

Problem 2:

In the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{M^2}{2} \Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\mu}{2} \Phi \phi^2, \quad (1)$$

the first 4 terms on the RHS describe two free scalar fields $\Phi(x)$ and $\phi(x)$, while the fifth term is the interaction that we treat as a perturbation. In Feynman rules, the propagators follow from the free part of the Lagrangian, so for the theory at hand there are two distinct propagators,

$$\Phi \text{---} \text{---} \Phi = \frac{i}{q^2 - m^2 + i0} \quad \text{and} \quad \phi \text{---} \text{---} \phi = \frac{i}{q^2 - M^2 + i0}. \quad (\text{S.1})$$

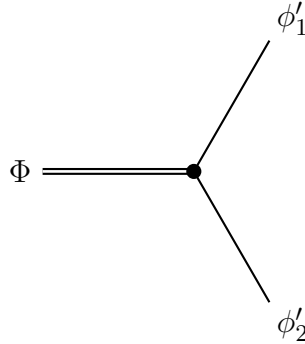
Likewise, there are two kinds of external lines according to the species of the incoming or outgoing particles for the process in question.

The Feynman vertices follow from the interaction part of the Lagrangian, which for the theory at hand is the cubic potential term $V_3 = \frac{\mu}{2} \Phi \phi^2$. Consequently, all vertices should be connected to three lines (net valence = 3), one double line for the one $\hat{\Phi}$ field, and two single lines for the two $\hat{\phi}$ fields,

$$\begin{array}{c} \phi \\ \diagup \\ \Phi \text{---} \text{---} \bullet \\ \diagdown \\ \phi \end{array} = -i \frac{\mu}{2} \times 2! = -i\mu \quad (\text{S.2})$$

where the $2!$ factor comes from the interchangeability of two identical $\hat{\phi}$ fields in the vertex.

Now consider the decay process $\Phi \rightarrow \phi + \phi$. To the lowest order of the perturbation theory, the decay amplitude follows from a single tree diagrams



$$(S.3)$$

This diagram has one vertex, one incoming double line, two outgoing single lines and no internal lines of either kind, hence

$$\langle \phi'_1 + \phi'_2 | i\hat{T} | \Phi \rangle \equiv i\mathcal{M} \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2) = -i\mu \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2), \quad (S.4)$$

or in other words

$$\mathcal{M}(\Phi \rightarrow \phi'_1 + \phi'_2) = -\mu. \quad (S.5)$$

This amplitude is related to the $\Phi \rightarrow \phi\phi$ decay rate as

$$\Gamma = \int |\mathcal{M}|^2 d\mathcal{P} \quad (S.6)$$

where the phase space factor for 1 particle \rightarrow 2 particles decays is

$$d\mathcal{P} = \frac{p'}{32\pi^2 M^2} d\Omega, \quad (S.7)$$

cf. [my notes on the phase space](#), eq. (65) on page 12. In this formula $p' = |\mathbf{p}'_1| = |\mathbf{p}'_2|$ in the rest frame of the original Φ particle, and $d\Omega = d\Omega(\mathbf{p}'_1) = d\Omega(\mathbf{p}'_2)$ in the same frame. For

decays to two particles of equal masses $m < \frac{M}{2}$,

$$E'_1 = E'_2 = \frac{M}{2} \implies p' = \sqrt{E'^2_1 - m^2} = \frac{M}{2} \times \sqrt{1 - \frac{4m^2}{M^2}}, \quad (\text{S.8})$$

hence

$$d\mathcal{P} = \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{1}{64\pi^2 M} \times d\Omega, \quad (\text{S.9})$$

and therefore the partial decay rate is

$$\frac{d\Gamma}{d^2\Omega} = \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{|\mathcal{M}|^2}{64\pi^2 M}. \quad (\text{S.10})$$

