Problem $\mathbf{1}$(a):

In the previous homework#7, problem (5.b–c), we saw that

$$u(p,s) = \begin{pmatrix} \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}, \quad v(p,s) = \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \tag{S.1}$$

where ξ_s and η_s are 2-component $SU(2)$ spinors normalized to

$$\xi_s^\dagger \xi_{s'} = \eta_s^\dagger \eta_{s'} = \delta_{s,s'}, \quad \eta_s = \sigma_2 \xi_s^*. \tag{S.2}$$

Before we check eqs. (2), let's check the normalization (1) conditions for the spinors (S.1):

$$\begin{aligned} u^\dagger(p,s)u(p,s') &= \xi_s^\dagger\Big((\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}})^2 + (\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}})^2\Big)\xi_{s'} = \xi_s^\dagger(2E)\xi_{s'} = 2E\delta_{s,s'}, \\ v^\dagger(p,s)v(p,s') &= \eta_s^\dagger\Big((+\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}})^2 + (-\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}})^2\Big)\eta_{s'} = \eta_s^\dagger(+2E)\eta_{s'} = 2E\delta_{s,s'}, \end{aligned} \tag{S.3}$$

because

$$(\pm\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}})^2 + (\pm'\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}})^2 = (E-\mathbf{p}\cdot\boldsymbol{\sigma}) + (E+\mathbf{p}\cdot\boldsymbol{\sigma}) = 2E. \tag{S.4}$$

And now that we have verified that the spinors (S.1) are properly normalized, let's consider the Lorentz invariant products $\bar{u}u$ and $\bar{v}v$. For the $u(p,s)$ and $v(p,s)$ as in eqs. (S.1), the $\bar{u}$ and $\bar{v}$ are given by

$$\begin{aligned} \bar{u}(p,s) = u^\dagger(p,s)\gamma^0 &= \begin{pmatrix} \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & 0 \end{pmatrix} \\ &= \left(\xi_s^\dagger \times \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}},\quad \xi_s^\dagger \times \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\right), \\ \bar{v}(p,s) = v^\dagger(p,s)\gamma^0 &= \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & 0 \end{pmatrix} \\ &= \left(-\eta_s^\dagger \times \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}},\quad +\eta_s^\dagger \times \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\right). \end{aligned} \tag{S.5}$$

Consequently,

$$\begin{aligned}\bar{u}(p,s)\,u(p,s') \;&=\; \xi_s^\dagger \times \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}} \times \xi_{s'} \\ &\quad+\; \xi_s^\dagger \times \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}} \times \xi_{s'} \\ &=\; 2m \times \xi_s^\dagger \xi_{s'} \;=\; 2m\delta_{s,s'}\end{aligned} \tag{S.6}$$

because

$$\begin{aligned}\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}} \;&=\; \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}} \\ &=\; \sqrt{E^2-(\mathbf{p}\cdot\boldsymbol{\sigma})^2} \;=\; \sqrt{E^2-\mathbf{p}^2} \;=\; m.\end{aligned} \tag{S.7}$$

Likewise,

$$\begin{aligned}\bar{v}(p,s)\,v(p,s') \;&=\; -\,\eta_s^\dagger \times \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}} \times \eta_{s'} \\ &\quad-\; \eta_s^\dagger \times \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}} \times \eta_{s'} \\ &=\; -\,2m \times \eta_s^\dagger \eta_{s'} \;=\; -2m\delta_{s,s'}\,.\end{aligned} \tag{S.8}$$

Problem **1**(b):

In matrix notations (column × row = matrix), we have

$$\begin{aligned}u(p,s)\times\overline{u}(p,s) \;&=\; \begin{pmatrix}\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\end{pmatrix} \times \left(\xi_s^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,,\; \xi_s^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\right) \\ &=\; \begin{pmatrix}\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\times\xi_s^\dagger\,\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\times\xi_s^\dagger\,\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\times\xi_s^\dagger\,\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\times\xi_s^\dagger\,\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\end{pmatrix},\end{aligned} \tag{S.9}$$

$$\begin{aligned}v(p,s)\times\overline{v}(p,s) \;&=\; \begin{pmatrix}+\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\eta_s\end{pmatrix} \times \left(-\eta_s^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,, +\eta_s^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\right) \\ &=\; \begin{pmatrix}-\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left(\eta_s\times\eta_s^\dagger\right)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left(\eta_s\times\eta_s^\dagger\right)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left(\eta_s\times\eta_s^\dagger\right)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left(\eta_s\times\eta_s^\dagger\right)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\end{pmatrix}.\end{aligned} \tag{S.10}$$

Summing over two spin polarizations replaces $\xi_s \times \xi_s^\dagger$ with $\sum_s \xi_s \times \xi_s^\dagger = \mathbf{1}_{2\times 2}$ and likewise

