Problem **2**(a):

For the ultra-relativistic electrons and positrons, the spinors $u(p,s)$ and $v(p,s)$ become chiral. Indeed, by inspection of eqs. (2), u has chirality matching the electron's helicity — left for $\lambda=-\frac12$ and right for $\lambda=+\frac12$ — while v has chirality opposite to the positron's helicity — left for $\lambda=+\frac12$ and right for $\lambda=-\frac12$. At the same time, the amplitude (1) depends on the electron's and positron's spin states via the 'Dirac sandwich' $\bar v(e^+)\gamma_\nu u(e^-)$ which does not mix helicities. Indeed,

$$\bar v\gamma_\nu u \;=\; v^\dagger\gamma^0\gamma_\nu u \;=\; v^\dagger\begin{pmatrix}\bar\sigma_\nu & 0\\ 0 & \sigma_\nu\end{pmatrix}u \;=\; v_L^\dagger\bar\sigma_\nu u_L \;+\; v_R^\dagger\sigma_\nu u_R\,, \tag{S.1}$$

so if u and v are chiral, then they should have the same chirality — both left or both right — or else $\bar v\gamma_\nu u=0$. In terms of helicities, $u(e^-)$ and $v(e^+)$ being both left-handed means $\lambda(e^-)=-\frac12$ while $\lambda(e^+)=+\frac12$; likewise, $u(e^-)$ and $v(e^+)$ being both right-handed means $\lambda(e^-)=+\frac12$ while $\lambda(e^+)=-\frac12$. Thus, a non-zero amplitude (3) requires the electron and the positron to have opposite helicities, $\lambda(e^+)=-\lambda(e^-)$; for similar helicities of the two initial particles, the amplitude vanishes. *Quod erat demonstrandum.*

For future reference, let me calculate the explicit Dirac sandwiches (S.1) for the ultra-relativistic electron and positron spinors (2):

$$\bar v(e_L^+)\gamma_\nu u(e_L^-) \;=\; -2E\begin{pmatrix}0\\ \eta_L\end{pmatrix}^\dagger\begin{pmatrix}\bar\sigma_\nu & 0\\ 0 & \sigma_\nu\end{pmatrix}\begin{pmatrix}\xi_L\\ 0\end{pmatrix} \;=\; 0, \tag{3}$$

$$\bar v(e_L^+)\gamma_\nu u(e_R^-) \;=\; -2E\begin{pmatrix}0\\ \eta_L\end{pmatrix}^\dagger\begin{pmatrix}\bar\sigma_\nu & 0\\ 0 & \sigma_\nu\end{pmatrix}\begin{pmatrix}0\\ \xi_R\end{pmatrix} \;=\; -2E\times\eta_L^\dagger\sigma_\nu\xi_R\,, \tag{S.2}$$

$$\bar v(e_R^+)\gamma_\nu u(e_L^-) \;=\; +2E\begin{pmatrix}\eta_R\\ 0\end{pmatrix}^\dagger\begin{pmatrix}\bar\sigma_\nu & 0\\ 0 & \sigma_\nu\end{pmatrix}\begin{pmatrix}\xi_L\\ 0\end{pmatrix} \;=\; +2E\times\eta_R^\dagger\bar\sigma_\nu\xi_L\,, \tag{S.3}$$

$$\bar v(e_R^+)\gamma_\nu u(e_R^-) \;=\; +2E\begin{pmatrix}\eta_R\\ 0\end{pmatrix}^\dagger\begin{pmatrix}\bar\sigma_\nu & 0\\ 0 & \sigma_\nu\end{pmatrix}\begin{pmatrix}0\\ \xi_R\end{pmatrix} \;=\; 0. \tag{3}$$

As promised, the sandwich — and hence the amplitude (1) vanishes for the same helicities.

Problem **2**(b):

For the ultra-relativistic muons, the $u(\mu-)$ and $v(\mu^+)$ are chiral, and the chiralities behave exactly similar to the electron and the positron in part (a): the Dirac sandwich $\bar{u}(\mu^-)\gamma^\nu v(\mu^+)$ vanishes unless u and v have the same chirality, which requires the μ^- and the μ^+ to have opposite helicities. Specifically,

$$\bar{u}(\mu_L^-)\gamma_\nu v(\mu_L^+) = -2E \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix} = 0, \tag{4}$$

$$\bar{u}(\mu_L^-)\gamma_\nu v(\mu_R^+) = -2E \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix} = +2E \times \xi_L^\dagger \bar{\sigma}_\nu \eta_R\,, \tag{S.4}$$

$$\bar{u}(\mu_R^-)\gamma_\nu v(\mu_L^+) = -2E \begin{pmatrix} 0 \\ \xi_L \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix} = -2E \times \xi_R^\dagger \sigma_\nu \eta_L\,, \tag{S.5}$$

$$\bar{u}(\mu_R^-)\gamma_\nu v(\mu_L^+) = -2E \begin{pmatrix} 0 \\ \xi_L \end{pmatrix}^\dagger \begin{pmatrix} \bar{\sigma}_\nu & 0 \\ 0 & \sigma_\nu \end{pmatrix} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix} = 0. \tag{4}$$

Eqs. (4) — and similar formulae for the other fermion-antifermion pairs produced with ultra-relativistic speeds in electron-positron collisions — assure that the fermion and the antifermion always have opposite helicities. Experimentally, this means that if for some event we are able to determine the helicity of one final particle, then we may infer the second final particle's helicity without any further experimental effort.