For the problem at hand, $\mathcal{M}_{\text{tree}} = -\mu$ regardless of directions of final particles, hence

$$\frac{d\Gamma_{\text{tree}}}{d^2\Omega} = \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{\mu^2}{64\pi^2 M}. \quad (\text{S.11})$$

Integrating this partial decay rate over the directions of \mathbf{p}' we must remember that the two final particles are identical bosons, so we cannot tell \mathbf{p}'_1 from $\mathbf{p}'_2 = -\mathbf{p}'_1$. Consequently, $\int d^2\Omega = 4\pi/2$ and therefore

$$\Gamma = \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{\mu^2}{32\pi M}. \quad (\text{S.12})$$

Problem 3:

Similar to the previous problem, the Feynman propagators of a theory follow from the free part of its Lagrangian. This time, we have N scalar fields $\phi^i(x)$ of similar mass m , hence in momentum space

$$\phi^j \text{---} \phi^k = \frac{i\delta^{jk}}{q^2 - m^2 + i0}. \quad (\text{S.13})$$

Note the δ^{jk} factor — the two fields connected by a propagator must be of the same species. Graphically, this means that both ends of the propagator carry the same species label $j = k$. Likewise, the external lines should also carry a species label of the incoming or outgoing particle in question. For the external lines, these labels are fixed (for a particular process), while for the internal lines we sum over $j = 1, 2, \dots, N$.

The Feynman vertices follow from the interactions between the fields; for the theory in question, they come from the quartic potential

$$V_4 = \frac{\lambda}{8} \left(\phi \cdot \phi = \sum_j \phi^j \phi^j \right)^2 = \sum_j \frac{\lambda}{8} (\hat{\phi}^j)^4 + \sum_{j < k} \frac{\lambda}{4} (\hat{\phi}^j)^2 (\hat{\phi}^k)^2. \quad (\text{S.14})$$

Consequently, all vertices have net valence = 4, but there are two vertex types with different indexologies: (1) a vertex involving 4 lines of the same field species ϕ^j , with the vertex factor $-i(\lambda/8) \times 4! = -3i\lambda$; and (2) a vertex involving 2 lines of one field species ϕ^j and 2 lines of a different species ϕ^k , with the vertex factor $-i(\lambda/4) \times (2!)^2 = -i\lambda$. (The combinatorial factors arise from the interchanges of the identical fields in the same vertex, thus $4!$ for the first vertex type and $(2!)^2$ for the second type.) Equivalently, we may use a single vertex type involving 4 fields of whatever species, with the species-dependent vertex factor

$$\begin{array}{c}
 \phi^j \quad \quad \phi^\ell \\
 \diagdown \quad \diagup \\
 \bullet \\
 \diagup \quad \diagdown \\
 \phi^k \quad \quad \phi^m
 \end{array}
 = -i\lambda (\delta^{jk} \delta^{\ell m} + \delta^{j\ell} \delta^{km} + \delta^{jm} \delta^{k\ell}). \quad (\text{S.15})$$

Now consider the scattering process $\phi^j + \phi^k \rightarrow \phi^\ell + \phi^m$. At the lowest order of the perturbation theory, there is just one Feynman diagram for this process; it has one vertex, 4 external legs and no internal lines. Consequently, at the lowest order,

$$\mathcal{M}(\phi^j + \phi^k \rightarrow \phi^\ell + \phi^m) = -\lambda (\delta^{jk} \delta^{\ell m} + \delta^{j\ell} \delta^{km} + \delta^{jm} \delta^{k\ell}) \quad (\text{S.16})$$

independent of the particles' momenta. Specifically,

$$\begin{aligned}
 \mathcal{M}(\phi^1 + \phi^2 \rightarrow \phi^1 + \phi^2) &= -\lambda, \\
 \mathcal{M}(\phi^1 + \phi^1 \rightarrow \phi^2 + \phi^2) &= -\lambda, \\
 \mathcal{M}(\phi^1 + \phi^1 \rightarrow \phi^1 + \phi^1) &= -3\lambda.
 \end{aligned} \quad (\text{S.17})$$