$\eta_s \times \eta_s^\dagger$ with $\sum_s \eta_s \times \eta_s^\dagger = \mathbf{1}_{2\times 2}$. Consequently,

$$\sum_s u(p,s) \times \overline{u}(p,s) \; =$$

$$= \begin{pmatrix} \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \xi_s \times \xi_s^\dagger\right]\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \xi_s \times \xi_s^\dagger\right]\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \xi_s \times \xi_s^\dagger\right]\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \xi_s \times \xi_s^\dagger\right]\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \end{pmatrix}$$

$$= \begin{pmatrix} m & E-\mathbf{p}\boldsymbol{\sigma} \\ E+\mathbf{p}\boldsymbol{\sigma} & m \end{pmatrix} \; = \; m \times \mathbf{1}_{4\times 4} \; + \; E \times \gamma^0 \; - \; \mathbf{p}\cdot\vec{\gamma}$$

$$= \; \not{p} + m. \tag{S.11}$$

$$\sum_s v(p,s) \times \overline{v}(p,s) \; =$$

$$= \begin{pmatrix} -\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \eta_s \times \eta_s^\dagger\right]\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \eta_s \times \eta_s^\dagger\right]\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \eta_s \times \eta_s^\dagger\right]\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\left[\sum_s \eta_s \times \eta_s^\dagger\right]\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \\ +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} \times \sqrt{E-\mathbf{p}\boldsymbol{\sigma}} \end{pmatrix}$$

$$= \begin{pmatrix} -m & E-\mathbf{p}\boldsymbol{\sigma} \\ E+\mathbf{p}\boldsymbol{\sigma} & -m \end{pmatrix} \; = \; -m \times \mathbf{1}_{4\times 4} \; + \; E \times \gamma^0 \; - \; \mathbf{p}\cdot\vec{\gamma}$$

$$= \; \not{p} - m. \tag{S.12}$$

Quod erat demonstrandum.

Problem **2**(a):

The γ^0 matrix commutes with itself but anticommutes with the space-indexed $\gamma^{1,2,3}$. At the same time, the parity reflects the space coordinates but not the time coordinate, $\mathbf{x} \to \mathbf{x}' = -\mathbf{x}$ but $t \to t' = +t$, hence the new space and time derivatives are related to the old derivatives as $\nabla' = -\nabla$ but $\partial_0' = +\partial$. Together, these two facts give us

$$\not{\partial}' \times \gamma^0 \; = \; \big(\gamma^0\partial_0' + \vec{\gamma}\cdot\nabla'\big)\gamma^0 \; = \; \gamma^0\big(\gamma^0\partial_0' - \vec{\gamma}\cdot\nabla'\big) \; = \; \gamma^0\big(+\gamma^0\partial_0 + \vec{\gamma}\cdot\nabla\big) \; = \; \gamma^0 \times \not{\partial} \tag{S.13}$$

and hence

$$(i\,\partial\!\!\!/' - m) \times \gamma^0 \;=\; \gamma^0 \times (i\,\partial\!\!\!/ - m). \tag{S.14}$$

Combining this formula with eq. (10) for the Dirac field, we find

$$(i\,\partial\!\!\!/ - m)'\Psi'(x') \;=\; (i\,\partial\!\!\!/' - m)\big(\pm\gamma^0\Psi(x)\big) \;=\; \pm(i\,\partial\!\!\!/' - m)\gamma^0\Psi(x) \;=\; \pm\gamma^0(i\,\partial\!\!\!/ - m)\Psi(x) \tag{S.15}$$

— the $(i\,\partial\!\!\!/ - m)\Psi(x)$ transforms under parity precisely like the $\Psi(x)$ field itself. In other words, the Dirac equation transforms *covariantly.*

Now consider the Dirac Lagrangian. Taking the Hermitian conjugate of eq. (10) we find

$$\Psi'^{\dagger}(-\mathbf{x},t) \;=\; \pm\Psi^{\dagger}(\mathbf{x},t)\gamma^{0\dagger} \;=\; \pm\Psi^{\dagger}(\mathbf{x},t)\gamma^0 \tag{S.16}$$

and hence

$$\overline{\Psi}'(-\mathbf{x},t) \;=\; \pm\overline{\Psi}(\mathbf{x},t)\gamma^0. \tag{S.17}$$

Consequently, the Dirac Lagrangian $\mathcal{L} = \overline{\Psi}(i\,\partial\!\!\!/ - m)\Psi$ transforms into

$$\begin{aligned} \mathcal{L}(x') \;&=\; \overline{\Psi}'(x') \times (i\,\partial\!\!\!/ - m)'\Psi'(x') \\ &=\; \pm\overline{\Psi}(x)\gamma^0 \times \pm\gamma^0(i\,\partial\!\!\!/ - m)\Psi(x) \\ &=\; +\overline{\Psi}(x) \times (i\,\partial\!\!\!/ - m)\Psi(x) \\ &=\; \mathcal{L}(x). \end{aligned} \tag{S.18}$$

In other words, the Dirac Lagrangian is invariant modulo $x \to x' = (-\mathbf{x}, +t)$, and the Dirac action $S = \int d^4x\,\mathcal{L}$ is invariant.