Problem **2**(c):

The electron moves in the positive z direction, so its helicity λ is the same as its S_z — the z component of its spin. Hence, the ξ spinors corresponding to the 2 helicities are

$$\xi(e_L^-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \xi(e_R^-) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{S.6}$$

The positron moves in the negative z direction, so its helicity is opposite from S_z, hence

$$\xi(e_L^+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi(e_R^+) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{S.7}$$

However, the η spinors in eqs. (2) for the positrons have opposite spins from these ξ spinors, specifically $\eta = \sigma_2 \xi^*$, thus

$$\eta(e^+_L) \;=\; \begin{pmatrix} 0 \\ +i \end{pmatrix} \quad\text{and}\quad \eta(e^+_R) \;=\; \begin{pmatrix} -i \\ 0 \end{pmatrix}. \tag{S.8}$$

Substituting these 2–component spinors into eqs. (S.2) and (S.3), we obtain

$$\begin{aligned}
\bar v(e^+_L)\gamma_\nu u(e^-_R) \;&=\; -2E \times (\,0 \quad -i\,)\,\sigma_\nu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&=\; +2iE \times (\sigma_\nu)_{21} \\
&=\; 2E \times (0, -i, +1, 0)_\nu, \\
\bar v(e^+_R)\gamma_\nu u(e^-_L) \;&=\; +2E \times (\,+i \quad 0\,)\,\bar\sigma_\nu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&=\; +2iE \times (\bar\sigma_\nu)_{12} \\
&=\; 2E \times (0, +i, +1, 0)_\nu.
\end{aligned} \tag{5}$$

When taking the 21 and 12 matrix elements of the σ_ν and $\bar\sigma_\nu$ matrices, please remember that $\sigma_\nu = (1, \sigma^x, \sigma^y, \sigma^z)$ while $\bar\sigma_\nu = (1, -\sigma^x, -\sigma^y, -\sigma^z)$.

Problem **2**(d):

Suppose for a moment $\theta = 0$ and the $\mu^\mp$ move in the same directions as $e^\mp$. Then the muons have exactly the same spinors u and v as the $e^\mp$ of the same charge and helicity, and similarly to eqs. (5) we have

$$\bar v(\mu^+_L)\gamma_\nu u(\mu^-_R) \;=\; 2E \times (0, -i, +1, 0)^\nu \quad\text{and}\quad \bar v(\mu^+_R)\gamma_\nu u(\mu^-_L) \;=\; 2E \times (0, +i, +1, 0)^\nu. \tag{S.9}$$

For the muons, the amplitude (1) involves the $\bar u(\mu^-)\gamma^\nu v(\mu^+)$ Dirac sandwich rather than the $\bar v(\mu^+)\gamma_\nu u(\mu^-)$, but these two sandwiches are related by complex conjugation

$$\bigl(\bar v \gamma^\nu u\bigr)^* \;=\; \bar u \overline{\gamma^\nu} v = \; \bar u \gamma^\nu v, \tag{S.10}$$

as well as raising the index ν, hence

$$\begin{aligned}
\bar u(\mu^-_R)\gamma^\nu v(\mu^+_L) \;&=\; 2E \times \bigl[(0, -i, +1, 0)_\nu = (0, +i, -1, 0)^\nu\bigr]^* \;=\; 2E \times (0, -i, -1, 0)^\nu, \\
\bar u(\mu^-_L)\gamma^\nu v(\mu^+_R) \;&=\; 2E \times \bigl[(0, +i, +1, 0)_\nu = (0, -i, -1, 0)^\nu\bigr]^* \;=\; 2E \times (0, +i, -1, 0)^\nu.
\end{aligned} \tag{S.11}$$

Eqs. (S.11) apply for $\theta = 0$. For other muon directions, we may simply rotate the

4–vectors (S.11) through angle θ in the xz plane, thus

$$\begin{aligned} \bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) &= 2E \times (0, -i\cos\theta, -1, +i\sin\theta)^\nu, \\ \bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) &= 2E \times (0, +i\cos\theta, -1, -i\sin\theta)^\nu. \end{aligned} \tag{7}$$

Problem **2**(e):

Substituting the Dirac sandwiches (5) and (7) into the pair production amplitude (1), we obtain

$$\begin{aligned} \left\langle \mu_L^-, \mu_R^+ \right| \mathcal{M} \left| e_L^-, e_R^+ \right\rangle &= \left\langle \mu_R^-, \mu_L^+ \right| \mathcal{M} \left| e_R^-, e_L^+ \right\rangle = -e^2 \times (1+\cos\theta), \\ \left\langle \mu_R^-, \mu_L^+ \right| \mathcal{M} \left| e_L^-, e_R^+ \right\rangle &= \left\langle \mu_L^-, \mu_R^+ \right| \mathcal{M} \left| e_R^-, e_L^+ \right\rangle = -e^2 \times (1-\cos\theta), \end{aligned} \tag{S.12}$$

while all the other polarized amplitudes vanish by eqs. (3) and (4):

$$\begin{aligned} \left\langle \mu_{\rm any}^-, \mu_{\rm any}^+ \right| \mathcal{M} \left| e_L^-, e_L^+ \right\rangle &= \left\langle \mu_{\rm any}^-, \mu_{\rm any}^+ \right| \mathcal{M} \left| e_R^-, e_R^+ \right\rangle = 0, \\ \left\langle \mu_L^-, \mu_L^+ \right| \mathcal{M} \left| e_{\rm any}^-, e_{\rm any}^+ \right\rangle &= \left\langle \mu_R^-, \mu_R^+ \right| \mathcal{M} \left| e_{\rm any}^-, e_{\rm any}^+ \right\rangle = 0. \end{aligned} \tag{S.13}$$

