Problem **1**(a):

By inspection,

$$\begin{aligned} W &= \mathrm{Re}(\Phi^2)\times\mathbf{1}_{2\times2} \;-\; i\,\mathrm{Im}(\Phi^2)\times\sigma_3 \;+\; i\,\mathrm{Im}(\Phi^1)\times\sigma_1 \;+\; i\,\mathrm{Re}(\Phi^1)\times\sigma_2 \\ &= (\text{a real number})\times(\text{an } SU(2) \text{ matrix}), \end{aligned}\tag{S.1}$$

and it is easy to see that any linear combination of $1, i\sigma_1, i\sigma_2, i\sigma_3$ with *real* coefficients obeys $\sigma^2 W^* \sigma^2 = W$. Conversely, any 2×2 matrix is a linear combination of $1, i\sigma_1, i\sigma_2, i\sigma_3$, and if W happens to obey $\sigma^2 W^* \sigma^2 = W$ then the coefficients of such linear combination must be real. Identifying these coefficients with (respectively) $\mathrm{Re}(\Phi^2)$, $\mathrm{Im}(\Pi^1)$, $\mathrm{Re}(\Phi^2)$, and $-\mathrm{Im}(\Phi^2)$ according to eq. (S.1), we immediately put W to the form (3) for the appropriate Φ^1 and Φ^2.

Problem **1**(b):

Let's split the W matrix into two columns $W^{(1)}$ and $W^{(2)}$ and treat each column as a column-vector. In terms of the Φ doublet,

$$W^{(2)} \;=\; \Phi \quad\text{while}\quad W^{(1)} \;=\; i\sigma_2\Phi^*. \tag{S.2}$$

Consequently, when Φ is gauge transformed by some $SU(2)$ matrix U, $\Phi' = U\Phi$, we have

$$\begin{aligned} W'^{(2)} &= \Phi' \;=\; U\times\Phi \;=\; U\times W^{(2)}, \\ W'^{(1)} &= i\sigma_2\Phi'^* \;=\; i\sigma_2 U^*\times\Phi^* \;=\; i\sigma_2 U^*\times(-i)\sigma_2 W^{(1)} \\ &= U\times W^{(1)}, \end{aligned}\tag{S.3}$$

where the last equality follows from $\sigma_2 U^* \sigma_2 = U$ for any $SU(2)$ matrix U, *cf.* part(a). Note that both columns of the W matrix transform in a similar way — they get multiplied from the left by the same matrix U, — so me may combine these transformation laws into matrix multiplication

$$W' \;=\; U\times W. \tag{S.4}$$

Finally, note that the matrix $W(x)$ follows from $\Phi(x)$ at the same point x, so if we have different $U(x) \in SU(2)$ at different points x, this would not affect the above argument. Thus,

$$\textbf{if} \quad \Phi'(x) \;=\; U(x) \times \Phi(x) \quad \textbf{then} \quad W'(x) \;=\; U(x) \times W(x), \tag{4}$$

quod erat demonstrandum.

Problem **1**(c):

Let's start by comparing the Lagrangians (1) and (5). Both Lagrangians have the same Yang–Mills terms for the $SU(2)$ gauge fields. The scalar potential terms are also the same since $\mathrm{tr}(W^\dagger W) = 2\Phi^\dagger\Phi$; indeed

$$\begin{aligned} \mathrm{tr}(W^\dagger W) \;&=\; |\Phi_2^*| \;+\; |-\Phi_1^*|^2 \;+\; \left|\Phi^1\right|^2 \;+\; \left|\Phi^2\right|^2 \\ &=\; 2\left|\Phi^1\right|^2 \;+\; 2\left|\Phi^2\right|^2 \;=\; 2\Phi^\dagger\Phi. \end{aligned} \tag{S.5}$$

hence

$$\frac{\lambda}{8}\bigl(\mathrm{tr}(W^\dagger W) \,-\, v^2\bigr)^2 \;=\; \frac{\lambda}{8}\bigl(2\Phi^\dagger\Phi \,-\, v^2\bigr)^2 \;=\; \frac{\lambda}{2}\left(\Phi^\dagger\Phi \,-\, \frac{v^2}{2}\right)^2. \tag{S.6}$$