Problem **2**(b):

The linear momentum $\mathbf{p}$ is a polar vector while the angular momentum — orbital, or spin, or whatever — is an axial vector. Therefore, when the parity symmetry acts on a particle state with momentum $\mathbf{p}$ and spin s, it reverses $\mathbf{p} \to -\mathbf{p}$ but leaves the spin state as it is

$s \to +s$. The same rules apply to the plane waves of definite momentum and spin, hence for the Dirac spinors (S.1):

$$\begin{aligned}
\mathbf{P} : u(\mathbf{p}, s) \;\to\; u(-\mathbf{p}, +s) \;&= \begin{pmatrix} +\sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ +\sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix} \times \begin{pmatrix} +\sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ +\sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \\
&= \gamma^0 \times u(\mathbf{p}, s), \qquad\qquad \text{(S.19)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{P} : v(\mathbf{p}, s) \;\to\; v(-\mathbf{p}, +s) \;&= \begin{pmatrix} +\sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ -\sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}_{2\times 2} & -\mathbf{1}_{2\times 2} \\ -\mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix} \times \begin{pmatrix} +\sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ -\sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \\
&= -\gamma^0 \times v(\mathbf{p}, s), \qquad\qquad \text{(S.20)}
\end{aligned}$$

quod erat demonstrandum.

Problem **2**(c):

Let's apply parity to the quantum Dirac field

$$\widehat{\Psi}(\mathbf{x}, t) \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s \left(e^{-itE_{\mathbf{p}}+i\mathbf{x}\cdot\mathbf{p}} \times u(\mathbf{p}, s) \times \hat{a}_{\mathbf{p},s} \;+\; e^{+itE_{\mathbf{p}}-i\mathbf{x}\cdot\mathbf{p}} \times v(\mathbf{p}, s) \times \hat{b}^\dagger_{\mathbf{p},s} \right). \qquad \text{(S.21)}$$

Since everything besides the $\hat{a}_{\mathbf{p},s}$ and $\hat{b}^\dagger_{\mathbf{p},s}$ operators in this expansion is a c-number, sandwiching the field between two parity operators gives us

$$\widehat{\mathbf{P}}\,\widehat{\Psi}(\mathbf{x}, t)\widehat{\mathbf{P}} \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s \begin{pmatrix} e^{-itE_{\mathbf{p}}+i\mathbf{x}\cdot\mathbf{p}} \times u(\mathbf{p}, s) \times \widehat{\mathbf{P}}\,\hat{a}_{\mathbf{p},s}\,\widehat{\mathbf{P}} \\ +\,e^{+itE_{\mathbf{p}}-i\mathbf{x}\cdot\mathbf{p}} \times v(\mathbf{p}, s) \times \widehat{\mathbf{P}}\,\hat{b}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} \end{pmatrix}. \qquad \text{(S.22)}$$

At the same time, this expansion should match the the right hand side of eq. (5), for which we have

$$\pm\gamma^0\widehat{\Psi}(-\mathbf{x},t) = \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s \begin{pmatrix} \pm\ e^{-itE_{\mathbf{p}}-i\mathbf{x}\cdot\mathbf{p}}\times\gamma^0 u(\mathbf{p},s)\times\hat a_{\mathbf{p},s} \\ \pm\ e^{+itE_{\mathbf{p}}+i\mathbf{x}\cdot\mathbf{p}}\times\gamma^0 v(\mathbf{p},s)\times\hat b^\dagger_{\mathbf{p},s}\end{pmatrix}$$

⟪ using part (b) ⟫

$$= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s \begin{pmatrix} \pm\ e^{-itE_{\mathbf{p}}-i\mathbf{x}\cdot\mathbf{p}}\times u(-\mathbf{p},s)\times\hat a_{\mathbf{p},s} \\ \mp\ e^{+itE_{\mathbf{p}}+i\mathbf{x}\cdot\mathbf{p}}\times v(-\mathbf{p},s)\times\hat b^\dagger_{\mathbf{p},s}\end{pmatrix} \tag{S.23}$$

⟪ changing $\int$ variable $\mathbf{p}\to-\mathbf{p}$ ⟫

$$= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s \begin{pmatrix} \pm\ e^{-itE_{\mathbf{p}}+i\mathbf{x}\cdot\mathbf{p}}\times u(\mathbf{p},s)\times\hat a_{-\mathbf{p},s} \\ \mp\ e^{+itE_{\mathbf{p}}-i\mathbf{x}\cdot\mathbf{p}}\times v(\mathbf{p},s)\times\hat b^\dagger_{-\mathbf{p},s}\end{pmatrix}.$$