The partial cross-sections (8) follow from these amplitudes according to

$$\frac{d\sigma}{d\Omega_{\rm c.m.}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \times \left(\frac{|\mathbf{p}'|}{|\mathbf{p}|} \approx 1\right). \tag{S.14}$$

Problem **2**(f):

Summing the polarized cross-sections (8) over the muons' helicities, we get

$$\begin{aligned} \frac{d\sigma(e_L^- + e_R^+ \to \mu_{\rm any}^- + \mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_{\rm any}^- + \mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} \\ &= \frac{\alpha^2}{4s} \times (1+\cos\theta)^2 + \frac{\alpha^2}{4s} \times (1-\cos\theta)^2 + 0 + 0 \\ &= \frac{\alpha^2}{2s} \times (1+\cos^2\theta) \end{aligned} \tag{S.15}$$

while

$$\frac{d\sigma(e_L^- + e_L^+ \to \mu_{\rm any}^- + \mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} = \frac{d\sigma(e_R^- + e_R^+ \to \mu_{\rm any}^- + \mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} = 0. \tag{S.16}$$

Averaging these cross-sections over the electron's and positron's helicities gives

$$\begin{aligned}\frac{d\sigma(e^-_{\rm avg}+e^+_{\rm avg}\to\mu^-_{\rm any}+\mu^+_{\rm any})}{d\Omega_{\rm c.m.}} &= \frac{1}{4}\left(\frac{\alpha^2}{2s}\times(1+\cos^2\theta)\,+\,\frac{\alpha^2}{2s}\times(1+\cos^2\theta)\,+\,0\,+\,0\right)\\ &= \frac{\alpha^2}{4s}\times(1+\cos^2\theta),\end{aligned} \tag{S.17}$$

which is exactly what we found in class for the un-polarized cross-section when $E \gg M_\mu$.

Problem **3**(a):

Using Feynman gauge for the photon propagator, the usual QED vertex for the electron, and the nuclear-photon vertex (9) (in the slow nucleus limit), the diagram (10) yields amplitude

$$i\mathcal{M} \;=\; \frac{-ig^{\mu\nu}}{q^2}\times\bar{u}(e')(ie\gamma_\mu)u(e)\times(-iZe)(2M_N)\delta_{\nu,0}\,, \tag{S.18}$$

and hence, after summing over the Lorentz indices

$$\mathcal{M} \;=\; \frac{-2M_N Ze^2}{q^2}\times\bar{u}(e')\gamma^0 u(e). \tag{S.19}$$

Problem **2**(b):

For any elastic scattering in the CM frame, the partial cross-section is

$$\frac{d\sigma}{d\Omega} \;=\; \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \tag{S.20}$$

where $\overline{|\mathcal{M}|^2}$ is the $|\mathcal{M}|^2$ summed over the final particle spins and averaged over the initial particle spins. For the Mott scattering, the net energy (in the CM frame) is

$$E_{\rm net} \;=\; E(N)\;+\;E(e)\;\approx\;M_N\;+\;E(e)\;\approx\;M_N \quad \text{since } E(e)\ll M_N\,, \tag{S.21}$$

hence $s\approx M_N^2$ and therefore

$$\frac{d\sigma}{d\Omega}\;\approx\;\frac{\overline{|\mathcal{M}|^2}}{64\pi^2 M_N^2}\,. \tag{S.22}$$

For the Mott amplitude (S.19), this translates to

$$\frac{d\sigma}{d\Omega} \approx \frac{Z^2e^4}{16\pi^2} \times \frac{1}{(q^2)^2} \times \frac{1}{2}\sum_{s,s'} \left|\bar{u}(p',s')\gamma^0 u(p,s)\right|^2. \tag{S.23}$$

To bring this formula to the form (11), note that in the $\hbar = c = 1$ units $(e^4/16\pi^2) = \alpha^2$; also in the CM frame $q^2 = -\mathbf{q}^2 = -2\mathbf{p}^2(1-\cos\theta)$.

Problem **3**(c):

As explained in class, for any Dirac matrix Γ,

$$\sum_{s,s'} \left|\bar{u}(p',s')\Gamma u(p,s)\right|^2 = \mathrm{tr}\big((\not{p}'+m)\Gamma(\not{p}+m)\overline{\Gamma}\big). \tag{S.24}$$

In particular, for the spin sum (12) we have

$$\begin{aligned}\frac{1}{2}\sum_{s,s'} \left|\bar{u}(p',s')\gamma^0 u(p,s)\right|^2 &= \tfrac{1}{2}\,\mathrm{tr}\big((\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\big)\\ &= \tfrac{1}{2}\,\mathrm{tr}(\not{p}'\gamma^0\,\not{p}\gamma^0) + \tfrac{1}{2}m^2\,\mathrm{tr}(\gamma^0\gamma^0)\\ &= 2\big(p'^0p^0 - (p'p)g^{00} + p'^0p^0\big) + 2m^2g^{00}\\ &= 2\big(2\times E'E - (p'p)\big) + 2m^2\\ &= 2\big(E'E + \mathbf{p}'\cdot\mathbf{p} + m^2\big).\end{aligned} \tag{12}$$

Problem **3**(d):

Combining eqs. (11) and (12), we immediately get

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{(\mathbf{q}^2)^2} \times 2\big(E'E + \mathbf{p}'\cdot\mathbf{p} + m^2\big). \tag{S.25}$$

