Problem **1**(a):

In the first diagram (1), the virtual photon has momentum $q = p_1' - p_1 = p_2 - p_2'$, hence $q^2 = t$. In the second diagram, the virtual photon's momentum is $\tilde{q} = p_1 + p_2 = p_1' + p_2'$, hence $\tilde{q}^2 = s$. Accordingly, the two diagrams are called the s-channel diagram and the t-channel diagram.

The t-channel diagram evaluates to

$$\begin{aligned} i\mathcal{M}_1 &= -\Big(\bar{v}(e^+)(ie\gamma_\mu)v(e^{+\prime})\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)u(e^-)\Big) \times \frac{-ig^{\mu\nu}}{q^2} \\ &= \frac{-ie^2}{t} \times \bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu u(e^-) \end{aligned} \tag{S.1}$$

where the overall minus sign is due to the positron-out to positron-in fermionic line. And the s-channel diagram evaluates to

$$\begin{aligned} i\mathcal{M}_2 &= +\Big(\bar{v}(e^+)(ie\gamma_\mu)u(e^-)\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)v(e^{+\prime}\Big) \times \frac{-ig^{\mu\nu}}{\tilde{q}^2} \\ &= \frac{+ie^2}{s} \times \bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \end{aligned} \tag{S.2}$$

where the overall sign is plus because both fermionic lines have an incoming or outgoing electron at one end.

Problem **1**(b):

Summing /averaging the $|\mathcal{M}_2|^2$ over spins works exactly as for the muon pair production discussed in class:

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}_2|^2 &= \left(\frac{e^2}{s}\right)^2 \sum_{\text{spins}} \Big[\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^-)\gamma_\nu v(e^+)\Big] \times \Big[\bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma^\nu u(e^{-\prime})\Big] \\ &= \left(\frac{e^2}{s}\right)^2 \operatorname{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right] \times \operatorname{tr}\left[(\not{p}_1' - m)\gamma^\mu(\not{p}_2' - m)\gamma^\nu\right] \\ &\qquad \langle\!\langle \text{ neglecting the mass relative to the momenta } \rangle\!\rangle \\ &\approx \left(\frac{e^2}{s}\right)^2 \operatorname{tr}\left[\not{p}_2\gamma_\mu \not{p}_1\gamma_\nu\right] \times \operatorname{tr}\left[\not{p}_1'\gamma^\mu \not{p}_2'\gamma^\nu\right] \end{aligned} \tag{S.3}$$

$$
\begin{aligned}
&= \left(\frac{e^2}{s}\right)^2 \times 4\left[p_{2\mu}p_{1\nu} + p_{2\nu}p_{1\mu} - g_{\mu\nu}(p_2p_1)\right] \times 4\left[p_2'^{\mu}p_1'^{\nu} + p_2'^{\nu}p_1'^{\mu} - g^{\mu\nu}(p_2'p_1')\right] \\
&= 16\left(\frac{e^2}{s}\right)^2 \Big[2(p_2'p_2)(p_1'p_1) + 2(p_2'p_1)(p_1'p_2). \\
&\qquad - 2(p_2'p_1')(p_2p_1) - 2(p_2'p_1')(p_2p_1) + 4(p_2'p_1')(p_2p_1)\Big] \\
&= 32\left(\frac{e^2}{s}\right)^2 \left[(p_2'p_2)(p_1'p_1) + (p_2'p_1)(p_1'p_2)\right] \\
&= 8\left(\frac{e^2}{s}\right)^2 \left[t^2 + u^2\right] \qquad \text{(S.3)}
\end{aligned}
$$

where the last equality follows from the kinematic relations (4). Altogether,

$$
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_2|^2 \;=\; 2e^4 \times \frac{t^2+u^2}{s^2}\,. \qquad (5)
$$

Problem **1**(c):

The two diagrams for Bhabha scattering are related by the *crossing symmetry*, so the amplitudes $\mathcal{M}_1$ and $\mathcal{M}_2$ are related to each other via analytic continuation of particle's momenta. In terms of the spin-summed $|\mathcal{M}|^2$ and Mandelstam's variables,

$$
\sum_{\text{spins}} |\mathcal{M}_1(s,t,u)|^2 \;=\; \sum_{\text{spins}} |\mathcal{M}_2(t,s,u)|^2, \qquad \text{(S.4)}
$$

hence eq. (5) for the second amplitude implies a similar equation for the first amplitude, but with s and t exchanged with each other — *i.e.*, eq. (6).

Alternatively, we may sum the $|\mathcal{M}_1|^2$ over all the spins in the same way as we summed the $|\mathcal{M}_2|^2$ in part (b):

$$
\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}_1|^2 &= \left(\frac{e^2}{t}\right)^2 \sum_{\text{spins}} \left[\bar{u}(e^{-\prime})\gamma^\mu u(e^-) \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime})\right] \times \left[\bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\right] \\
&= \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[(\not{p}_1' + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\right] \times \operatorname{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_2' - m)\gamma_\nu\right]
\end{aligned}
$$

$$\approx \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[\not{p}_1'\gamma^\mu\,\not{p}_1\gamma^\nu\right] \times \operatorname{tr}\left[\not{p}_2\gamma_\mu\,\not{p}_2'\gamma_\nu\right] \tag{S.5}$$

$$= \left(\frac{e^2}{t}\right)^2 \times 4\left[p_1'^\mu p_1^\nu + p_1'^\nu p_1^\mu - g^{\mu\nu}(p_1'p_1)\right] \times 4\left[p'_{2\mu}p_{2\nu} + p'_{2\nu}p_{2\mu} - g_{\mu\nu}(p_2'p_2)\right]$$

$$= 16\left(\frac{e^2}{t}\right)^2 \Big[2(p_1'p_2')(p_1p_2) + 2(p_1'p_2)(p_1p_2')$$
$$- 2(p_1'p_1)(p_2'p_2) - 2(p_1'p_1)(p_2'p_2) + 4(p_1'p_1)(p_2'p_2)\Big]$$

$$= 32\left(\frac{e^2}{t}\right)^2 \left[(p_1'p_2')(p_1p_2) + (p_1'p_2)(p_1p_2')\right]$$

$$= 8\left(\frac{e^2}{t}\right)^2 \left[s^2 + u^2\right] \tag{S.5}$$

and hence

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1|^2 = 2e^4 \times \frac{s^2+u^2}{t^2}\,. \tag{6}$$