By eq. (5), the right hand sides of eqs. (S.22) and (S.23) must be equal to each other. Since the Dirac plane waves $e^{-ipx}u(p,s)$ and $e^{+ipx}v(p,s)$ are linearly independent from each other, this means

$$\widehat{\mathbf{P}}\,\hat a_{\mathbf{p},s}\,\widehat{\mathbf{P}} = \pm\hat a_{-\mathbf{p},s} \quad\text{and}\quad \widehat{\mathbf{P}}\,\hat b^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} = \mp\hat b^\dagger_{-\mathbf{p},s}\,. \tag{7a}$$

The rest of eq. (7) follows by hermitian conjugation: Since $\widehat{\mathbf{P}}^\dagger=\widehat{\mathbf{P}}^{-1}=\widehat{\mathbf{P}}$,

$$\begin{aligned}\widehat{\mathbf{P}}\,\hat a^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \left(\widehat{\mathbf{P}}\,\hat a_{\mathbf{p},s}\,\widehat{\mathbf{P}}\right)^\dagger = \pm\hat a^\dagger_{-\mathbf{p},s}\,,\\ \widehat{\mathbf{P}}\,\hat b_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \left(\widehat{\mathbf{P}}\,\hat b^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}}\right)^\dagger = \mp\hat b_{-\mathbf{p},s}\,.\end{aligned} \tag{7b}$$

Finally, eqs. (8) follow from eqs. (7) and from parity-invariance of the vacuum state, $\widehat{\mathbf{P}}\,|0\rangle = |0\rangle$. Indeed,

$$\begin{aligned}\widehat{\mathbf{P}}\,|F(\mathbf{p},s)\rangle &= \widehat{\mathbf{P}}\times\hat a^\dagger_{\mathbf{p},s}\,|0\rangle = \widehat{\mathbf{P}}\hat a^\dagger_{\mathbf{p},s}\widehat{\mathbf{P}}\times\widehat{\mathbf{P}}\,|0\rangle\\ &= \pm\hat a^\dagger_{-\mathbf{p},+s}\times|0\rangle = \pm\,|F(-\mathbf{p},+s)\rangle\,,\end{aligned} \tag{S.24}$$

$$\begin{aligned}\widehat{\mathbf{P}}\,\big|\overline{F}(\mathbf{p},s)\big\rangle &= \widehat{\mathbf{P}}\times\hat b^\dagger_{\mathbf{p},s}\,|0\rangle = \widehat{\mathbf{P}}\hat b^\dagger_{\mathbf{p},s}\widehat{\mathbf{P}}\times\widehat{\mathbf{P}}\,|0\rangle\\ &= \mp\hat b^\dagger_{-\mathbf{p},+s}\times|0\rangle = \mp\,\big|\overline{F}(-\mathbf{p},+s)\big\rangle\,.\end{aligned} \tag{S.25}$$

Problem $\mathbf{3}$(a):

Consider a state $\hat{a}^\dagger(+\mathbf{p}_{\rm red},s_1)\hat{b}^\dagger(-\mathbf{p}_{\rm red},s_2)\left|0\right\rangle$ of one fermion and one antifermion with definite reduced momentum and spins. The charge conjugation operator $\widehat{\mathbf{C}}$ turns this state into

$$\begin{aligned}\widehat{\mathbf{C}}\times\hat{a}^\dagger(+\mathbf{p}_{\rm red},s_1)\hat{b}^\dagger(-\mathbf{p}_{\rm red},s_2)\left|0\right\rangle &= \widehat{\mathbf{C}}\,\hat{a}^\dagger(+\mathbf{p}_{\rm red},s_1)\widehat{\mathbf{C}}\times\widehat{\mathbf{C}}\,\hat{b}^\dagger(-\mathbf{p}_{\rm red},s_2)\widehat{\mathbf{C}}\times\left|0\right\rangle \\ &= \hat{b}^\dagger(+\mathbf{p}_{\rm red},s_1)\times\hat{a}^\dagger(-\mathbf{p}_{\rm red},s_2)\left|0\right\rangle \\ &= -\hat{a}^\dagger(-\mathbf{p}_{\rm red},s_2)\times\hat{b}^\dagger(+\mathbf{p}_{\rm red},s_1)\left|0\right\rangle .\end{aligned}\tag{S.26}$$

