqrt{m}\,M_L \;=\; \sqrt{E\,-\,\mathbf{p}\cdot\boldsymbol{\sigma}}\,,\quad \sqrt{m}\,M_R \;=\; \sqrt{E\,+\,\mathbf{p}\cdot\boldsymbol{\sigma}}\,. \tag{S.92}$$

Plugging these formulae into eq. (S.90) immediately gives us

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}. \tag{28}$$

Problem **5**(d):

The negative-frequency solutions $e^{+ipx}v_\alpha(p,s)$ have Dirac spinors v_α satisfying $(\not{p}+m)v=0$. For a particle at rest, $p^\mu=(+m,\mathbf{0})$, this equation becomes $m(\gamma^0+1)v=0$, or in the Weyl basis

$$\begin{pmatrix} \mathbf{1}_{2\times2} & \mathbf{1}_{2\times2} \\ \mathbf{1}_{2\times2} & \mathbf{1}_{2\times2} \end{pmatrix} v(\mathbf{p}=\mathbf{0},s) \;=\; 0 \quad\Longrightarrow\quad v(\mathbf{p}=\mathbf{0},s) \;=\; \sqrt{m}\begin{pmatrix} +\eta_s \\ -\eta_s \end{pmatrix} \tag{S.93}$$

for some two-component spinor η_s. As in part (b), the $\sqrt{m}$ factor translates the normalization $v^\dagger(p,s)v(p,s')=2E_p\delta_{s,s'}=2m\delta_{s,s'}$ (for $\mathbf{p}=\mathbf{0}$) to $\eta_s^\dagger\eta_{s'}=\delta_{s,s'}$.

For $\mathbf{p}\neq\mathbf{0}$ we proceed similarly to part (c), namely Lorentz-boost the rest-frame solution (S.93) to the frame where $p^\mu=(+E_{\mathbf{p}},\mathbf{p})$:

$$\begin{aligned} v(p,s) \;&=\; M_D(B)\,v(p_0,s) \;=\; \begin{pmatrix} M_L & 0 \\ 0 & M_R \end{pmatrix}\begin{pmatrix} +\sqrt{m}\,\eta_s \\ -\sqrt{m}\,\eta_s \end{pmatrix} \\ &=\; \begin{pmatrix} +\sqrt{m}\,M_L\,\eta_s \\ -\sqrt{m}\,M_R\,\eta_s \end{pmatrix} \;=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix}. \end{aligned} \tag{S.94}$$

precisely as in eq. (29).

Problem **5**(e):

Physically, a hole in the Fermi sea has opposite free energy, opposite momentum, opposite spin, *etc.*, from the missing fermion (see my notes for the explanation). A positron is a hole in the Dirac sea of electrons, so it should have opposite $p^\mu = (E, \mathbf{p})$ from the missing electron state — that's why the $v(p,s)$ spinors accompany the $e^{+ipx} = e^{+iEt-i\mathbf{px}}$ plane waves instead of the $e^{-ipx} = E^{-iEt+i\mathbf{px}}$ factors of the $u(p,s)$ spinors. Likewise, the positron should have the opposite spin state from the missing electron, and that's why the η_s should have the opposite spin from the ξ_s. More accurately, the η_s should carry the opposite spin vector $\eta_s^\dagger \mathbf{S} \eta_s$ from the $\xi_s^\dagger \mathbf{S} \xi_s$.

The solution to this spin relation is $\eta_s = \sigma_2 \xi_s^*$ (where $*$ denotes complex conjugation). Indeed, let

$$\eta = \sigma_2 \xi^* \quad \Longrightarrow \quad \eta^\dagger = \xi^\top \sigma_2 \,, \tag{S.95}$$

and let's use the formula

$$\sigma_2 \boldsymbol{\sigma}^* \sigma_2 = -\boldsymbol{\sigma} \tag{S.96}$$

from problem 2(c). Consequently,

$$\eta^\dagger \boldsymbol{\sigma} \eta = \xi^\top \sigma_2 \boldsymbol{\sigma} \sigma_2 \xi^* = \left(\xi^\dagger \sigma_2 \boldsymbol{\sigma}^* \sigma_2 \xi \right)^* = \left(-\xi^\dagger \boldsymbol{\sigma} \xi \right)^* = -\xi^\dagger \boldsymbol{\sigma} \xi, \tag{S.97}$$

or in other words

$$\eta^\dagger \mathbf{S} \eta = \tfrac{1}{2} \eta^\dagger \boldsymbol{\sigma} \eta = -\xi^\dagger \mathbf{S} \xi \,. \tag{S.98}$$

And this is why we set $\eta_s = \sigma_2 \xi_s^*$.