The *partial* cross sections in the CM frame follow from these amplitudes via eq. (4.85) of the

textbook or eq. (60) on page 12 of [my notes on phase space](#): For the elastic scattering,

$$\frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2}, \quad (\text{S.18})$$

hence

$$\begin{aligned} \frac{d\sigma(\phi^1 + \phi^2 \rightarrow \phi^1 + \phi^2)}{d\Omega_{\text{cm}}} &= \frac{\lambda^2}{64\pi^2 E_{\text{cm}}^2}, \\ \frac{d\sigma(\phi^1 + \phi^1 \rightarrow \phi^2 + \phi^2)}{d\Omega_{\text{cm}}} &= \frac{\lambda^2}{64\pi^2 E_{\text{cm}}^2}, \\ \frac{d\sigma(\phi^1 + \phi^1 \rightarrow \phi^1 + \phi^1)}{d\Omega_{\text{cm}}} &= \frac{9\lambda^2}{64\pi^2 E_{\text{cm}}^2}. \end{aligned} \quad (\text{S.19})$$

To calculate the total cross sections, we integrate over $d\Omega$, which gives the factor of 4π when the two final particles are of distinct species, but for the same species, we only get 2π because of Bose statistics. Thus,

$$\begin{aligned} \sigma_{\text{tot}}(\phi^1 + \phi^2 \rightarrow \phi^1 + \phi^2) &= \frac{\lambda^2}{16\pi E_{\text{cm}}^2}, \\ \sigma_{\text{tot}}(\phi^1 + \phi^1 \rightarrow \phi^2 + \phi^2) &= \frac{\lambda^2}{32\pi E_{\text{cm}}^2}, \\ \sigma_{\text{tot}}(\phi^1 + \phi^1 \rightarrow \phi^1 + \phi^1) &= \frac{9\lambda^2}{32\pi E_{\text{cm}}^2}. \end{aligned} \quad (\text{S.20})$$

Problem 4(a):

In perturbation theory, the Feynman propagators follow from the quadratic part of the Lagrangian (and hence free Hamiltonian), while the vertices follow from the cubic, quartic, *etc.*, terms treated as perturbation. For the linear sigma model's Lagrangian (3),

$$\mathcal{L} = \mathcal{L}_{\text{free}} - V_{\text{pert}}, \quad (\text{S.21})$$

$$\mathcal{L}_{\text{free}} = \frac{1}{2} \sum_i (\partial_\mu \phi_i)^2 + \frac{1}{2} (\partial_\mu \sigma)^2 - \frac{M_\sigma^2 = \lambda f^2}{2} \times \sigma^2, \quad (\text{S.22})$$

$$\begin{aligned}
V_{\text{pert}} = & \frac{3\lambda f}{6} \times \sigma^3 + \frac{\lambda f}{2} \times \sum_i \sigma \phi_i^2 \\
& + \frac{3\lambda}{24} \times \sigma^4 + \frac{\lambda}{4} \times \sum_i \sigma^2 \phi_i^2 + \frac{\lambda}{8} \times \left(\sum_i \phi_i^2 \right)^2.
\end{aligned} \tag{S.23}$$

The free Lagrangian (S.22) describes one massive field σ plus N massless fields π_i , hence two types of scalar propagators,

$$\sigma \text{ --- } \sigma = \frac{i}{q^2 - M_\sigma^2 + i0} \quad \text{and} \quad \pi^j \text{ --- } \pi^k = \frac{i\delta^{jk}}{q^2 + i0}, \tag{S.24}$$

and the $\pi\pi$ propagator carries a label $j = k = 1, 2, \dots, N$ specifying a particular species of the pion field.