Let's plug this formula into eq. (9):

$$\begin{aligned}\widehat{\mathbf{C}}\times\left|B(\mathbf{p}_{\rm tot}=0)\right\rangle &= \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\rm red},s_1,s_2)\times\widehat{\mathbf{C}}\,\hat{a}^\dagger(+\mathbf{p}_{\rm red},s_1)\hat{b}^\dagger(-\mathbf{p}_{\rm red},s_2)\left|0\right\rangle \\ &= \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\rm red},s_1,s_2)\times-\hat{a}^\dagger(-\mathbf{p}_{\rm red},s_2)\hat{b}^\dagger(+\mathbf{p}_{\rm red},s_1)\left|0\right\rangle \\ &\qquad\langle\!\langle\text{ change variables }\mathbf{p}_{\rm red}\to-\mathbf{p}_{\rm red}\text{ and }s_1\leftrightarrow s_2\ \rangle\!\rangle \\ &= \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}-\psi(-\mathbf{p}_{\rm red},s_2,s_1)\times\hat{a}^\dagger(+\mathbf{p}_{\rm red},s_1)\hat{b}^\dagger(-\mathbf{p}_{\rm red},s_2)\left|0\right\rangle .\end{aligned}\tag{S.27}$$

In terms of the bound state's wave function ψ, this action of the C-parity operator $\widehat{\mathbf{C}}$ is equivalent to

$$\widehat{\mathbf{C}}\,\psi(\mathbf{p}_{\rm red},s_1,s_2) = -\psi(-\mathbf{p}_{\rm red},s_2,s_1).\tag{S.28}$$

For a bound state with a definite orbital angular momentum L,

$$\psi(-\mathbf{p}_{\rm red},s_1,s_2) = (-1)^L\times\psi(+\mathbf{p}_{\rm red},s_1,s_2).\tag{S.29}$$

Likewise, for a bound state with a definite net spin S,

$$\psi(\mathbf{p}_{\rm red},s_2,s_1) = (-1)^{1-S}\psi(\mathbf{p}_{\rm red},s_1,s_2).\tag{S.30}$$

Plugging these two formulae into eq. (S.28) for the C-parity, we obtain

$$\begin{aligned}\widehat{\mathbf{C}}\psi(\mathbf{p}_{\rm red},s_1,s_2) &= -\psi(-\mathbf{p}_{\rm red},s_2,s_1)\\ &= -(-1)^L\psi(+\mathbf{p}_{\rm red},s_2,s_1)\\ &= -(-1)^L(-1)^{1-S}\psi(+\mathbf{p}_{\rm red},s_1,s_2).\end{aligned}\tag{S.31}$$

In other words, the bound state has definite C-parity

$$C = -(-1)^L(-1)^{1-S} = (-1)^L\times(-1)^S,\tag{S.32}$$

Quod erat demonstrandum.

Problem **3**(b):

Now consider how the P-parity (reflection of space) acts on the one-fermion+one-antifermions state $\hat a^\dagger(+\mathbf{p}_{\rm red},s_1)\hat b^\dagger(-\mathbf{p}_{\rm red},s_2)\,|0\rangle$:

$$\begin{aligned}\widehat{\mathbf{P}}\times\hat a^\dagger(+\mathbf{p}_{\rm red},s_1)\hat b^\dagger(-\mathbf{p}_{\rm red},s_2)\,|0\rangle &= \widehat{\mathbf{P}}\,\hat a^\dagger(+\mathbf{p}_{\rm red},s_1)\widehat{\mathbf{P}}\times\widehat{\mathbf{P}}\,\hat b^\dagger(-\mathbf{p}_{\rm red},s_2)\widehat{\mathbf{P}}\times|0\rangle\\ &= (\pm1)\hat a^\dagger(-\mathbf{p}_{\rm red},s_1)\times(\mp1)\hat b^\dagger(+\mathbf{p}_{\rm red},s_2)\,|0\rangle\\ &= -\hat a^\dagger(-\mathbf{p}_{\rm red},s_1)\hat b^\dagger(+\mathbf{p}_{\rm red},s_2)\,|0\rangle\,.\end{aligned}\tag{S.33}$$

where the overall $-$ sign comes from the opposite intrinsic parities of the fermion and the antifermion. Again, we plug this formula into eq. (9) and then change the integration variable $\mathbf{p}_{\rm red}\to-\mathbf{p}_{\rm red}$ — but this time we do not swap the spins s_1 and s_2:

$$\begin{aligned}\widehat{\mathbf{P}}\times|B(\mathbf{p}_{\rm tot})=0\rangle &= \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\rm red},s_1,s_2)\times\widehat{\mathbf{P}}\,\hat a^\dagger(+\mathbf{p}_{\rm red},s_1)\hat b^\dagger(-\mathbf{p}_{\rm red},s_2)\,|0\rangle\\ &= \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\rm red},s_1,s_2)\times-\hat a^\dagger(-\mathbf{p}_{\rm red},s_1)\hat b^\dagger(+\mathbf{p}_{\rm red},s_2)\,|0\rangle\\ &= \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}-\psi(-\mathbf{p}_{\rm red},s_1,s_2)\times\hat a^\dagger(+\mathbf{p}_{\rm red},s_1)\hat b^\dagger(-\mathbf{p}_{\rm red},s_2)\,|0\rangle\,,\end{aligned}\tag{S.34}$$