Now consider implication of this relation between the ξ_s and η_s spinors for the plane-wave factors $u_\alpha(p,s)$ and $v_\alpha(p,s)$. Thanks to eq. (S.96) we have

$$\sigma_2 \times \sqrt{E \mp \mathbf{p} \cdot \boldsymbol{\sigma}} \times \sigma_2 = \left(\sqrt{E \pm \mathbf{p} \cdot \boldsymbol{\sigma}} \right)^* , \tag{S.99}$$

hence in light of the explicit formulae (28) and (29),

$$\begin{aligned}
v(p,s) &= \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} = \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\sigma_2\xi_s^* \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\sigma_2\xi_s^* \end{pmatrix} \\
&= \begin{pmatrix} +\sigma_2\sigma_2\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\sigma_2\xi_s^* \\ -\sigma_2\sigma_2\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\sigma_2\xi_s^* \end{pmatrix} = \begin{pmatrix} +\sigma_2\left(\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\right)^*\,\xi_s^* \\ -\sigma_2\left(\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\right)^*\,\xi_s^* \end{pmatrix} \\
&= \begin{pmatrix} 0 & +\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \times \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ +\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}^* \\
&= \gamma^2 u^*(p,s).
\end{aligned} \tag{S.100}$$

This verifies the first eq. (30). We may verify the second eq. (30) in a similar manner, but it's easier to use γ^2 being imaginary and squaring to -1, hence $\gamma^2(\gamma^2)^* = -\gamma^2\gamma^2 = +1$, and therefore

$$\gamma^2 v^*(p,s) \;=\; \gamma^2\left(\gamma^2 u^*(p,s)\right)^* \;=\; \gamma^2\left(\gamma^2\right)^* u(p,s) \;=\; +u(p,s). \tag{S.101}$$

Problem **5**(f):

The 3D spinors ξ_λ of definite helicity $\lambda = \mp\frac{1}{2}$ satisfy

$$(\mathbf{p}\cdot\boldsymbol{\sigma})\xi_\mp \;=\; \mp|\mathbf{p}|\times\xi_\mp\,. \tag{S.102}$$

Plugging these ξ_λ into the positive-energy Dirac spinors (28), we obtain

$$u(p,\lambda=\mp\tfrac{1}{2}) \;=\; \begin{pmatrix} \sqrt{E\pm|\mathbf{p}|}\,\times\xi_\mp \\ \sqrt{E\mp|\mathbf{p}|}\,\times\xi_\mp \end{pmatrix}. \tag{S.103}$$

In the ultra-relativistic limit $E\approx|\mathbf{p}|\gg m$, the square roots here simplify to $\sqrt{E+|\mathbf{p}|}\approx\sqrt{2E}$ and $\sqrt{E-|\mathbf{p}|}\approx 0$ (in comparison with the other root). Consequently, eq. (S.103) simplifies to

$$u(p,L) \;\approx\; \sqrt{2E}\begin{pmatrix}\xi_L \\ 0\end{pmatrix}, \qquad u(p,R) \;\approx\; \sqrt{2E}\begin{pmatrix}0 \\ \xi_R\end{pmatrix}. \tag{S.104}$$

In other words, the ultra-relativistic positive-energy Dirac spinors of definite helicity are chiral — dominated by the LH Weyl components for the left helicity or by the RH Weyl components for the right helicity.

Now consider the negative-energy Dirac spinors (29). The η_s spinors have exactly opposite spins from the ξ_s, so their helicities are also opposite from the ξ_s. Thus,

$$(\mathbf{p}\cdot\boldsymbol{\sigma})\eta_{\mp} \;=\; \pm|\mathbf{p}|\times\eta_{\mp} \tag{S.105}$$

— note the opposite sign from eq. (S.102). Therefore, the negative-energy Dirac spinors v of definite helicity are

$$v(p,\lambda=\mp\tfrac{1}{2}) \;=\; \begin{pmatrix} +\sqrt{E\mp|\mathbf{p}|}\times\eta_{\mp} \\ -\sqrt{E\pm|\mathbf{p}|}\times\eta_{\mp} \end{pmatrix}, \tag{S.106}$$

and in the ultra-relativistic limit they become

$$v(p,L) \;\approx\; -\sqrt{2E}\begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, \qquad v(p,R) \;\approx\; +\sqrt{2E}\begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \tag{S.107}$$

Again, the ultra-relativistic negative-energy spinors are chiral, but this time the chirality is opposite from the helicity — the left-helicity spinor has dominant RH Weyl components while the right-helicity spinor has dominant LH Weyl components.