Problem **1**(d):

The interference term between the two diagrams is more complicated:

$$\mathcal{M}_1^* \times \mathcal{M}_2 = -\frac{e^2}{t}\Big(\bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\Big) \times$$
$$\times \frac{e^2}{s}\Big(\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime})\Big)$$
$$= -\frac{e^4}{st} \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+) \times \bar{v}(e^+)\gamma_\mu u(e^-) \tag{S.6}$$

where on the last line I have re-ordered the factors so that each $\bar{u}$ is followed by u of the same electron and each $\bar{v}$ is followed by v for the same positron. After summing over all the spins, each $u \times \bar{u}$ becomes $(\not{p} + m)$, each $v \times \bar{v}$ becomes $(\not{p} - m)$, and the whole product becomes a single big trace rather than a product of two traces,

$$\sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = -\frac{e^4}{st} \times \operatorname{tr}\Big[(\not{p}_1 + m)\gamma^\nu(\not{p}_1' + m)\gamma^\mu(\not{p}_2' - m)\gamma_\nu(\not{p}_2 - m)\gamma_\mu\Big]$$
$$\approx -\frac{e^4}{st} \times \operatorname{tr}\Big[\not{p}_1\gamma^\nu\,\not{p}_1'\gamma^\mu\,\not{p}_2'\gamma_\nu\,\not{p}_2\gamma_\mu\Big]. \tag{S.7}$$

This trace looks more complicated than it is, and we may drastically simplify it by summing over ν and μ before taking the trace. Back in homework#7 we saw that

$$\gamma^\alpha \not{a}\not{b}\not{c}\gamma_\alpha \;=\; -2\not{c}\not{b}\not{a} \quad\text{and}\quad \gamma^\alpha \not{a}\not{b}\gamma_\alpha \;=\; 4(ab). \tag{S.8}$$

For the problem at hand, this gives us $\gamma^\nu \not{p}_1'\gamma^\mu \not{p}_2'\gamma_\nu = -2\not{p}_2'\gamma^\mu \not{p}_1'$ and hence

$$\begin{aligned}
\mathrm{tr}\Big[\not{p}_1\times\gamma^\nu\not{p}_1'\gamma^\mu\not{p}_2'\gamma_\nu\times\not{p}_2\gamma_\mu\Big] &= -2\,\mathrm{tr}\Big[\not{p}_1\times\not{p}_2'\gamma^\mu\not{p}_1'\times\not{p}_2\gamma_\mu\Big] \;=\; -2\,\mathrm{tr}\Big[\not{p}_1\not{p}_2'\times\gamma^\mu\not{p}_1'\not{p}_2\gamma_\mu\Big]\\
&= -2\,\mathrm{tr}\Big[\not{p}_1\not{p}_2'\times 4(p_1'p_2)\Big] \;=\; -8(p_1'p_2)\times\mathrm{tr}\Big[\not{p}_1\not{p}_2'\Big]\\
&= -8(p_1'p_2)\times 4(p_1p_2')\\
&= -8u^2.
\end{aligned} \tag{S.9}$$

Plugging this trace back into eq. (S.6), we arrive at

$$\frac{1}{4}\sum_{\text{spins}}\mathcal{M}_1^*\times\mathcal{M}_2 \;=\; +2e^4\times\frac{u^2}{st}\,. \tag{7}$$

Problem **1**(e):

Assembling the spin sums / averages (5–7) together according to eq. (3), we get

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &\stackrel{\text{def}}{=} \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_1+\mathcal{M}_2|^2\\
&= \frac{1}{4}\sum_{\text{spins}}\Big(|\mathcal{M}_1|^2 \,+\, |\mathcal{M}_2|^2 \,+\, 2\,\mathrm{Re}\,\mathcal{M}_1^*\mathcal{M}_2\Big)\\
&= 2e^4\times\frac{s^2+u^2}{t^2} \;+\; 2e^4\times\frac{t^2+u^2}{s^2} \;+\; 4e^4\times\frac{u^2}{st}\\
&= 2e^4\left(\frac{s^2}{t^2}\,+\,\frac{t^2}{s^2}\,+\,\frac{u^2}{s^2t^2}\times\Big(s^2+t^2+2st \,=\, (s+t)^2 \,=\, u^2\Big)\right)\\
&= 2e^4\times\frac{s^4+t^4+u^4}{s^2\times t^2}\,.
\end{aligned} \tag{S.10}$$

Consequently, the un-polarized partial cross-section for the Bhabha scattering is

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} \;=\; \frac{\overline{|\mathcal{M}|^2}}{64\pi^2\,s} \;=\; \frac{\alpha^2}{2s}\times\frac{s^4+t^4+u^4}{s^2\times t^2}\,. \tag{8.a}$$

To complete the problem, let's work out the kinematics in the center of mass frame:

$$\begin{aligned} s &= 4E^2 \approx 4\mathbf{p}^2, \\ t &= -(\mathbf{p}'_1-\mathbf{p}_1)^2 = -2\mathbf{p}^2(1-\cos\theta), \\ u &= -(\mathbf{p}'_2-\mathbf{p}_1)^2 = -2\mathbf{p}^2(1+\cos\theta), \end{aligned} \tag{S.11}$$

hence

$$\begin{aligned} \frac{s^4+t^4+u^4}{s^2\,t^2} &= \frac{(4\mathbf{p}^2)^4 + (2\mathbf{p}^2)^4\times(1-\cos\theta)^4 + (2\mathbf{p}^2)^4\times(1+\cos\theta)^4}{(4\mathbf{p}^2)^2\times(2\mathbf{p}^2)^2(1-\cos\theta)^2} \\ &= \frac{16+(1-\cos\theta)^4+(1+\cos\theta)^4}{4\times(1-\cos\theta)^2} = \frac{18+12\cos^2\theta+2\cos^4\theta}{4\times(1-\cos\theta)^2} \\ &= \frac{(3+\cos^2\theta)^2}{2(1-\cos\theta)^2}\,. \end{aligned} \tag{S.12}$$