As to the gauge-covariant kinetic term for the scalars, note that eq. (4) for the gauge transformation of the $W(x)$ matrix implies

$$D_\mu W(x) \;=\; \partial_\mu W(x) \;+\; \frac{ig}{2} A_\mu^a(x)\sigma^a \times W(x). \tag{S.7}$$

In terms of the two columns of the W matrix, this means

$$\begin{aligned} D_\mu W^{(2)} \;&=\; \partial_\mu W^{(2)} \;+\; \frac{ig}{2}A_\mu^a\sigma^a \times W^{(2)} \;=\; \partial_\mu\Phi \;+\; \frac{ig}{2}A_\mu^a\sigma^a \times \Phi \\ &=\; D_\mu\Phi, \end{aligned} \tag{S.8}$$

$$\begin{aligned} D_\mu W^{(1)} \;&=\; \partial_\mu W^{(1)} \;+\; \frac{ig}{2}A_\mu^a\sigma^a \times W^{(1)} \;=\; \partial_\mu(i\sigma_2\Phi^*) \;+\; \frac{ig}{2}A_\mu^a\sigma^a \times (i\sigma^2\Phi^*) \\ &=\; i\sigma_2\left(\partial_\mu\Phi \;-\; \frac{ig}{2}A_\mu^a(\sigma_2\sigma^a\sigma_2)^* \times \Phi\right)^* \\ &=\; i\sigma_2\left(\partial_\mu\Phi \;+\; \frac{ig}{2}A_\mu^a\sigma^a \times \Phi\right)^* \quad \langle\!\langle \text{ because } (\sigma_2\sigma^a\sigma_2)^* = -\sigma^a \,\rangle\!\rangle \\ &=\; i\sigma_2\bigl(D_\mu\Phi\bigr)^*. \end{aligned} \tag{S.9}$$

Consequently,

$$\begin{aligned}
\operatorname{tr}\big(D_\mu W^\dagger D^\mu W\big) &= \operatorname{tr}\big((D_\mu W)^\dagger (D^\mu W)\big) \\
&= \big(D_\mu W^{(1)}\big)^\dagger \big(D^\mu W^{(1)}\big) \;+\; \big(D_\mu W^{(2)}\big)^\dagger \big(D^\mu W^{(2)}\big) \\
&= \big(i\sigma_2 (D_\mu \Phi)^*\big)^\dagger \big(i\sigma_2 (D^\mu \Phi)^*\big) \;+\; \big(D_\mu \Phi\big)^\dagger \big(D^\mu \Phi\big) \\
&= (D_\mu \Phi)^\top \times (-i\sigma_2 \times i\sigma_2 \,=\, 1) \times (D^\mu \Phi)^* + \; (D_\mu \Phi)^\dagger (D^\mu \Phi) \\
&= (D_\mu \Phi)^\top (D^\mu \Phi)^* \;+\; (D_\mu \Phi)^\dagger (D^\mu \Phi) = \; 2 \times (D_\mu \Phi)^\dagger (D^\mu \Phi) \\
&= 2(D_\mu \Phi^\dagger)(D^\mu \Phi),
\end{aligned} \tag{S.10}$$

and that's how the Lagrangians (1) and (5) have similar gauge-covariant kinetic terms for the scalars.

Altogether, the Lagrangian (5) is indeed the same as the Lagrangian (1) rephrased in terms of the $W(x)$ instead of the $\Phi(x)$.