The rest is relativistic kinematics: in $\hbar = c = 1$ units,

$$\mathbf{q}^2 = \mathbf{p}^2 \times 2(1-\cos\theta) = \mathbf{p}^2 \times 4\sin^2(\theta/2) = m^2\gamma^2\beta^2 \times 4\sin^2(\theta/2), \tag{S.26}$$

while in the numerator of eq. (S.25)

$$\mathbf{p}'\cdot\mathbf{p} \;=\; \mathbf{p}^2\cos\theta \;=\; \mathbf{p}^2 \;-\; 2\mathbf{p}^2\sin^2(\theta/2), \tag{S.27}$$

hence

$$E'E \,+\, \mathbf{p}'\cdot\mathbf{p} \,+\, m^2 \;=\; 2E^2 \;-\; 2\mathbf{p}^2\sin^2(\theta/2) \;=\; 2m^2\gamma^2(1-\beta^2\sin^2(\theta/2)). \tag{S.28}$$

Plugging all these formulae into eq. (S.25), we get

$$\begin{aligned}\frac{d\sigma}{d\Omega}_{\rm Mott} &= \frac{(Z\alpha)^2}{16m^4\gamma^4\beta^4\sin^2(\theta/2)}\times 4m^2\gamma^2(1-\beta^2\sin^2(\theta/2))\\ &= \frac{(Z\alpha)^2}{4m^2\beta^4\sin^2(\theta/2)}\times\frac{1-\beta^2\sin^2(\theta/2)}{\gamma^2}\\ &= \frac{d\sigma}{d\Omega}_{\rm Rutherford}\times\frac{1-\beta^2\sin^2(\theta/2)}{\gamma^2}\,.\end{aligned} \tag{13}$$

Quod erat demonstrandum.

Problem **4**, preamble:

A point of notation: In the solutions to this problem, the indices μ, e, $\nu\equiv\nu_\mu$, and $\bar\nu\equiv\bar\nu_e$ denote the particles. For the Lorentz indices, I shall use α, β, γ, δ, κ, λ, σ, ρ, but never μ or ν. Thus, $p_{\mu\alpha}$ denotes the α component of the muon's 4–momentum, *etc.*, *etc.*

Problem **4**(a):

Since the muon is so much lighter than the $W^\pm$ bosons, we have $q^2 \ll M_W^2$ which justifies using Fermi's effective low-energy theory instead of the full Glashow–Weinberg–Salam theory of the weak interactions. In terms of the diagram (15), this means approximating the W propagator as $ig_{\kappa\lambda}/M_W^2$, hence combining this propagator with the two vertices and the external line factors of the diagram, we get

$$\begin{aligned} i\mathcal{M} &= \frac{ig_{\kappa\lambda}}{M_W^2}\times\bar u(\nu_\mu)\left(\frac{-ig_2}{\sqrt2}\gamma_\kappa\frac{1-\gamma_5}{2}\right)u(\mu)\times\bar u(e)\left(\frac{-ig_2}{\sqrt2}\gamma_\lambda\frac{1-\gamma_5}{2}\right)v(\bar\nu_e)\\ &= -\frac{ig_2^2}{8M_W^2}\times\bar u(\nu_\mu)\gamma^\lambda(1-\gamma_5)u(\mu)\times\bar u(e)\gamma_\lambda(1-\gamma_5)v(\bar\nu_e).\end{aligned} \tag{S.29}$$

Eq. (18) obtains from this formula by identifying the overall factor here as the Fermi constant, or rather

$$\frac{G_F}{\sqrt{2}} \stackrel{\rm def}{=} \frac{g_2^2}{8M_W^2}\,. \tag{19}$$

<u>Problem **4**(b)</u>:

Let's start with the muon decay amplitude

$$\mathcal{M}(\mu^-\to e^-\nu_\mu\bar\nu_e) \;=\; -\frac{G_F}{\sqrt{2}}\left[\bar u(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-)\right]\times\left[\bar u(e^-)\gamma_\alpha(1-\gamma^5)v(\bar\nu_e)\right]. \tag{18}$$

Since the Dirac conjugate of the $\gamma^\alpha(1-\gamma^5)$ matrix is the same

$$\overline{\gamma^\alpha(1-\gamma^5)} \;=\; (1-\bar\gamma^5)\bar\gamma^\alpha \;=\; (1+\gamma^5)\gamma^\alpha \;=\; \gamma^\alpha(1-\gamma^5), \tag{S.30}$$

the complex conjugate of the amplitude (18) is

$$\mathcal{M}^* \;=\; \frac{G_F}{\sqrt{2}}\left[\bar u(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\right]\times\left[\bar v(\bar\nu_e)\gamma_\beta(1-\gamma^5)u(e^-)\right]. \tag{S.31}$$

Note: I changed the summed-over Lorentz index here from α to β, so that in the product $\mathcal{M}\times\mathcal{M}^*$ below I can separately sum over α and β. Thus,

$$\begin{aligned}
|\mathcal{M}|^2 \;&=\; \tfrac12 G_F^2\times\left[\bar u(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-)\right]\times\left[\bar u(e^-)\gamma_\alpha(1-\gamma^5)v(\bar\nu_e)\right]\times\\
&\qquad\times\left[\bar u(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\right]\times\left[\bar v(\bar\nu_e)\gamma_\beta(1-\gamma^5)u(e^-)\right]\\
&=\; \tfrac12 G_F^2\times\left[\bar u(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-)\times\bar u(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\right]\times\\
&\qquad\times\left[\bar u(e^-)\gamma_\alpha(1-\gamma^5)v(\bar\nu_e)\times\bar v(\bar\nu_e)\gamma_\beta(1-\gamma^5)u(e^-)\right].
\end{aligned}$$