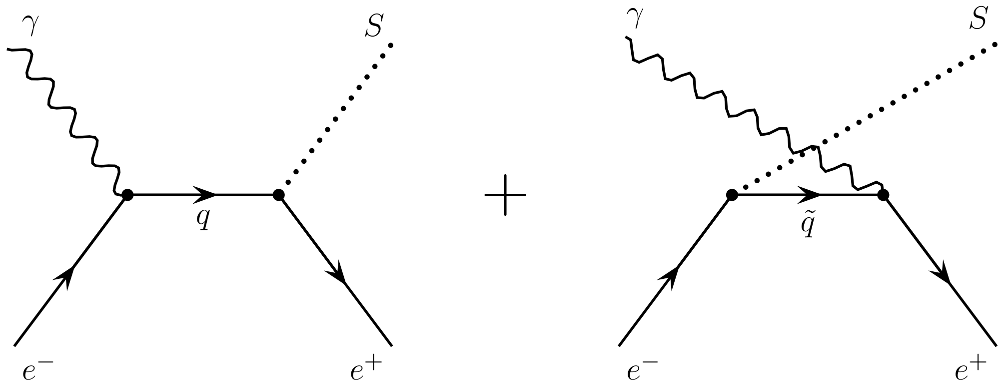
Plugging this formula into eq. (8.a), we finally obtain

$$\frac{d\sigma^{\rm Bhabha}}{d\Omega_{\rm cm}} = \frac{\alpha^2}{4s}\times\frac{(3+\cos^2\theta)^2}{(1-\cos\theta)^2}\,. \tag{8.b}$$

Quod erat demonstrandum.

Problem **3**(a):

There are two tree diagrams for the $e^-e^+\to S\gamma$ process, namely



(S.13)

These two diagrams are related by the $t\leftrightarrow u$ crossing, and also by the charge conjugation (which exchanges the initial e^- and e^+). The net tree-level amplitude is

$$\mathcal{M}_{\rm tree} = \mathcal{E}^{*\mu}_{\mathbf{k},\lambda}(\gamma)\times\mathcal{M}_\mu\,, \tag{S.14.a}$$

$$\mathcal{M}^\mu \;=\; \mathcal{M}_1^\mu \;+\; \mathcal{M}_2^\mu\,, \tag{S.14.b}$$

$$\begin{aligned} i\mathcal{M}_1^\mu &= -\bar{v}(e^+)(-ig)\frac{i}{\not{q}-m_e}\,(ie\gamma^\mu)\,u(e^-) \\ &= \frac{ieg}{t-m_e^2}\times\bar{v}(\not{q}+m_e)\gamma^\mu u\,, \end{aligned} \tag{S.14.c}$$

$$\begin{aligned} i\mathcal{M}_2^\mu &= \bar{v}(e^+)\,(ie\gamma^\mu)\,\frac{i}{\not{\tilde{q}}-m_e}(-ig)u(e^-) \\ &= \frac{ieg}{u-m_e^2}\times\bar{v}\gamma^\mu(\not{\tilde{q}}+m_e)u\,, \end{aligned} \tag{S.14.d}$$

where

$$\begin{aligned} q \;&=\; p_- \,-\, k_\gamma \;=\; k_s \,-\, p_+\,, \quad q^2 \;=\; t, \\ \text{and}\quad \tilde{q} \;&=\; p_- \,-\, k_s \;=\; k_\gamma \,-\, p_+\,, \quad \tilde{q}^2 \;=\; u. \end{aligned} \tag{S.15}$$

<u>Problem **3**(b)</u>:

The Ward identity for the one-photon amplitude (S.14.a) says $k_\gamma^\mu\times\mathcal{M}_\mu=0$. To verify it, let's start with the first diagram:

$$\begin{aligned} k_\gamma^\mu\times\bar{v}(\not{q}+m_e)\gamma_\mu u &= \bar{v}(\not{q}+m_e)\,\not{k}_\gamma u \\ &= \bar{v}(\not{p}_- - \not{k}_\gamma + m_e)\,\not{k}_\gamma u \\ &= \bar{v}(\not{p}_- + m_e)\,\not{k}_\gamma u \qquad \langle\!\langle\,\text{because } \not{k}_\gamma\,\not{k}_\gamma = k_\gamma^2 = 0\,\rangle\!\rangle \\ &= \bar{v}\Big(2(p_-k_\gamma) - \not{k}_\gamma(\not{p}_- - m_e)\Big)u \quad \langle\!\langle\,\text{anticommuting } \not{p}_- \text{ and } \not{k}_\gamma\,\rangle\!\rangle \\ &= 2(p_-k_\gamma)\times\bar{v}u \;-\; 0 \qquad \langle\!\langle\,\text{because } (\not{p}_- - m_e)\times u(e^-) = 0\,\rangle\!\rangle \\ &= (m_e^2 - t)\times\bar{v}u\,, \end{aligned} \tag{S.16}$$

and hence

$$k_\gamma^\mu\times\mathcal{M}_{1\mu} \;=\; -eg\times\bar{v}u\,. \tag{S.17}$$

We see that *by itself*, the first diagram does not satisfy the Ward entity. Instead, we need to add the second diagram's contribution

$$\begin{aligned}
k_\gamma^\mu \times \bar{v}\gamma_\mu(\tilde{q}\!\!/ + m_e)u &= \bar{v}\, k\!\!\!/_\gamma(\tilde{q}\!\!/ + m_e)u \\
&= \bar{v}\, k\!\!\!/_\gamma(k\!\!\!/_\gamma - p\!\!\!/_+ + m_e)u \\
&= \bar{v}\, k\!\!\!/_\gamma(-p\!\!\!/_+ + m_e)u \qquad \langle\!\langle \text{because } k\!\!\!/_\gamma\, k\!\!\!/_\gamma = k_\gamma^2 = 0 \rangle\!\rangle \\
&= \bar{v}\Big(-2(p_+k_\gamma) \;+\; (p\!\!\!/_+ + m_e)\, k\!\!\!/_\gamma\Big)u \quad \langle\!\langle \text{anticommuting } p\!\!\!/_+ \text{ and } k\!\!\!/_\gamma \rangle\!\rangle \\
&= -2(p_+k_\gamma) \times \bar{v}u \;+\; 0 \qquad \langle\!\langle \text{because } \bar{v}(e^+) \times (p\!\!\!/_+ + m_e) = 0 \rangle\!\rangle \\
&= (u - m_e^2) \times \bar{v}u\,,
\end{aligned} \tag{S.18}$$

and hence

$$k_\gamma^\mu \times \mathcal{M}_{2\mu} \;=\; +eg \times \bar{v}u\,. \tag{S.19}$$