Now consider the symmetry (6) of the $W(x)$. The second line of eq. (6) is the usual non-abelian gauge symmetry of the vector field $A_\mu(x) = A^a_\mu(x) \times \frac{1}{2}\sigma^a$, which assures that the $D_\mu W$ transforms covariantly,

$$D'_\mu W(x) \;=\; U_L(x) \times D_\mu W(x) \times U_G^\dagger. \tag{S.11}$$

Consequently,

$$D'_\mu W^\dagger(x) \;=\; \big(D'_\mu W(x)\big)^\dagger \;=\; U_G \times D_\mu W^\dagger(x) \times U_L^\dagger(x), \tag{S.12}$$

and therefore

$$\begin{aligned}
\big(\operatorname{tr}\big(D_\mu W^\dagger(x) D^\mu W(x)\big)\big)' &= \operatorname{tr}\big(U_G D_\mu W^\dagger(x) U_L^\dagger(x) \times U_L(x) D^\mu W(x) U_G^\dagger\big) \\
&= \operatorname{tr}\big(U_G D_\mu W^\dagger(x) D^\mu W(x) U_G^\dagger\big) \\
&\qquad \langle\!\langle \text{ by the cyclic symmetry of the trace } \rangle\!\rangle \\
&= \operatorname{tr}\big(D_\mu W^\dagger(x) D^\mu W(x) \times U_G^\dagger U_G\big) \\
&= \operatorname{tr}\big(D_\mu W^\dagger(x) D^\mu W(x)\big).
\end{aligned} \tag{S.13}$$

This makes the covariant kinetic term for the $W(x)$ invariant under both $SU(2)_{\text{local}}$ and $SU(2)_{\text{global}}$ symmetries.

The scalar potential is invariant in the similar way:

$$\begin{aligned}\big(\mathrm{tr}(W^\dagger W)\big)' \;&=\; \mathrm{tr}\big(U_G W^\dagger U_L^\dagger \times U_L W U_G^\dagger\big)\\ &=\; \mathrm{tr}\big(U_G W^\dagger W U_G^\dagger\big) \;=\; \mathrm{tr}\big(W^\dagger W \times U_G^\dagger U_G\big)\\ &=\; \mathrm{tr}\big(W^\dagger W\big),\end{aligned} \tag{S.14}$$

and therefore

$$V' \;=\; \frac{\lambda}{8}\big(\mathrm{tr}(W'^\dagger W') \,-\, v^2\big)^2 \;=\; \frac{\lambda}{8}\big(\mathrm{tr}(W^\dagger W) \,-\, v^2\big)^2 \;=\; V. \tag{S.15}$$

Finally, the Yang–Mills term $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ is gauge-invariant under the $SU(2)_{\mathrm{local}}$ symmetries and is completely unaffected by the $SU(2)_{\mathrm{global}}$ symmetries, so it also invariant.

Problem **1**(d–e):

Since $\langle W\rangle$ is proportional to a unit matrix, it is obviously invariant under transforms

$$\langle W\rangle \;\to\; U \times \langle W\rangle \times U^\dagger \tag{S.16}$$

for any unitary matrix U. In terms of the symmetries (6), this means

$$\text{any } U_G \in SU(2), \quad \text{but} \quad \forall x:\; U_L(x) \;=\; U_G\,. \tag{S.17}$$

Thus, we still have an unbroken $SU(2)'_{\mathrm{global}}$ symmetry, but now each $U_G \in SU(2)$ has to be accompanied by a matching gauge transform with x-independent $U_L = U_G$. Consequently, the second line of eq.(6) becomes

$$A'^a_\mu(x) \times \tfrac{1}{2}\sigma^a \;=\; A^b_\mu(x) \times U_G \tfrac{1}{2}\sigma^b U_G^\dagger, \tag{S.18}$$

which makes the vector fields $A^{a=1,2,3}_\mu(x)$ a triplet of the unbroken global symmetry $SU(2)'_{\mathrm{global}}$. And since this global symmetry remains unbroken by the Higgs mechanism, all members of the same multiplet must acquire exactly the same mass. In particular, all three vector fields acquire exactly the same mass

$$M_v \;=\; \tfrac{1}{2}gv \;+\; \text{perturbations.} \tag{S.19}$$

Note: our analysis in class was semiclassical, so our value of the vector fields' mass is subject to perturbative (and even non-perturbative) corrections. However, thanks to unbroken $SU(2)'_{\rm global}$ symmetry, all such corrections to the vector mass would be exactly the same for all 3 vector fields.