Consequently, when this $|\mathcal{M}|^2$ is summed over the final fermions spins and averaged over the spin of the initial muon, it becomes a product of two traces,

$$\begin{aligned}
\overline{|\mathcal{M}|^2} \stackrel{\rm def}{=} \tfrac12\sum_{\substack{\rm all\\ \rm spins}}|\mathcal{M}|^2 \;&=\; \tfrac14 G_F^2\times\sum_{s_\mu,s_\nu}\left[\bar u(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-)\times\bar u(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\right]\times\\
&\qquad\times\sum_{s_e,s_{\bar\nu}}\left[\bar u(e^-)\gamma_\alpha(1-\gamma^5)v(\bar\nu_e)\times\bar v(\bar\nu_e)\gamma_\beta(1-\gamma^5)u(e^-)\right]\\
&=\; \tfrac14 G_F^2\times\operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)(\not p_\mu+M_\mu)\gamma^\beta(1-\gamma^5)(\not p_\nu+m_\nu)\Big)\\
&\qquad\times\operatorname{tr}\Big(\gamma_\alpha(1-\gamma^5)(\not p_{\bar\nu}-m_{\bar\nu})\gamma_\beta(1-\gamma^5)(\not p_e+m_e)\Big).
\end{aligned} \tag{S.32}$$

Problem **4**(c):

Our next task is to evaluate the traces in eq. (S.32). For the first trace, we have

$$\begin{aligned}
&\operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)(\not p_\mu+M_\mu)\gamma^\beta(1-\gamma^5)(\not p_\nu+m_\nu)\Big) \;= \\
&\quad=\; \operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)\not p_\mu\gamma^\beta(1-\gamma^5)\not p_\nu\Big) \;+\; M_\mu m_\nu\times\operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)\gamma^\beta(1-\gamma^5)\Big) \\
&\qquad\quad+\;\text{vanishing traces of odd numbers of }\gamma^\lambda\text{ matrices} \\
&\qquad\quad\langle\!\langle\text{ next, move the }(1-\gamma^5)\text{ factors left using }\gamma^\lambda(1\mp\gamma^5)=(1\pm\gamma^5)\gamma^\lambda\,\rangle\!\rangle \\
&\quad=\; \operatorname{tr}\Big((1+\gamma^5)\gamma^\alpha\not p_\mu\gamma^\beta\,(1-\gamma^5)\not p_\nu\Big) \;+\; M_\mu m_\nu\times\operatorname{tr}\Big((1+\gamma^5)\gamma^\alpha\gamma^\beta(1-\gamma^5)\Big) \\
&\quad=\; \operatorname{tr}\Big((1+\gamma^5)^2\gamma^\alpha\not p_\mu\gamma^\beta\not p_\nu\Big) \;+\; M_\mu m_e\times\operatorname{tr}\Big((1+\gamma^5)(1-\gamma^5)\gamma^\alpha\gamma^\beta\Big) \\
&\qquad\quad\langle\!\langle\text{ use }(1+\gamma^5)^2=2(1+\gamma^5)\text{ while }(1+\gamma^5)(1-\gamma^5)=0\,\rangle\!\rangle \\
&\quad=\; 2\operatorname{tr}\Big((1+\gamma^5)\gamma^\alpha\not p_\mu\gamma^\beta\not p_\nu\Big) \;+\; 0 \\
&\quad=\; 2\operatorname{tr}\Big(\gamma^\alpha\not p_\mu\gamma^\beta\not p_\nu\Big) \;+\; 2\operatorname{tr}\Big(\gamma^5\gamma^\alpha\not p_\mu\gamma^\beta\not p_\nu\Big)\,.
\end{aligned} \tag{S.33}$$

Furthermore, the traces on the last line here were explicitly evaluated in my notes on Dirac traces:

$$\begin{aligned}
\operatorname{tr}\Big(\gamma^\alpha\not p_\mu\gamma^\beta\not p_\nu\Big) \;&=\; 4p_\mu^\alpha p_\nu^\beta \;+\; 4p_\mu^\beta p_\nu^\alpha \;-\; 4g^{\alpha\beta}(p_\mu\cdot p_\nu), \\
\operatorname{tr}\Big(\gamma^5\gamma^\alpha\not p_\mu\gamma^\beta\not p_\nu\Big) \;&=\; -4i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\,,
\end{aligned} \tag{S.34}$$

hence

$$\begin{aligned}
&\operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)(\not p_\mu+M_\mu)\gamma^\beta(1-\gamma^5)(\not p_\nu+m_\nu)\Big) \;= \\
&\qquad\qquad=\; 8\Big[p_\mu^\alpha p_\nu^\beta \;+\; p_\mu^\beta p_\nu^\alpha \;-\; g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \;-\; 8i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}.
\end{aligned} \tag{S.35}$$

In exactly the same way, the second big trace in eq. (S.32) evaluates to

$$\begin{aligned}
&\operatorname{tr}\big(\gamma_\alpha(1-\gamma^5)(\not p_e+m_e)\gamma_\beta(1-\gamma^5)(\not p_{\bar\nu}-m_{\bar\nu}\big) \;= \\
&\qquad\qquad=\; 8\left[(p_{e\alpha}p_{\bar\nu\beta} \;+\; p_{e\beta}p_{\bar\nu\alpha} \;-\; g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\right] \;-\; 8i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\,.
\end{aligned} \tag{S.36}$$

Problem **4**(d):