In terms of the wave-function ψ, this action of the P-parity operator means

$$\widehat{\mathbf{P}}\,\psi(\mathbf{p}_{\rm red},s_1,s_2) = -\psi(-\mathbf{p}_{\rm red},s_1,s_2).\tag{S.35}$$

For a bound state with a definite angular momentum, this gives us

$$\widehat{\mathbf{P}}\,\psi(\mathbf{p}_{\rm red},s_1,s_2)\ =\ -\psi(-\mathbf{p}_{\rm red},s_1,s_2)\ =\ -(-1)^L\times\psi(\mathbf{p}_{\rm red},s_1,s_2) \tag{S.36}$$

and hence definite P-parity

$$P\ =\ -(-1)^L, \tag{S.37}$$

Quod erat demonstrandum.

Problem **3**(c):

Finally, consider the positronium atom decaying into photons. Since the EM interactions respect the charge conjugation symmetry, the EM processes such as $e^-+e^+\to$ photons conserve C-parity. A photon of any momentum or polarization has $C=-1$, so the net C-parity of an n–photon final state is $(-1)^n$. Consequently, if the initial electron and positron are in a bound state with $C=+1$ they must annihilate into an even number of photons, $e^-+e^+\to 2\gamma,4\gamma,6\gamma,\ldots$. But if the bound state has $C=-1$, the electron and the positron must annihilate into an odd number of photons, $e^-+e^+\to 3\gamma,5\gamma,\ldots$. (Annihilation into a single photon is forbidden because of $\mathbf{p}^2_{\rm net}<E^2_{\rm net}$.)

The ground state of a hydrogen-like positronium 'atom' is 1S, meaning $n_{\rm rad}=1$ and $L=0$. Due to spins, there are actually 4 almost-degenerate 1S states; the hyperfine structure splits them into a $1\mathrm{S}_3$ triplet and a $1\mathrm{S}_1$ singlet of the net spin. According to eq. (S.32), the triplet states have $C=(-1)^L(-1)^S=(-1)^0(-1)^1=-1$ while the singlet state has $C=(-1)^L(-1)^S=(-1)^0(-1)^0=+1$. Consequently, the singlet $S=0$ state decays into an even number of photons,

$$(e^-+e^+)@1\mathrm{S}_1\ \to\ 2\gamma,4\gamma,\ldots, \tag{S.38}$$

while the triplet $S=1$ states decay into odd numbers of photons,

$$(e^-+e^+)@1\mathrm{S}_3\ \to\ 3\gamma,5\gamma,\ldots. \tag{S.39}$$

This difference affects the net decay rate of each state because QED (Quantum ElectroDynamics) has a rather small coupling constant $\alpha=(e^2/4\pi)\approx 1/137$. For each photon in the final state, the decay amplitude carries a factor of e, so the decay rate of a positronium atom into n photons $\Gamma(e^-e^+ \to n\gamma)$ is $O(\alpha^n)$. Consequently, the $S = 0$ positronium state usually decays into just 2 photons while decays into 4, 6, or more photons are allowed but much less common. Likewise, the $S = 1$ positronium states usually decays into 3 photons while decays into 5 or more photons are allowed but rare. More over, the decay rate into 3 photons is much slower than the decay rate into just 2 photons,

$$\frac{\Gamma\big((e^- + e^+)@1\mathrm{S}_3 \to 3\gamma\big)}{\Gamma\big((e^- + e^+)@1\mathrm{S}_1 \to 2\gamma\big)} \;=\; \frac{O(\alpha^3)}{O(\alpha^2)} \;=\; O(\alpha), \tag{S.40}$$

hence the net decay rate of an $S = 1$ state into anything it can decay to — *i.e.*, into any odd number of photons — is much slower then the net decay rate of the $S = 0$ state,

$$\frac{\Gamma\big((e^- + e^+)@1\mathrm{S}_3 \to \text{anything}\big)}{\Gamma\big((e^- + e^+)@1\mathrm{S}_1 \to \text{anything}\big)} \;=\; O(\alpha) \;\ll\; 1. \tag{S.41}$$

And that's why the $S = 1$ states have mush longer lifetimes than the $S = 0$ state.

Problem **4**(a):

Despite anticommutativity of the fermionic fields, the Hermitian conjugation of an operator product reverses the order of operators without any extra sign factors, thus $(\Psi^\dagger_\alpha \Psi_\beta)^\dagger = +\Psi^\dagger_\beta \Psi_\alpha$. Consequently, for any 4×4 matrix Γ, $(\Psi^\dagger \Gamma \Psi)^\dagger = +\Psi^\dagger \Gamma^\dagger \Psi$, and hence $(\overline{\Psi}\Gamma\Psi)^\dagger = \overline{\Psi}\overline{\Gamma}\Psi$ where $\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$ is the Dirac conjugate of Γ.