Again, the second diagram does not satisfy the Ward identity *by itself*, but the net amplitude does:

$$k_\gamma^\mu \times (\mathcal{M}_\mu \;=\; \mathcal{M}_{1\mu} \;+\; \mathcal{M}_{2\mu}) \;=\; 0. \tag{S.20}$$

Problem **3**(c):

Thanks to the Ward identity, summing $|\mathcal{M}|^2$ over the photon's polarizations is easy:

$$\begin{aligned}
\sum_\lambda |\mathcal{M}|^2 &= -\mathcal{M}^\mu \mathcal{M}_\mu^* \qquad \langle\!\langle \text{see my notes on Ward identities} \rangle\!\rangle \\
&= -\mathcal{M}_1^\mu \mathcal{M}_{1\mu}^* \;-\; \mathcal{M}_2^\mu \mathcal{M}_{2\mu}^* \;-\; 2\,\mathrm{Re}\left(\mathcal{M}_1^\mu \mathcal{M}_{2\mu}^*\right) \\
&= -\frac{e^2g^2}{(t-m_e^2)^2} \times \bar{v}(q\!\!\!/ + m_e)\gamma^\mu u \times \bar{u}\gamma_\mu(q\!\!\!/ + m_e)v \\
&\quad - \frac{e^2g^2}{(u-m_e^2)^2} \times \bar{v}\gamma^\mu(\tilde{q}\!\!/ + m_e)u \times \bar{u}(\tilde{q}\!\!/ + m_e)\gamma_\mu v \\
&\quad - \frac{2e^2g^2}{(t-m_e^2)(u-m_e^2)} \times \mathrm{Re}\Big(\bar{v}(q\!\!\!/ + m_e)\gamma^\mu u \times \bar{u}(\tilde{q}\!\!/ + m_e)\gamma_\mu v\Big).
\end{aligned} \tag{S.21}$$

Consequently, averaging this formula over the electron's and the positron's spins yields

$$\overline{|\mathcal{M}|^2} \;=\; \frac{1}{4}\sum_{s_-,s_+}\sum_\lambda |\mathcal{M}|^2 \;=\; e^2g^2\left(\frac{A_{11}}{(t-m_2^2)^2} \;+\; \frac{A_{22}}{(u-m_e^2)^2} \;+\; \frac{2\,\mathrm{Re}\,A_{12}}{(t-m_e^2)(u-m_e^2)}\right) \tag{10}$$

where

$$
\begin{aligned}
A_{11} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m_e)v, \\
A_{22} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}\gamma^\mu(\tilde{\not{q}}+m_e)u \times \bar{u}(\tilde{\not{q}}+m_e)\gamma_\mu v, \qquad \text{(S.22)} \\
A_{12} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}(\tilde{\not{q}}+m_e)\gamma_\mu v.
\end{aligned}
$$

At this point, we use the spin sums

$$
\sum_{s_-} u \times \bar{u} \;=\; (\not{p}_- + m_e), \quad \sum_{s_+} v \times \bar{v} \;=\; (\not{p}_+ - m_e) \qquad \text{(S.23)}
$$

to convert eqs. (S.22) to Dirac traces (11):

$$
\begin{aligned}
A_{11} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)(\not{q}+m_e)\gamma^\mu\left(\sum\nolimits_{s_-} u\times\bar{u}\right)\gamma_\mu(\not{q}+m_e)\right) \\
&= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q}+m_e)\gamma^\mu(\not{p}_- + m_e)\gamma_\mu(\not{q}+m_e)\Big), \qquad \text{(S.24)}
\end{aligned}
$$

and likewise

$$
\begin{aligned}
A_{22} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)\gamma^\mu(\tilde{\not{q}}+m_e)\left(\sum\nolimits_{s_-} u\times\bar{u}\right)(\tilde{\not{q}}+m_e)\gamma_\mu\right) \\
&= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)\gamma^\mu(\tilde{\not{q}}+m_e)(\not{p}_- + m_e)(\tilde{\not{q}}+m_e)\gamma_\mu\Big), \qquad \text{(S.25)} \\
A_{12} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)(\not{q}+m_e)\gamma^\mu\left(\sum\nolimits_{s_-} u\times\bar{u}\right)(\tilde{\not{q}}+m_e)\gamma_\mu\right) \\
&= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q}+m_e)\gamma^\mu(\not{p}_- + m_e)(\tilde{\not{q}}+m_e)\gamma_\mu\Big). \qquad \text{(S.26)}
\end{aligned}
$$

Quod erat demonstrandum.