Let's plug the traces (S.35) and (S.36) back into eq. (S.32) and contract the Lorentz indices α and β:

$$\begin{aligned}\overline{|\mathcal{M}|^2} \;=\; 16G_F^2 \times \Big(\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \,-\, i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\Big) \times \\ \times \Big(\Big[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] \,-\, i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\Big).\end{aligned} \tag{S.37}$$

Note that inside each pair of () here, the first term is symmetric WRT $\alpha\leftrightarrow\beta$ while the second term is antisymmetric, so when we multiply the two factors together and contract the indices, only the symmetric × symmetric and antisymmetric × antisymmetric products contribute to the

$$\overline{|\mathcal{M}|^2} \;=\; 16G_F^2\left(\begin{aligned}\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \times \Big[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] \\ -\; i0 \;-i0 \;-\; \epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\end{aligned}\right). \tag{S.38}$$

Moreover, when we open the brackets on the top line here, several terms cancel each other:

$$\begin{aligned}\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \times \Big[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] \;=& \\ =\, 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) \;+\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)& \\ -\; \cancel{2\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})} \;-\; \cancel{2\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})}& \\ +\; \cancel{(g^{\alpha\beta}g_{\alpha\beta}=4)\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})}& \\ =\, 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) \;+\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e).&\end{aligned} \tag{S.39}$$

As to the second line of eq. (S.38), contracting the two Levi–Civita tensors gives us

$$\epsilon^{\alpha\gamma\beta\delta}\times\epsilon_{\alpha\rho\beta\sigma} \;=\; -2\delta^\gamma_\rho\delta^\delta_\sigma \;+\; 2\delta^\gamma_\sigma\delta^\delta_\rho\,, \tag{S.40}$$

hence

$$-\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma \;=\; +2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \;-\; 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}). \tag{S.41}$$

Finally, combining the two lines of eq. (S.38) gives us one more cancellation, thus

$$\overline{|\mathcal{M}|^2} \;=\; 16G_F^2 \begin{pmatrix} \cancel{2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})} \;+\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \\ +\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \;-\; \cancel{2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})} \end{pmatrix} \;=\; 64G_F^2\times(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e). \tag{21}$$

Quod erat demonstrandum.

Problem **4**(e):

As explained in my notes on phase space — and in much more detail in §4.5 of the *Peskin & Schroeder* textbook, — the partial rate of a decay process (in the rest frame of the initial particle) is given by

$$d\Gamma \;=\; \overline{|\mathcal{M}|^2}\times d\mathcal{P} \tag{S.42}$$

where $\mathcal{M}$ is the decay's amplitude, $\overline{|\mathcal{M}|^2}$ is $|\mathcal{M}|^2$ averaged over the unknown initial spins and summed over the unmeasured final spins, and $d\mathcal{P}$ is the infinitesimal phase space factor. For three final-state particles,

$$\begin{aligned} d\mathcal{P} \;=\; \frac{1}{2M_0}\times\frac{d^3\mathbf{p}_1}{(2\pi)^3(2E_1)}\,\frac{d^3\mathbf{p}_2}{(2\pi)^3(2E_2)}\,\frac{d^3\mathbf{p}_3}{(2\pi)^3(2E_3)}\,\times \\ \times\,(2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\times(2\pi)\delta(E_1+E_2+E_3-M_0) \end{aligned} \tag{S.43}$$

where the energy-momentum conservation laws apply in the rest frame, thus $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=\mathbf{p}_{\rm tot}=\mathbf{0}$ and $E_1+E_2+E_3=E_{\rm tot}=M_0$.

We start by using the momentum-conservation δ–function to eliminate the $\mathbf{p}_3$ as independent variable, thus

$$d\mathcal{P} \;=\; \frac{d^3\mathbf{p}_1\,d^3\mathbf{p}_2}{512\pi^5}\times\frac{\delta(E_1+E_2+E_3-E_{\rm tot})}{M_0E_1E_2E_3}\bigg|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}. \tag{S.44}$$

Next, we use spherical coordinates for the two remaining momenta,

$$d^3\mathbf{p}_1 \;=\; p_1^2\,dp_1\,d^2\Omega_1\,, \qquad d^3\mathbf{p}_2 \;=\; p_2^2\,dp_2\,d^2\Omega_2\,, \tag{S.45}$$

and then replace the $d^2\Omega_2$ describing the direction of the second particle's momentum relative to the fixed external frame with

$$d^2\Omega_2^{(1)} \;=\; d\theta_{12}\,\sin\theta_{12}\,d\phi_2^{(1)}$$

describing the same direction of $\mathbf{p}_2$ relative to the frame centered on the $\mathbf{p}_1$. Consequently,

$$d^2\Omega_1\,d^2\Omega_2 \;=\; d^2\Omega_1\,d^2\Omega_2^{(1)} \;=\; \left[d^2\Omega_1\,d\phi_2^{(1)}\right]\,d\theta_{12}\,\sin\theta_{12} \;\equiv\; d^3\Omega\times d(\cos\theta_{12}) \tag{S.46}$$

and hence

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{512\pi^5}\times\frac{p_1^2p_2^2}{M_0E_1E_2E_3}\,dp_1\,dp_2\times d(\cos\theta_{12})\,\delta(E_1+E_2+E_3-E_{\rm tot})\bigg|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}. \tag{S.47}$$

Next, we use the cosine theorem

$$p_3^2 \;=\; (\mathbf{p}_1+\mathbf{p}_2)^2 \;=\; p_1^2\,+\,p_2^2\,+\,2p_1p_2\cos\theta_{12}$$

which gives

$$d(\cos\theta_{12}) \;=\; \frac{p_3\,dp_3}{p_1\,p_2}$$

(for fixed p_1, p_2), and therefore

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{512\pi^5M_0}\times\frac{p_1p_2p_3}{E_1E_2E_3}\times dp_1\,dp_2\,dp_3\times\delta(E_1+E_2+E_3-E_{\rm tot}). \tag{S.48}$$