As to the perturbation (S.23), it has two cubic terms and 3 quartic terms, hence two types of valence = 3 vertices and three types of valence = 4 vertices: the $\sigma\sigma\sigma$ and $\sigma\pi\pi$ vertices

$$\begin{aligned}
\sigma \text{ --- } \sigma \text{ --- } \sigma & = -3i\lambda f \quad \text{and} \quad \sigma \text{ --- } \pi^j \text{ --- } \pi^k = -i\lambda f \delta^{jk},
\end{aligned} \tag{S.25}$$

the $\sigma\sigma\sigma\sigma$ and $\sigma\sigma\pi\pi$ vertices

$$\begin{aligned}
\sigma \text{ --- } \sigma \text{ --- } \sigma \text{ --- } \sigma & = -3i\lambda \quad \text{and} \quad \pi^j \text{ --- } \sigma \text{ --- } \sigma \text{ --- } \pi^k = -i\lambda \delta^{jk}
\end{aligned} \tag{S.26}$$

and finally the $\pi\pi\pi\pi$ vertex similar to what we had in problem 3,

$$\begin{aligned}
\pi^j \text{ --- } \pi^\ell \text{ --- } \pi^k \text{ --- } \pi^m & = -i\lambda (\delta^{jk} \delta^{\ell m} + \delta^{j\ell} \delta^{km} + \delta^{jm} \delta^{k\ell}).
\end{aligned} \tag{S.27}$$

This completes the Feynman rules of the linear sigma model.

Problem 4(b):

As explained in class, a tree diagram ($L = 0$) with $E = 4$ external lines has either (A) one valence = 4 vertex and no propagators, or else (B) two valence = 3 vertices and one propagator. Topologically, there are three diagrams of type (B) with different arrangements of incoming versus outgoing external lines, so altogether there are 4 tree diagrams.

Specifically for the $\pi^j + \pi^k \rightarrow \pi^\ell + \pi^m$ scattering, the diagrams are

$$\begin{array}{c}
 \pi^j(p_1) \quad \pi^\ell(p'_1) \\
 \diagdown \quad \diagup \\
 \bullet \\
 \diagup \quad \diagdown \\
 \pi^k(p_2) \quad \pi^m(p'_2)
 \end{array}
 = -i\lambda(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}), \quad (\text{S.28})$$

$$\begin{array}{c}
 \pi^j(p_1) \quad \pi^\ell(p'_1) \\
 \diagdown \quad \diagup \\
 \bullet \text{---} \bullet \\
 \diagup \quad \diagdown \\
 \pi^k(p_2) \quad \pi^m(p'_2)
 \end{array}
 = (-i\lambda f \delta^{jk}) \frac{i}{s - M_\sigma^2} (-i\lambda f \delta^{\ell m}), \quad (\text{S.29})$$

$$\begin{array}{c}
 \pi^j(p_1) \quad \pi^\ell(p'_1) \\
 \diagdown \quad \diagup \\
 \bullet \\
 \parallel \\
 \bullet \\
 \diagup \quad \diagdown \\
 \pi^k(p_2) \quad \pi^m(p'_2)
 \end{array}
 = (-i\lambda f \delta^{j\ell}) \frac{i}{t - M_\sigma^2} (-i\lambda f \delta^{km}), \quad (\text{S.30})$$

$$\begin{array}{c}
 \pi^j(p_1) \quad \pi^\ell(p'_1) \\
 \diagdown \quad \diagup \\
 \bullet \\
 \parallel \\
 \bullet \\
 \diagup \quad \diagdown \\
 \pi^k(p_2) \quad \pi^m(p'_2)
 \end{array}
 = (-i\lambda f \delta^{jm}) \frac{i}{u - M_\sigma^2} (-i\lambda f \delta^{k\ell}), \quad (\text{S.31})$$

where s, t, u are the Mandelstam variables

$$\begin{aligned} s &\stackrel{\text{def}}{=} (p_1 + p_2)^2 \equiv (p'_1 + p'_2)^2, \\ t &\stackrel{\text{def}}{=} (p'_1 - p_1)^2 \equiv (p'_2 - p_2)^2, \\ u &\stackrel{\text{def}}{=} (p'_1 - p_2)^2 \equiv (p'_2 - p'_1)^2. \end{aligned} \tag{S.32}$$

Note that in the diagrams (S.29), (S.30), and (S.31), the internal line belongs to the σ field rather than to any π^i fields since there are no $\pi\pi\pi$ vertices but only $\pi\pi\sigma$.