Problem **3**(d):

Evaluating the Dirac traces (11) is straightforward but tedious. Fortunately, it becomes much simpler when we neglect the electron's mass. In that limit, the first trace becomes

$$\begin{aligned} A_{11} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not p_+ \not q \gamma^\mu \not p_- \gamma_\mu \not q) \\ &= +\tfrac{1}{2}\,\mathrm{Tr}(\not p_+ \not q \not p_- \not q) \qquad \langle\!\langle \text{ using } \gamma^\mu \not p_- \gamma_\mu = -2 \not p_- \rangle\!\rangle \\ &= 2(p_+ q)(p_- q)\times 2 \;-\; 2(p_+ p_-)\, q^2 \\ &\approx (M_s^2 - t)\times t \;-\; s\times t \;=\; (M_s^2 - t - s)\times t \\ &\approx u \times t\,, \end{aligned} \tag{S.27}$$

where the last two lines follow from

$$\begin{aligned} q^2 &= t, \\ 2(p_+p_-) &= (p_- + p_+)^2 - (p_-^2 + p_+^2) \;=\; s - 2m_e^2 \;\approx\; s, \\ 2(p_-q) &= 2p_-(p_- - k_\gamma) \;=\; (p_- - k_\gamma)^2 + p_-^2 - k_\gamma^2 \;=\; t + m_e^2 - 0 \;\approx\; t, \\ 2(p_+q) &= 2p_+(k_s - p_+) \;=\; -(k_s - p_+)^2 - p_+^2 + k_s^2 \;=\; -t - m_e^2 + M_s^2 \;\approx\; M_s^2 - t, \\ s + t + u &= M_s^2 + 2m_e^2 \;\approx\; M_s^2\,. \end{aligned} \tag{S.28}$$

Likewise, the second trace becomes

$$\begin{aligned} A_{22} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not p_+ \gamma^\mu \not{\tilde q} \not p_- \not{\tilde q} \gamma_\mu) \\ &= -\tfrac{1}{4}\,\mathrm{Tr}(\gamma_\mu \not p_+ \gamma^\mu \not{\tilde q} \not p_- \not{\tilde q}) \\ &= +\tfrac{1}{2}\,\mathrm{Tr}(\not p_+ \not{\tilde q} \not p_- \not{\tilde q}) \qquad \langle\!\langle \text{ using } \gamma_\mu \not p_+ \gamma^\mu = -2 \not p_+ \rangle\!\rangle \\ &= 2(p_+\tilde q)(p_-\tilde q)\times 2 - \;2(p_+p_-)\,\tilde q^2 \\ &\approx (M_s^2 - u)\times u \;-\; s\times u \;=\; (M_s^2 - u - s)\times u \\ &\approx t\times u\,, \end{aligned} \tag{S.29}$$

where the last two lines follow from (S.28) and

$$\begin{aligned} \tilde q^2 &= u, \\ 2(p_+\tilde q) &= 2p_+(k_\gamma - p_+) \;=\; -(k_\gamma - p_+)^2 - p_+^2 - + k_\gamma^2 \;=\; -u - m_e^2 + 0 \;\approx\; -u\,, \\ 2(p_-\tilde q) &= 2p_-(p_- - k_s) \;=\; (p_- - k_s)^2 - k_s^2 + p_-^2 \;=\; u - M_s^2 + m_e^2 \;\approx\; u - M_s^2\,. \end{aligned} \tag{S.30}$$

Finally, the third trace becomes

$$\begin{aligned}
A_{22} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\,\not{q}\gamma^\mu\,\not{p}_-\,\tilde{\not{q}}\gamma_\mu) \\
&= -(p_-\tilde{q})\times\mathrm{Tr}(\not{p}_+\,\not{q}) \qquad \langle\!\langle\,\text{using } \gamma^\mu\,\not{p}_-\,\tilde{\not{q}}\gamma_\mu = +4(p_-\tilde{q})\,\rangle\!\rangle \\
&= -4(p_-\tilde{q})(p_+q) \\
&\approx +(u-M_s^2)(t-M_s^2).
\end{aligned} \tag{S.31}$$

Quad erat demonstrandum.

Problem **3**(e):

Now let's evaluate eq. (10) for the spin summed/averaged $\overline{|\mathcal{M}|^2}$. Neglecting the m_e^2 terms in the denominators and plugging in eqs. (12) for the A_{11}, A_{22}, and A_{12}, we have

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= e^2g^2\left(\frac{tu}{t^2}\,+\,\frac{ut}{u^2}\,+\,\frac{2(t-M_s^2)(u-M_s^2)}{tu}\right) \\
&= \frac{e^2g^2}{tu}\times\Bigl(u^2\,+\,t^2\,+\,2(t-M_s^2)(u-M_s^2)\Bigr) \\
&= \frac{e^2g^2}{tu}\times\Bigl((t+u-M_s^2)^2\,+\,M_s^4\Bigr) \\
&= e^2g^2\times\frac{s^2+M_s^4}{tu}\,.
\end{aligned} \tag{S.32}$$

Now let's work out the kinematics in the center of mass frame. The initial electron and positron have 4–momenta $p_\mp^\mu = (E_e,\pm\mathbf{p})$ where $E_e\approx|\mathbf{p}|$. But since the scalar and the photon produced in the collision have different masses, they have equal and opposite 3-momenta (in the CM frame) but different energies: $k_\gamma^\mu = (\omega,+\mathbf{k})$ while $k_S^\mu = (E_s,-\mathbf{k})$, where $\omega = |\mathbf{k}| \neq E_s = \sqrt{\mathbf{k}^2+M_s^2}$. By energy conservation

$$\omega\,+\,E_s \;=\; 2E_e \;=\; \sqrt{s}\,. \tag{S.33}$$

To solve this equation, we rewrite it as

$$\omega^2\,+\,M_s^2 \;=\; E_s^2 \;=\; (\sqrt{s}\,-\omega)^2 \;=\; s\,-\,2\sqrt{s}\times\omega\,+\omega^2, \tag{S.34}$$

which gives us

$$\omega \;=\; \frac{s-M_s^2}{2\sqrt{s}} \quad\Longrightarrow\quad E_s \;=\; \frac{s+M_s^2}{2\sqrt{s}}\,. \tag{S.35}$$

Given all these momenta, Mandelstam's t and u obtain as

$$\begin{aligned} t \;&\approx\; -2(p_-k_\gamma) \;=\; -2E_e\omega \,+\, 2\mathbf{p}\cdot\mathbf{k} \approx\; -2E_e\omega\times(1-\cos\theta) \\ &=\; -\tfrac{1}{2}(s-M_s^2)\times(1-\cos\theta), \end{aligned} \tag{S.36}$$

$$\begin{aligned} u \;&\approx\; -2(p_+k_\gamma) \;=\; -2E_e\omega \,-\, 2\mathbf{p}\cdot\mathbf{k} \approx\; -2E_e\omega\times(1+\cos\theta) \\ &=\; -\tfrac{1}{2}(s-M_s^2)\times(1+\cos\theta). \end{aligned} \tag{S.37}$$