Finally, we notice that for a relativistic particle of any mass, $pdp = EdE$, hence

$$\frac{p_1p_2p_3}{E_1E_2E_3}\times dp_1\,dp_2\,dp_3 \;=\; dE_1\,dE_2\,dE_3 \tag{S.49}$$

and therefore

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{512\pi^5M_0}\times dE_1\,dE_2\,dE_3\,\delta(E_1+E_2+E_3-E_{\rm tot}). \tag{S.50}$$

Plugging this formula into eq. (S.42) immediately gives us eq. (22) for the partial 3-body decay rate, *quod erat demonstrandum.*

Problem **4**(f):

The kinematic limits on the final particles' energies follow from the triangle inequalities for the magnitudes of three momentum vectors which add up to zero:

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0} \quad \Longrightarrow \quad p_1 \le p_2 + p_3 \quad \textbf{and} \quad p_2 \le p_1 + p_3 \quad \textbf{and} \quad p_3 \le p_1 + p_2 . \tag{S.51}$$

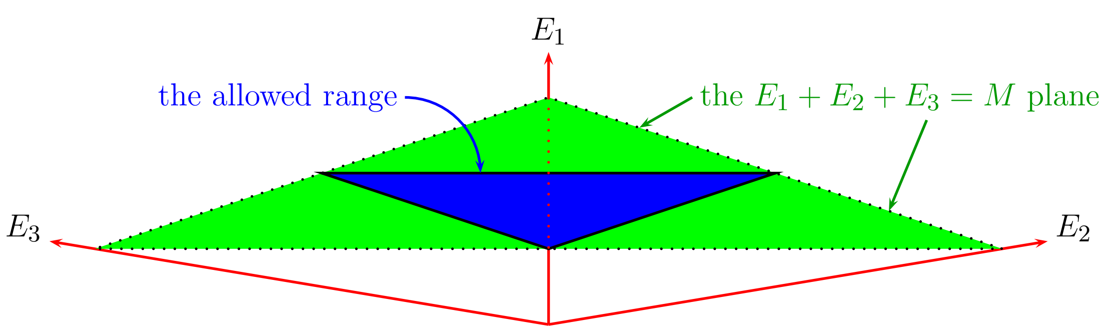
These inequalities look simple in terms of momenta but generally produce rather complicated inequalities for the energies $E_1 = \sqrt{p_1^2 + m_1^2}$, $E_2 = \sqrt{p_2^2 + m_2^2}$, and $E_3 = \sqrt{p_3^2 + m_3^2}$. However, when all three final particles are massless, the kinematic restrictions become simply

$$\begin{aligned} E_1 &\le E_2 + E_3 = M - E_1 , \\ E_2 &\le E_1 + E_3 = M - E_2 , \\ E_3 &\le E_1 + E_2 = M - E_3 , \end{aligned} \tag{S.52}$$

where the second expression on each right hand side follows from the net energy conservation $E_1 + E_2 + E_3 = M$. In other words, the kinematically allowed energies of the three final particles' range over

$$0 \le E_1, E_2, E_3 \le \tfrac{1}{2} M_0 , \qquad \text{while} \quad E_1 + E_2 + E_3 = M_0 . \tag{23}$$

The picture below shows this range in the (E_1, E_2, E_3) space:



(S.53)

Problem **4**(g):

In the muon's rest frame

$$(p_\mu \cdot p_{\bar\nu}) \;=\; M_\mu E_{\bar\nu} \tag{S.54}$$

while

$$\begin{aligned}
(p_e \cdot p_\nu) \;&=\; E_e E_\nu \;-\; p_e p_\nu \cos\theta_{e\nu} \\
\langle\!\langle \text{ by the cosine theorem } \rangle\!\rangle \quad & \\
&=\; E_e E_\nu \;+\; \tfrac12 p_e^2 \;+\; \tfrac12 p_\nu^2 \;-\; \tfrac12 p_{\bar\nu}^2 \\
\langle\!\langle \text{ neglecting } m_e, m_\nu, m_{\bar\nu} \rangle\!\rangle \quad & \\
&\approx\; E_e E_\nu \;+\; \tfrac12 E_e^2 \;+\; \tfrac12 E_\nu^2 \;-\; \tfrac12 E_{\bar\nu}^2 \\
&=\; \tfrac12 (E_e + E_\nu)^2 \;-\; \tfrac12 E_{\bar\nu}^2 \\
\langle\!\langle \text{ using } E_e + E_\nu = M_\mu - E_{\bar\nu} \rangle\!\rangle \quad & \\
&=\; \tfrac12 (M_\mu - E_{\bar\nu})^2 \;-\; \tfrac12 E_{\bar\nu}^2 \\
&=\; \tfrac12 M_\mu (M_\mu - 2E_{\bar\nu}).
\end{aligned} \tag{S.55}$$

Consequently, the spin-averaged muon decay amplitude^2 (21) becomes

$$\overline{|\mathcal{M}|^2} \;=\; 32 G_F^2 M_\mu^2 E_{\bar\nu} (M_\mu - 2E_{\bar\nu}). \tag{S.56}$$

Plugging this formula into eq. (22) for the decay rate gives us

$$d\Gamma(\mu^- \to e^- \nu_\mu \bar\nu_e) \;=\; \frac{G_F^2}{16\pi^5}\, M_\mu E_{\bar\nu} (M_\mu - 2E_{\bar\nu}) \times dE_e\, dE_\nu\, dE_{\bar\nu}\, d^3\Omega\, \delta(E_e + E_\nu + E_{\bar\nu} - M_\mu), \tag{S.57}$$

and all we need to do now is to integrate this formula over the final-state variables.