Finally, under infinitesimal $SU(2)_{\rm local}\times SU(2)_{\rm global}$ symmetries, various fields transform according to

$$\delta\varphi(x) \;=\; i\epsilon^a_L(x)\times\hat T^a_{\rm local}\varphi(x) \;+\; i\epsilon^a_G\times\hat T^a_{\rm global}\varphi(x). \tag{S.20}$$

where $\hat T^a_{\rm local}$ and $\hat T^a_{\rm global}$ are the generators of the two $SU(2)$ symmetry groups while $\epsilon^a_L(x)$ and ϵ_G are the infinitesimal parameters of the transform in question. For the surviving $SU(2)'_{\rm global}$ symmetry we have $U_L(x)\equiv U_G$, hence for infinitesimal symmetries $\epsilon^a_L(x)\equiv\epsilon^a_G$, and therefore

$$\delta\varphi(x) \;=\; i\epsilon^a_G\times\Big(\hat T^a_{\rm local}\varphi(x)\;+\;\hat T^a_{\rm global}\varphi(x)\Big) \;=\; i\epsilon^a_G\times\hat T^a_{\rm global'}\varphi(x) \tag{S.21}$$

where

$$\hat T^a_{\rm global'} \;=\; \hat T^a_{\rm global} \;+\; \hat T^a_{\rm local}\,. \tag{9}$$

In other words, the combined charges (9) act as generators of the surviving $SU(2)'_{\rm global}$ symmetry group.

Problem **2**(a):

$$\text{given}\quad \Phi \;\to\; e^{+i\theta}U_L\Phi U_R^\dagger\,, \tag{11}$$

$$\text{we have}\quad \Phi^\dagger \;\to\; e^{-i\theta}U_R\Phi^\dagger U_L^\dagger\,, \tag{S.22}$$

$$\text{hence}\quad \Phi^\dagger\Phi \;\to\; U_R(\Phi^\dagger\Phi)U_R^\dagger\,, \tag{S.23}$$

$$\left(\Phi^\dagger\Phi\right)^2 \;\to\; U_R\Phi^\dagger\Phi U_R^\dagger\, U_R\Phi^\dagger\Phi U_R^\dagger \;=\; U_R\left(\Phi^\dagger\Phi\right)^2 U_R^\dagger\,, \tag{S.24}$$

$$\text{likewise}\quad \left(\Phi^\dagger\Phi\right)^n \;\to\; U_R\left(\Phi^\dagger\Phi\right)^n U_R^\dagger \quad \forall n=1,2,3,\ldots, \tag{S.25}$$

and therefore

$$\text{all traces}\quad \mathrm{tr}\left(\left(\Phi^\dagger\Phi\right)^n\right)\quad \text{are invariant under symmetries (12),} \tag{S.26}$$

thanks to the cyclic invariance rule for traces, $\mathrm{tr}\big(U_R X U_R^\dagger\big) = \mathrm{tr}\big(X U_R^\dagger U_R\big) = \mathrm{tr}\big(X\big)$ for any $X = \left(\Phi^\dagger\Phi\right)^n$. Consequently, the scalar potential (11) is invariant under the symmetries (12).