Also, each of the diagrams (S.29), (S.30), and (S.31) yields a different combination of Kronecker $\delta\delta$ for the j, k, ℓ, m indices of the four pions, while the first diagram (S.28) yields all three combinations. So when we total up the four tree diagrams' amplitudes, it's convenient to reorganize the net tree amplitude by the j, k, ℓ, m indexology, thus

$$\begin{aligned} \mathcal{M}(\pi^j(p_1) + \pi^k(p_2) \rightarrow \pi^\ell(p'_1) + \pi^m(p'_2)) &= -\delta^{jk}\delta^{\ell m} \left(\lambda + \frac{\lambda^2 f^2}{s - M_\sigma^2} \right) \\ &\quad - \delta^{j\ell}\delta^{km} \left(\lambda + \frac{\lambda^2 f^2}{t - M_\sigma^2} \right) \\ &\quad - \delta^{jm}\delta^{k\ell} \left(\lambda + \frac{\lambda^2 f^2}{u - M_\sigma^2} \right). \end{aligned} \tag{S.33}$$

Problem 4(c-d):

The Lagrangian (3) of the linear sigma models has a very important relation between the quartic coupling λ , the cubic coupling $\kappa = \lambda f$, and the σ particle's mass $M_\sigma^2 = \lambda f^2$, thus

$$(M_\sigma^2 = \lambda f^2) \times \lambda = (\kappa = \lambda f)^2. \tag{S.34}$$

Thanks to this relation,

$$\lambda + \frac{(\lambda f)^2}{s - M_\sigma^2} = \frac{\lambda s - \cancel{\lambda M_\sigma^2} + \cancel{(\lambda f)^2}}{s - M_\sigma^2} = \frac{\lambda s}{s - M_\sigma^2} \tag{S.35}$$

and likewise

$$\lambda + \frac{(\lambda f)^2}{t - M_\sigma^2} = \frac{\lambda t}{t - M_\sigma^2} \quad \text{and} \quad \lambda + \frac{(\lambda f)^2}{u - M_\sigma^2} = \frac{\lambda u}{u - M_\sigma^2}. \tag{S.36}$$

Thanks to these formulae, the scattering amplitude (S.33) simplifies to

$$\mathcal{M} = -\lambda \left(\delta^{jk} \delta^{\ell m} \times \frac{s}{s - M_\sigma^2} + \delta^{j\ell} \delta^{km} \times \frac{t}{t - M_\sigma^2} + \delta^{jm} \delta^{k\ell} \times \frac{u}{u - M_\sigma^2} \right). \quad (\text{S.37})$$

Now consider the low-energy limit of this amplitude. In the CM frame, all 4 pions have the same energy E , hence

$$s = (E_{\text{cm}}^{\text{tot}})^2 = 4E^2, \quad t = -4E^2 \times \sin^2(\theta/2), \quad u = -4E^2 \times \cos^2(\theta/2), \quad (\text{S.38})$$

and therefore

$$s, t, u = O(E^2). \quad (\text{S.39})$$

Consequently, when the pion's energies are much smaller than the σ particle's mass, the denominators in the amplitude (S.37) may be approximated as

$$\frac{1}{s - M_\sigma^2} \approx \frac{1}{t - M_\sigma^2} \approx \frac{1}{u - M_\sigma^2} \approx \frac{-1}{M_\sigma^2} = \frac{-1}{\lambda f^2}. \quad (\text{S.40})$$

Consequently, the scattering amplitude (S.37) simplifies to

$$\mathcal{M} = \left(\frac{+\lambda}{M_\sigma^2} = \frac{+1}{f^2} \right) \times \left(\delta^{jk} \delta^{\ell m} \times s + \delta^{j\ell} \delta^{km} \times t + \delta^{jm} \delta^{k\ell} \times u + O\left(\frac{E^4}{M_\sigma^2}\right) \right). \quad (\text{S.41})$$

The magnitude of this amplitude is generally $O(E^2/f^2)$, so in the low-energy limit it becomes quite small.