The integration variables comprise 3 angles $d^3\Omega$ — which integrate to $\int d^3\Omega = 8\pi^2$ — and 3 particles' energies subject to the constraint $E_e + E_\nu + E_{\bar\nu} = M_\mu$ and the kinematic limits (23). Integrating the decay rate (S.57) over these variables, we have

$$\begin{aligned}
\Gamma\ &=\ \frac{G_F^2 M_\mu}{2\pi^3}\int\limits_0^{M_\mu/2} dE_e \int\limits_0^{M_\mu/2} dE_{\bar\nu} E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \times \int\limits_0^{M_\mu/2} dE_\nu\,\delta(E_e + E_\nu + E_{\bar\nu} - M_\mu)\\
&=\ \frac{G_F^2 M_\mu}{2\pi^3}\int\limits_0^{M_\mu/2} dE_e \int\limits_0^{M_\mu/2} dE_{\bar\nu} E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \times \text{restrict to}(E_\nu = M - E_e - E_{\bar\nu} \le \tfrac12 M)\\
&=\ \frac{G_F^2 M_\mu}{2\pi^3}\int\limits_0^{\frac12 M_\mu} dE_e \int\limits_{\frac12 M_\mu - E_e}^{\frac12 M_\mu} dE_{\bar\nu}\, E_{\bar\nu}(M_\mu - 2E_{\bar\nu})\\
&\qquad \langle\!\langle \text{ where the lower limit of the } \textstyle\int dE_{\bar\nu} \text{ comes from } E_\nu \le \tfrac12 M_\mu \Longrightarrow E_e + E_{\bar\nu} \ge \tfrac12 M_\mu \,\rangle\!\rangle\\
&=\ \frac{G_F^2 M_\mu}{2\pi^3}\int\limits_0^{\frac12 M_\mu} dE_e\,\bigl(\tfrac12 M_\mu E_e^2\ -\ \tfrac23 E_e^3\bigr).
\end{aligned} \tag{S.58}$$

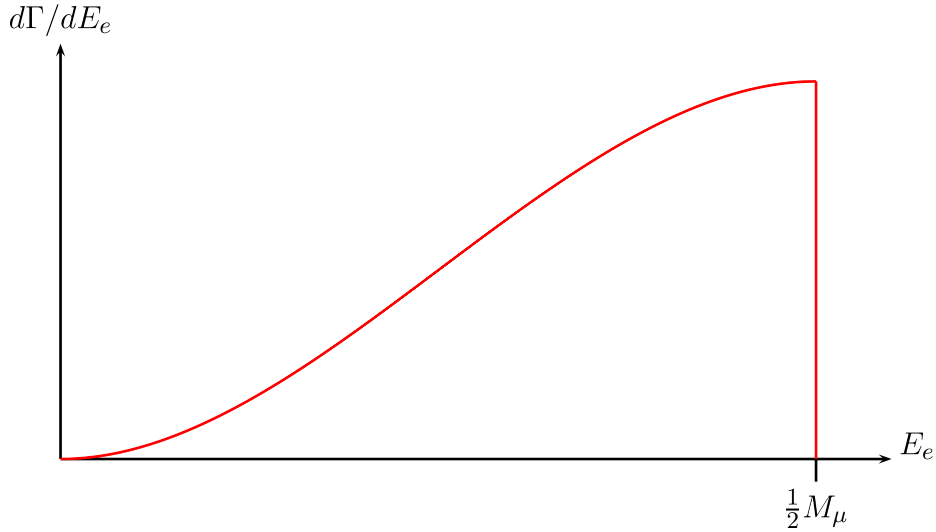
In other words, the partial muon decay rate with respect to the final electron's energy is given by

$$\frac{d\Gamma}{dE_e}\ =\ \frac{G_F^2 M_\mu}{12\pi^3} \times E_e^2(3M_\mu - 4E_e) \tag{S.59}$$

or rather

$$\frac{d\Gamma}{dE_e}\ \approx \begin{cases} \frac{G_F^2}{12\pi^3}\, M_\mu E_e^2(3M_\mu - 4E_e) & \text{for } E_e < \frac12 M_\mu,\\ 0 & \text{for } E_e > \frac12 M_\mu. \end{cases} \tag{S.60}$$

Graphically,



Note how this curve smoothly reaches its maximum at $E_e = \frac{1}{2}M_\mu$ and then abruptly falls down to zero.

It remains to calculate the total decay rate of the muon by integrating the partial rate (S.60) over the electron's energy. The result is

$$\Gamma_{\rm tot}(\mu \to e\nu\bar{\nu}) \;=\; \frac{G_F^2 M_\mu}{12\pi^3} \times \int_0^{\frac{1}{2}M_\mu} dE_e\, E_e^2(3M_\mu - 4E_e) \;=\; \frac{G_F^2 M_\mu^5}{192\pi^3}\,, \tag{S.61}$$

or numerically $\Gamma = 3.01 \cdot 10^{-19}$ GeV $= 4.57 \cdot 10^5$ s^{-1}. This tree level result is in good agreement with the experimental muon lifetime $\tau = 2.197 \cdot 10^{-6}$ s or $\Gamma = 4.55 \cdot 10^5\ s^{-1}$, the small discrepancy being due to QED loop corrections.