Hence, plugging these values into eq. (S.32) gives us

$$\overline{|\mathcal{M}|^2} \;=\; 4e^2g^2\times\frac{s^2+M_S^4}{(s-M_S^2)^2}\times\frac{1}{\sin^2\theta}\,. \tag{S.38}$$

Finally, the partial cross-section for a 2 *particles* $\to$ 2 *particles* inelastic scattering in the CM frame is given by

$$\frac{d\sigma}{d\Omega_{\rm cm}} \;=\; \frac{|\mathcal{M}|^2}{64\pi^2 s}\times\frac{|\mathbf{p}'|}{|\mathbf{p}|}\,. \tag{S.39}$$

For the problem at hand, the inelasticity factor $|\mathbf{p}'|/|\mathbf{p}|$ is

$$\frac{|\mathbf{k}|}{|\mathbf{p}|} \;\approx\; \frac{\omega}{E_e} \;=\; \frac{s-M_s^2}{s}\,. \tag{S.40}$$

Combining this factor with eq. (S.38), we finally arrive at the following formula for the partial cross-section:

$$\frac{d\sigma(e^-e^+\to\gamma S)}{d\Omega_{\rm c.m.}} \;=\; \frac{\alpha g^2}{4\pi}\times\frac{s^2+M_s^4}{s^2(s-M_s^2)}\times\frac{1}{\sin^2\theta}\,. \tag{S.41}$$

Note the forward-backward symmetry $\theta\leftrightarrow\pi-\theta$ of this cross section. Physically, it is due to the charge-conjugation symmetry which exchanges the initial electron and positron.

As usual for annihilation processes in the ultra-relativistic limit, the cross-section (S.41) has divergent peaks in forward and backward directions, $\theta \to 0$ or $\theta \to \pi$. The divergence here is an artefact of the $m_e^2 = 0$ approximation, which becomes inaccurate at very small angles $\theta \lesssim (m_e/E)$ (or $\pi - \theta \lesssim (m_e/E)$).

A more careful analysis — which was not a required part of this homework — leads to

$$\text{for } \theta \lesssim \gamma^{-1}, \quad \frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}} \approx \frac{\alpha g^2}{4\pi s} \times \left(\frac{s - M_s^2}{s} \times \frac{1}{\theta^2 + \gamma^{-2}} + \frac{M_s^2}{s - M_s^2} \times \frac{2\theta^2}{(\theta^2 + \gamma^{-2})^2} \right) \tag{S.42}$$

— where $\gamma^{-1} = m_e/E \ll 1$ — instead of eq. (S.41). Consequently, the total cross-section turns out to be finite rather then divergent, namely

$$\sigma_{\text{tot}}(e^-e^+ \to \gamma S) = \alpha g^2 \times \frac{(s^2 + M_s^4)}{s^2(s - M_s^2)} \left(\log \frac{2E_e}{m_e} - \frac{sM_s^2}{s^2 + M_s^4} + O\left(\frac{m_e^2}{E_e^2} \right) \right). \tag{S.43}$$

Problem **4**(a):

The scalar potential part of the linear sigma model's Lagrangian (13) is

$$V(\phi) = \frac{\lambda}{8} \left(\sum\nolimits_i \phi_i^2 - f^2 \right)^2 - \beta\lambda f^2 \times \phi_{N+1}, \tag{S.44}$$

where the last term explicitly breaks the $O(N+1)$ symmetry of the first term down to the $O(N)$. To find the minimum of this potential, let's first find the stationary points where all the first derivatives $\partial V/\partial\phi_i$ are zero:

$$\text{for } i = 1, \ldots, N, \quad \frac{\partial V}{\partial \phi_i} = \frac{\lambda}{2} \left(\sum\nolimits_j \phi_j^2 - f^2 \right) \times \phi_i = 0, \tag{S.45}$$

$$\text{and} \quad \frac{\partial V}{\partial \phi_{N+1}} = \frac{\lambda}{2} \left(\sum\nolimits_j \phi_j^2 - f^2 \right) \times \phi_{N+1} - \beta\lambda f^2 = 0. \tag{S.46}$$

From eq. (S.46) we immediately see that at any stationary point $\left(\sum \phi^2 - f^2 \right) \neq 0$, hence eqs. (S.45) tell us that $\phi_1 = \cdots = \phi_N = 0$. In other words, all the stationary points lie on the ϕ_{N+1} axis in the $(N+1)$ dimensional space of the scalar field values. And in this space, eq. (S.46) becomes a simple cubic equation

$$\phi_{N+1}^3 \;-\; f^2 \times \Phi_{N+1} \;-\; 2\beta f^2 \;=\; 0. \tag{S.47}$$

For small $\beta \ll f$, this cubic equation has 3 real solutions, approximately

$$\langle\phi_{N+1}\rangle_1 \;\approx\; -2\beta, \qquad \langle\phi_{N+1}\rangle_2 \;\approx\; -f \;+\; \beta, \qquad \langle\phi_{N+1}\rangle_3 \;\approx\; +f \;+\; \beta. \tag{S.48}$$

Now let's find out which of the three stationary points is a minimum (or at least a local minimum) by looking at the second derivatives of the potential (S.44). Along the ϕ_{N+1} axis in the field space, the second derivatives amount to

$$\frac{\partial^2 V}{\partial\phi_i \partial\phi_j} \;=\; \frac{\lambda}{2} \times \begin{cases} \left(3\phi_{N+1}^2 - f^2\right) & \text{for } i = j = N+1, \\ 0 & \text{for } i \le N,\ j = N+1 \text{ or } j \le N,\ i = N+1, \\ \left(\phi_{N+1}^2 - f^2\right) \times \delta_{ij} & \text{for } i, j \le N. \end{cases} \tag{S.49}$$