Now consider the $\pi\pi \rightarrow \pi\pi$ scattering in a completely general frame of reference. Since the pions are massless, Mandelstam's s, t, u variables may be written as

$$\begin{aligned} s &\stackrel{\text{def}}{=} (p_1 + p_2)^2 \equiv (p'_1 + p'_2)^2 = +2(p_1 p_2) = +2(p'_1 p'_2), \\ t &\stackrel{\text{def}}{=} (p'_1 - p_1)^2 \equiv (p'_2 - p_2)^2 = -2(p'_1 p_1) = -2(p'_2 p_2), \\ u &\stackrel{\text{def}}{=} (p'_1 - p_2)^2 \equiv (p'_2 - p'_1)^2 = -2(p_1 p'_2) = -2(p'_1 p_2), \end{aligned} \quad (\text{S.42})$$

so whenever any one of the four momenta becomes small, all 3 of the s, t, u become small. In particular, when 3 of the momenta are $O(M_\sigma)$ or smaller while the fourth momentum

becomes much smaller, we have

$$s, t, u = O(M_\sigma \times p_{\text{smallest}}) \ll M_\sigma^2. \quad (\text{S.43})$$

Consequently, the scattering amplitude becomes as in eq. (S.41), and its magnitude is generally

$$\mathcal{M} \sim (s, t, u) \times \frac{\lambda}{M_\sigma^2} \lesssim P_{\text{smallest}} \times \frac{\lambda}{M_\sigma}. \quad (\text{S.44})$$

The physical reason for this behavior is the **Goldstone theorem**: Among other things, it says that *all scattering amplitudes involving Goldstone particles — such as the pions in this problem — become small as $O(p_\pi)$ when **any** Goldstone particle's momentum p_π becomes small*. A few lines above we saw how this works for the tree-level $\langle \pi, \pi | \mathcal{M} | \pi, \pi \rangle$ amplitude (S.37); the same behavior persists at all the higher orders of the perturbation theory, but seeing how *that* works is waaay beyond the scope of this exercise.

Problem 4(e):

In the low-energy limit $E \ll M_\sigma$, the tree-level $\pi\pi \rightarrow \pi\pi$ amplitudes may be approximated as in eq. (S.41). In particular,

$$\begin{aligned} \mathcal{M}(\pi^1 + \pi^2 \rightarrow \pi^1 + \pi^2) &= \frac{\lambda t}{M_\sigma^2} + O\left(\frac{\lambda E^4}{M_\sigma^4}\right) \approx \frac{t}{f^2}, \\ \mathcal{M}(\pi^1 + \pi^1 \rightarrow \pi^2 + \pi^2) &= \frac{\lambda s}{M_\sigma^2} + O\left(\frac{\lambda E^4}{M_\sigma^4}\right) \approx \frac{s}{f^2}, \\ \mathcal{M}(\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1) &= \frac{\lambda(s+t+u)}{M_\sigma^2} + O\left(\frac{\lambda E^4}{M_\sigma^4}\right) = O\left(\frac{\lambda E^4}{M_\sigma^4}\right) \\ &\quad \ll \text{since } s+t+u = 4m_\pi^2 = 0 \gg \end{aligned} \quad (\text{S.45})$$