Now consider the 16 matrices which appear in the bilinears (10). Obviously $\overline{1} = +1$ and this gives us $S^\dagger = +S$. We saw in class that $\overline{\gamma^\mu} = +\gamma^\mu$ for all $\mu = 0, 1, 2, 3$ (*cf.* my notes on Dirac spinor fields), and this gives us $(V^\mu)^\dagger = +V^\mu$. We also saw that $\overline{\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}} = -\frac{i}{2}\gamma^{[\nu}\gamma^{\mu]} = +\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}$, and this gives us $(T^{\mu\nu})^\dagger = +T^{\mu\nu}$. As to the γ^5 matrix, we saw in the last homework#7 (problem 3(d–e)) that it's Hermitian and anticommutes with all the γ^μ. Hence $\overline{\gamma^5} = \gamma^0(\gamma^5)^\dagger\gamma^0 = +\gamma^0\gamma^5\gamma^0 = -\gamma^5 \implies \overline{i\gamma^5} = +i\gamma^5$, which gives us $P^\dagger = +P$. Finally, $\overline{\gamma^5\gamma^\mu} = \overline{\gamma^\mu}\,\overline{\gamma^5} = -\gamma^\mu\gamma^5 = +\gamma^5\gamma^\mu$, which gives us $(A^\mu)^\dagger = +A^\mu$. Thus, by inspection, all the bilinears (10) are Hermitian. *Quod erat demonstrandum.*

Problem $\mathbf{4}$(b):

Under a continuous Lorentz symmetry $x \mapsto x' = Lx$, the Dirac spinor field and its conjugate transform according to

$$\Psi'(x') \;=\; M(L)\Psi(x = L^{-1}x'), \qquad \overline{\Psi}'(x') \;=\; \overline{\Psi}(x = L^{-1}x')M^{-1}(L), \tag{S.42}$$

hence any bilinear $\overline{\Psi}\Gamma\Psi$ transforms according to

$$\overline{\Psi}'(x')\Gamma\Psi(x') \;=\; \overline{\Psi}(x)\Gamma'\Psi(x) \tag{S.43}$$

where

$$\Gamma' = M^{-1}(L)\Gamma M(L). \tag{S.44}$$

So the Lorentz transformation properties of the Dirac bilinears (10) follow from this transformation rule for the 16 Γ matrices in question.

Obviously for $\Gamma = 1$, $\Gamma' = M^{-1}M = 1$, which makes S a Lorentz scalar.

For $\Gamma = \gamma^\mu$, we saw in class that $\Gamma' = M^{-1}\gamma^\mu M = L^\mu_{\ \nu}\gamma^\nu$ — see my notes on Dirac spinors, eq. (22). Consequently $V'^\mu = L^\mu_{\ \nu}V^\nu$, which makes V^μ a Lorentz vector.

For $\Gamma = \gamma^\mu\gamma^\nu$, $M^{-1}\gamma^\mu\gamma^\nu M = (M^{-1}\gamma^\mu M)(M^{-1}\gamma^\nu M) = L^\mu_{\ \kappa}\gamma^\kappa \times L^\nu_{\ \lambda}\gamma^\lambda$. Similar transformation works for $\Gamma = \frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}$: $\Gamma' = L^\mu_{\ \kappa}L^\nu_{\ \lambda} \times \frac{i}{2}\gamma^{[\kappa}\gamma^{\lambda]}$. This makes $T^{\mu\nu}$ a Lorentz tensor (with two antisymmetric indices).

Next, the γ^5 commutes with even products of the γ^μ matrices and hence with $M(L) = \exp\bigl(\frac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu\bigr)$. Consequently, $M^{-1}\gamma^5 M = \gamma^5$, which makes P a Lorentz scalar.

Finally, $M^{-1}(\gamma^\mu\gamma^5)M = \bigl(M^{-1}\gamma^\mu M\bigr)\gamma^5 = L^\mu_{\ \nu}\gamma^\nu\gamma^5$, which makes A^μ a Lorentz vector.

Problem **4**(c):

In problem (2) we saw that the Dirac fields transform under parity as

$$\Psi'(x') \;=\; \pm\gamma^0\Psi(x), \qquad \overline{\Psi}'(x') \;=\; \pm\overline{\Psi}(x)\gamma^0. \tag{S.45}$$

Consequently, the Dirac bilinears transform as

$$\mathcal{P} : \overline{\Psi}\Gamma\Psi\Big|_x \;\mapsto\; \overline{\Psi}'\Gamma\Psi'\Big|_{x'} \;=\; \overline{\Psi}(\gamma^0\Gamma\gamma^0)\Psi\Big|_x. \tag{S.46}$$

By inspection, out of 16 possible Γ matrices, 1, γ^0, $\gamma^{[i}\gamma^{j]}$, and $\gamma^5\gamma^i$ commute with the γ^0, while γ^i, $\gamma^0\gamma^i$, $\gamma^5\gamma^0$, and γ^5 anticommute with the γ^0. Therefore,

- the bilinears S, V^0, T^{ij}, and A^i are P-even, *i.e.* remain invariant under parity, while
- the bilinears V^i, T^{0i}, A^0, and P are P-odd — the parity flips their signs.