Evaluating these derivatives for the 3 stationary points (S.48) — while assuming small $\beta > 0$ — gives us

$$\begin{array}{llllll} @\ \langle\phi_{N+1}\rangle_1 : & \dfrac{\partial^2 V}{(\partial\phi_{N+1})^2} < 0 & \text{while other} & \dfrac{\partial^2 V}{(\partial\phi_i)^2} < 0 & \Longrightarrow & \text{maximum}, \\ @\ \langle\phi_{N+1}\rangle_2 : & \dfrac{\partial^2 V}{(\partial\phi_{N+1})^2} > 0 & \text{while other} & \dfrac{\partial^2 V}{(\partial\phi_i)^2} < 0 & \Longrightarrow & \text{saddle point}, \\ @\ \langle\phi_{N+1}\rangle_3 : & \dfrac{\partial^2 V}{(\partial\phi_{N+1})^2} > 0 & \text{while other} & \dfrac{\partial^2 V}{(\partial\phi_i)^2} > 0 & \Longrightarrow & \text{minimum}. \end{array} \tag{S.50}$$

Thus, the potential (S.44) has a unique minimum at

$$\langle\phi_1\rangle \;=\; \cdots \;=\; \langle\phi_N\rangle \;=\; 0, \quad \langle\phi_{N+1}\rangle \;=\; +f \;+\; \beta \;+\; O(\beta^2/f). \tag{14}$$

Quod erat demonstrandum.

Problem $\mathbf{4}$(b–c):

Let's shift the fields as in eq. (15). In terms of the shifted fields,

$$T \stackrel{\text{def}}{=} \sum_i \phi_i^2 - f^2 \;=\; \underset{\sim}{\pi}^2 + \bigl(\sigma + \langle\phi_{N+1}\rangle\bigr)^2 - f^2 \;=\; \underset{\sim}{\pi}^2 + \sigma^2 + 2\langle\phi_{N+1}\rangle \times \sigma + \bigl(\langle\phi_{N+1}\rangle^2 - f^2\bigr), \tag{S.51}$$

where $\underset{\sim}{\pi}$ is a short-hand for N-vector $(\pi^1, \ldots, \pi^N)$ of the pion fields, thus $\underset{\sim}{\pi}^2 = (\pi^1)^2 + \cdots + (\pi^N)^2$. Therefore, expanding the scalar potential (S.44) into powers of the shifted fields, we obtain

$$\begin{aligned} V \;&=\; \frac{\lambda}{8} \times T^2 \;-\; \beta\lambda f^2 \times \bigl(\sigma + \langle\phi_{N+1}\rangle\bigr) \\ &=\; \frac{\lambda}{8} \times \bigl(\underset{\sim}{\pi}^2 \,+\, \sigma^2\bigr)^2 \;+\; \frac{\lambda\langle\phi_{N+1}\rangle}{2} \times \bigl(\sigma\underset{\sim}{\pi}^2 \,+\, \sigma^3\bigr) \\ &\quad+\; \frac{\lambda\langle\phi_{N+1}\rangle^2}{2} \times \sigma^2 \;+\; \frac{\lambda(\langle\phi_{N+1}\rangle^2 - f^2)}{4} \times \bigl(\underset{\sim}{\pi}^2 + \sigma^2\bigr) \\ &\quad+\; \left(\frac{\lambda\langle\phi_{N+1}\rangle}{2} \times \bigl(\langle\phi_{N+1}\rangle^2 - f^2\bigr) \;-\; \beta\lambda f^2\right) \times \sigma \;+\; \text{const.} \end{aligned} \tag{S.52}$$

On the last line here, the coefficient of σ vanishes thanks to $\langle\phi_{N+1}\rangle$ obeying the cubic equation (S.47). For the same reason, the coefficient of $(\underset{\sim}{\pi}^2 + \sigma^2)$ on the line before the last may be simplified as

$$\frac{\lambda(\langle\phi_{N+1}\rangle^2 - f^2)}{4} \;=\; \frac{\beta\lambda f^2}{2\langle\phi_{N+1}\rangle}\,. \tag{S.53}$$

Altogether, we have

$$V(\sigma, \underset{\sim}{\pi}) \;=\; \frac{\lambda}{8} \times \bigl(\underset{\sim}{\pi}^2 \,+\, \sigma^2\bigr)^2 \;+\; \frac{\kappa}{2} \times \bigl(\sigma^3 \,+\, \sigma\underset{\sim}{\pi}^2\bigr) \;+\; \frac{M_\sigma^2}{2} \times \sigma^2 \;+\; \frac{M_\pi^2}{2} \times \underset{\sim}{\pi}^2 \;+\; \text{const}, \tag{S.54}$$

where

$$\begin{aligned} \text{quartic coupling} \quad \lambda \;&=\; \lambda, \\ \text{cubic coupling} \quad \kappa \;&=\; \lambda \times \langle\phi_{N+1}\rangle \;\approx\; \lambda(f + \beta), \\ \text{pion mass}^2 \quad M_\pi^2 \;&=\; \frac{\beta\lambda f^2}{\langle\phi_{N+1}\rangle} \;\approx\; \beta\lambda f, \\ \text{sigma mass}^2 \quad M_\sigma^2 \;&=\; M_\pi^2 \;+\; \lambda\langle\phi_{N+1}\rangle^2 \;\approx\; \lambda f(f + 3\beta). \end{aligned} \tag{S.55}$$

Note that

$$\kappa^2 \;=\; \lambda^2\langle\phi_{N+1}\rangle^2 \;=\; \lambda \times \lambda\langle\phi_{N+1}\rangle^2 \;=\; \lambda \times (M_\sigma^2 - M_\pi^2), \tag{S.56}$$

precisely as in eq. (16).

Finally, let's take a closer look at the pion's mass^2, $M_\pi^2 \approx \beta \times \lambda f$. In the $\beta = 0$ limit, the pions are massless in accordance with the Goldstone theorem. Indeed, for $\beta = 0$ the sigma model's Lagrangian has an exact $SO(N+1)$ symmetry which is spontaneously broken down to an $SO(N)$ subgroup; there are N spontaneously broken generators, so there should be N massless Goldstone bosons. But for $\beta \neq 0$, the $SO(N+1)$ symmetry of the Lagrangian is only approximate, and its *explicit* breaking by the $\beta\lambda f^2 \times \phi_{N+1}$ term spoils the Goldstone theorem. Thus, instead of exactly massless Goldstone bosons we should get light but not quite massless pseudo-Goldstone bosons; to the first order in β, their mass^2 should be proportional to β. And indeed, in the linear sigma model $M_\pi^2 \approx \beta \times \lambda f$.