Translating these amplitudes into the partial and the total scattering cross-sections, we

obtain

$$\begin{aligned}
\frac{d\sigma(\pi^1 + \pi^2 \rightarrow \pi^1 + \pi^2)}{d\Omega_{\text{cm}}} &= \frac{1}{64\pi^2 s} \times \frac{t^2}{f^4} = \frac{E_{\text{cm}}^2}{64\pi^2 f^4} \times \sin^4 \frac{\theta_{\text{cm}}}{2}, \\
\sigma_{\text{tot}}(\pi^1 + \pi^2 \rightarrow \pi^1 + \pi^2) &= \frac{E_{\text{cm}}^2}{48\pi f^4}, \\
\frac{d\sigma(\pi^1 + \pi^1 \rightarrow \pi^2 + \pi^2)}{d\Omega_{\text{cm}}} &= \frac{1}{64\pi^2 s} \times \frac{s^2}{f^4} = \frac{E_{\text{cm}}^2}{64\pi^2 f^4}, \\
\sigma_{\text{tot}}(\pi^1 + \pi^1 \rightarrow \pi^2 + \pi^2) &= \frac{E_{\text{cm}}^2}{32\pi f^4}, \\
\frac{d\sigma(\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1)}{d\Omega_{\text{cm}}} &= \frac{1}{64\pi^2 s} \times O\left(\frac{\lambda^2 E^8}{M_\sigma^8}\right) = O\left(\frac{E_{\text{cm}}^6}{f^4 M_\sigma^4}\right), \\
\sigma(\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1) &= O\left(\frac{E_{\text{cm}}^6}{f^4 M_\sigma^4}\right) \ll \frac{E_{\text{cm}}^2}{f^4}.
\end{aligned} \tag{S.46}$$

For a more accurate approximation to the same-species elastic scattering like $\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1$, we need to go back to the amplitude (S.37) and expand it to second powers in s, t, u . Thus,

$$\begin{aligned}
-\frac{\lambda s}{s - M_\sigma^2} &\approx \frac{\lambda s}{M_\sigma^2} + \frac{\lambda s^2}{M_\sigma^4}, \\
-\frac{\lambda t}{t - M_\sigma^2} &\approx \frac{\lambda t}{M_\sigma^2} + \frac{\lambda t^2}{M_\sigma^4}, \\
-\frac{\lambda u}{u - M_\sigma^2} &\approx \frac{\lambda u}{M_\sigma^2} + \frac{\lambda u^2}{M_\sigma^4},
\end{aligned} \tag{S.47}$$

and therefore

$$\begin{aligned}
\mathcal{M}(\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1) &= -\frac{\lambda s}{s - M_\sigma^2} - \frac{\lambda t}{t - M_\sigma^2} - \frac{\lambda u}{u - M_\sigma^2} \\
&\approx \frac{\lambda}{M_\sigma^2} \times (s + t + u = 0) + \frac{\lambda}{M_\sigma^4} \times (s^2 + t^2 + u^2).
\end{aligned} \tag{S.48}$$

In the center of mass frame,

$$\begin{aligned}
s^2 + t^2 + u^2 &= 16E^4 + 16E^4 \times \sin^4(\theta/2) + 16E^4 \times \cos^4(\theta/2) \\
&= 8E^4 \times (3 + \cos^2 \theta) = (E_{\text{cm}}^{\text{tot}})^4 \times \frac{3 + \cos^2 \theta}{2},
\end{aligned} \tag{S.49}$$

hence

$$\frac{d\sigma(\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1)}{d\Omega_{\text{cm}}} \approx \frac{\lambda^2 E_{\text{cm}}^6}{256\pi^2 M_\sigma^8} \times (3 + \cos^2 \theta)^2, \quad (\text{S.50})$$

and therefore

$$\sigma(\pi^1 + \pi^1 \rightarrow \pi^1 + \pi^1) \approx \frac{7\lambda^2 E_{\text{cm}}^6}{80\pi M_\sigma^8}. \quad (\text{S.51})$$