From the 3D point of view, this means that S and V^0 are true scalars, P and A^0 are pseudo-scalars, $\mathbf{V}$ is a true or polar vector, $\mathbf{A}$ is a pseudo-vector or axial vector, and the tensor T contains one true vector T^{0i} and one axial vector $\frac{1}{2}\epsilon^{ijk}T^{jk}$. In space-time terms, we call S a true (Lorentz) scalar, P a (Lorentz) pseudoscalar, V^μ a true (Lorentz) vector, and A^μ an axial (Lorentz) vector. Finally, the tensor $T^{\mu\nu}$ is a true Lorentz tensor. However, a physically equivalent tensor $\widetilde{T}^{\kappa\lambda} = \frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}T_{\mu\nu}$ — for which $\tilde{T}^{0i} = -\frac{1}{2}\epsilon^{ijk}T^{jk}$ is an axial 3–vector while $\frac{1}{2}\epsilon^{ijk}\tilde{T}^{jk} = +T^{0i}$ is a polar 3–vector — is a Lorentz pseudo-tensor.

Problem **4**(d):

In class we saw that in the Weyl convention, the charge conjugation symmetry acts on Dirac fields as

$$\begin{aligned} \mathbf{C} : \Psi(x) \;&\to\; \Psi'(x) \;=\; \gamma^2\Psi^*(x) \;=\; \gamma^2\big(\Psi^\dagger(x)\big)^\top, \\ \mathbf{C} : \overline{\Psi}(x) \;&\to\; \overline{\Psi}'(x) \;=\; \overline{\Psi}^*(x)\gamma^2 \;=\; \Psi^\top(x)\gamma^0\gamma^2 \;=\; -\Psi^\top(x)\gamma^2\gamma^0. \end{aligned} \tag{S.47}$$

Consequently, for any Dirac bilinear $\overline{\Psi}\Gamma\Psi$,

$$\overline{\Psi}'\Gamma\Psi' \;=\; -\Psi^\top\gamma^2\gamma^0\Gamma\gamma^2\big(\Psi^\dagger\big)^\top \;=\; +\Psi^\dagger(\gamma^2\gamma^0\Gamma\gamma^2)^\top\Psi \;=\; +\overline{\Psi}\gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2\Psi \;\equiv\; \overline{\Psi}\Gamma^c\Psi. \tag{S.48}$$

The second equality here follows by transposition of the Dirac "sandwich" $\Psi^\top\cdots\big(\Psi^\dagger\big)^\top$, which carries an extra minus sign because the fermionic fields Ψ and Ψ^* anticommute with each other (in the classical limit). The third equality follows from $(\gamma^0)^\top = +\gamma^0$, $(\gamma^2)^\top = +\gamma^2$, and $\Psi^\dagger = \overline{\Psi}\gamma^0$.

Problem **4**(e):

By inspection, $\mathbf{1}^c \equiv \gamma^0\gamma^2\gamma^0\gamma^2 = +\mathbf{1}$. The γ_5 matrix is symmetric and commutes with the $\gamma^0\gamma^2$, hence $\gamma_5^c = +\gamma_5$. Among the four γ_μ matrices, the γ_1 and γ_3 are anti-symmetric and commute with the $\gamma^0\gamma^2$ while the γ_0 and γ_2 are symmetric but anti-commute with the $\gamma^0\gamma^2$; hence, for all four γ_μ, $\gamma_\mu^c = -\gamma_\mu$. Finally, because of the transposition involved, $(\gamma_\mu\gamma_\nu)^c = \gamma_\nu^c\gamma_\mu^c = +\gamma_\nu\gamma_\mu$, hence $(\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]})^c = +\frac{i}{2}\gamma^{[\nu}\gamma^{\mu]} = -\frac{i}{2}\gamma^{[\mu}\gamma^{\nu]}$. Likewise, $(\gamma^5\gamma^\mu)^c = (\gamma^\mu)^c(\gamma^5)^c = -\gamma^\mu\gamma^5 = +\gamma^5\gamma^\mu$.

Therefore, according to eq. (S.48), the scalar S, the pseudoscalar P, and the axial vector A_μ are C–even, while the vector V_μ and the tensor $T_{\mu\nu}$ are C–odd.