Still, for $\beta \ll f$, the pions should be much lighter than the sigma particle. And indeed, according to eqs. (S.55),

$$\frac{M_\pi^2}{M_\sigma^2} \approx \frac{\beta\lambda f}{\lambda f^2} = \frac{\beta}{f} \ll 1. \tag{S.57}$$

Problem **4**(d): Back in homework#9 (problem 4), we wrote the Lagrangian (HW9.3) for the fields $\sigma(x)$ and $\pi^i(x)$ without explaining where it came from. Now we see that it came from the shifted fields of the linear sigma model with $\beta = 0$ and hence exact $O(N+1)$ symmetry spontaneously broken down to $O(N)$. In the present set up for $\beta \neq 0$, the pions are not exactly massless but are merely much lighter than the sigma. Also, eq. (16) relates the cubic and the quartic couplings to the difference $M_\sigma^2 - M_\pi^2$ rather than just the M_σ^2 as we had in the homework#9.

Consequently, when we calculate the tree-level pion-pion scattering amplitudes in the present setup, we get exactly the same formula in terms of λ, κ, and M_σ as in eq. (S9.33) from the solutions to homework#9, namely

$$\begin{aligned} \mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) \;=\; & -\,\delta^{jk}\delta^{\ell m}\left(\lambda + \frac{\kappa^2}{s - M_\sigma^2}\right) \\ & -\,\delta^{j\ell}\delta^{km}\left(\lambda + \frac{\kappa^2}{t - M_\sigma^2}\right) \\ & -\,\delta^{jm}\delta^{k\ell}\left(\lambda + \frac{\kappa^2}{u - M_\sigma^2}\right). \end{aligned} \tag{S.58}$$

Furthermore, in light of eq. (16),

$$\lambda + \frac{\kappa^2}{s - M_\sigma^2} = \frac{\lambda s - \lambda M_\sigma^2 + \kappa^2}{s - M_\sigma^2} = \frac{\lambda(s - M_\pi^2)}{s - M_\sigma^2}\,, \tag{S.59}$$

and likewise

$$\lambda + \frac{\kappa^2}{t - M_\sigma^2} = \frac{\lambda(t - M_\pi^2)}{t - M_\sigma^2}\,, \qquad \lambda + \frac{\kappa^2}{u - M_\sigma^2} = \frac{\lambda(u - M_\pi^2)}{u - M_\sigma^2}\,, \tag{S.60}$$

so the amplitude (S.58) becomes

$$\begin{aligned}\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) = -\frac{\lambda(s - M_\pi^2)}{s - M_\sigma^2} \times \delta^{jk}\delta^{\ell m} &- \frac{\lambda(t - M_\pi^2)}{t - M_\sigma^2} \times \delta^{j\ell}\delta^{km} \\ &- \frac{\lambda(u - M_\pi^2)}{u - M_\sigma^2} \times \delta^{jm}\delta^{k\ell}.\end{aligned} \tag{S.61}$$

Moreover, when the pions' energies become low compared to M_σ — or in Lorentz-invariant terms, when $s, t, u \ll M_\sigma^2$ — we may simplify the amplitude (S.61) by approximating all the denominators as $-M_\sigma^2$, thus

$$\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) \approx \left(\frac{\lambda}{M_\sigma^2} \approx \frac{1}{f^2}\right) \times \left(\begin{aligned}(s - M_\pi^2) \times \delta^{jk}\delta^{\ell m} + (t - M_\pi^2) \times \delta^{j\ell}\delta^{km} \\ + (u - M_\pi^2) \times \delta^{jm}\delta^{k\ell}\end{aligned}\right), \tag{S.62}$$

exactly as in eq. (17).

Note: in the $\beta \to 0$ and hence $M_\pi^2 \to 0$ limit, the amplitude (17) becomes exactly as in homework#9. But for $\beta \neq 0$ we have extra M_π^2 terms in the numerators, and these M_π^2 terms significantly change the very low-energy limit $s, t, u \sim M_\pi^2$ of pion scattering.

Also, since the pions become massive for $\beta \neq 0$, we cannot take all 4 components of a pion's p^μ to zero. The best we can do is to take $\mathbf{p} \to 0$ while $p^0 \to m$, which is the *non-relativistic limit*. However, if only one pion is non-relativistic while the other 3 pions have $E \gg M_\pi$ (but $E \ll M_\sigma$), we generally have $s, t, u = O(E \times M_\pi) \gg M_\pi^2$ (albeit $s, t, u \ll M_\sigma^2$), and the scattering amplitude becomes

$$\mathcal{M} \;=\; O\left(\frac{E\times M_\pi}{f^2}\right) \;\not\to\; 0. \tag{S.63}$$

The strongest low-energy limit we an take for massive pions is to make all four pions non-relativistic. In this limit, $s=E_{\rm cm}^2\approx 4M_\pi^2$ while $u,t=O(\mathbf{p}^2)\ll M_\pi^2$, so the scattering amplitude (17) becomes

$$\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m) \;\approx\; \left(\frac{\lambda M_\pi^2}{M_\sigma^2}\approx\frac{\beta\lambda}{f}\right)\times\Bigl(3\delta^{jk}\delta^{\ell m}\,-\,\delta^{j\ell}\delta^{km}\,-\,\delta^{jm}\delta^{k\ell}\Bigr). \tag{18}$$

This amplitude is suppressed by the factor β/f, but it does not vanish! And even if all 4 pions belong to the same species, the scattering amplitude does not vanish in the non-relativistic limit,

$$\mathcal{M}(\pi^1+\pi^1\to\pi^1+\pi^1) \;\approx\; \frac{\lambda\beta}{f} \;\neq\; 0, \tag{S.64}$$

unlike what we had back in homework#9.