For the global symmetries where $e^{i\theta}$, U_L, and U_R do not depend on x, the kinetic term in the Lagrangian (10) is also invariant. Indeed,

$$\begin{aligned}
&\text{for constant } e^{i\theta},\ U_L,\ U_R,\\
&\qquad \partial_\mu\Phi \;\to\; e^{+i\theta}U_L(\partial_\mu\Phi)U_R^\dagger\,,\\
&\qquad \partial_\mu\Phi^\dagger \;\to\; e^{-i\theta}U_R(\partial_\mu\Phi^\dagger)U_L^\dagger\,,\\
&\qquad \partial^\mu\Phi^\dagger\,\partial_\mu\Phi \;\to\; U_R(\partial^\mu\Phi^\dagger\,\partial_\mu\Phi)U_R^\dagger\,,\\
&\text{and}\quad \mathrm{tr}\big(\partial^\mu\Phi^\dagger\,\partial_\mu\Phi\big)\quad \text{is invariant.}
\end{aligned} \tag{S.27}$$

Altogether, the whole Lagrangian (10) is invariant, $\mathcal{Q.E.D.}$

Problem **2**(b):

Given the eigenvalues $(\kappa_1,\ldots,\kappa_N)$ of the $\Phi^\dagger\Phi$ matrix, the invariant traces (S.26) obtain as

$$\mathrm{tr}\left(\left(\Phi^\dagger\Phi\right)^n\right) \;=\; \sum_{i=1}^N \kappa_i^n\,. \tag{S.28}$$

Consequently, the scalar potential is

$$V \;=\; \frac{\alpha}{2}\sum_i \kappa_i^2 \;+\; \frac{\beta}{2}\left(\sum_i \kappa_i\right)^2 \;+\; m^2\sum_i \kappa_i\,. \tag{S.29}$$

Now let's minimize this potential. Since the matrix $\Phi^\dagger\Phi$ cannot have any negative eigenvalues, we are looking for a minimum of $V(\kappa_1,\ldots,\kappa_N)$ *under constraints* $\kappa_i \geq 0$. This requires

$$\forall i = 1,\ldots,N, \quad \textbf{either } \kappa_i \geq 0 \textbf{ and } \frac{\partial V}{\partial \kappa_i} \;=\; 0, \quad \textbf{or else } \kappa_i = 0 \textbf{ and } \frac{\partial V}{\partial \kappa_i} \;>\; 0, \tag{S.30}$$

where

$$\frac{\partial V}{\partial \kappa_i} \;=\; \alpha\kappa_i \;+\; m^2 \;+\; \beta\sum_j \kappa_j\,. \tag{S.31}$$

These derivatives are linear functions of the eigenvalues κ_i, so all the non-zero eigenvalues must obey the same linear equation

$$\alpha\times\kappa_i \;=\; -m^2 \;-\; \beta\times\sum_j \kappa_j\,, \quad \text{same for all } \kappa_i \neq 0,$$

which means that all non-zero κ_i have the same value. Thus, up to a permutation of eigenvalues,

$$\kappa_1 \;=\; \cdots \;=\; \kappa_k \;=\; C^2, \quad \kappa_{k+1} \;=\; \cdots \;=\; \kappa_N \;=\; 0, \tag{S.32}$$

for some $k = 0,1,2,\ldots,N$, and C^2 obtains from

$$\alpha\times C^2 \;+\; m^2 \;+\; \beta\times kC^2 \;=\; 0 \quad\longrightarrow\quad C^2 \;=\; \frac{-m^2}{\alpha+k\beta}\,. \tag{S.33}$$

To make sure that the solution (S.32) is a minimum rather that a maximum or a saddle point, we need

$$\begin{aligned} C^2 \;&=\; \frac{-m^2}{\alpha+k\beta} \;>\; 0 \quad \text{unless } k=0,\\ m^2 \;+\; \beta k C^2 \;&=\; \frac{\alpha m^2}{\alpha+k\beta} \;>\; 0 \quad \text{unless } k=N. \end{aligned} \tag{S.34}$$

Depending on the signs of α, β and m^2 parameters, this limits the solutions to the following:

- For $\alpha > 0$, $\beta > 0$, and $m^2 > 0$, the only solution is $k = 0$, which means $\kappa_1 = \cdots \kappa_N = 0$ and hence $\langle\Phi\rangle = 0$.

- For $\alpha > 0$, $\beta > 0$, and $m^2 < 0$, the only solutions is $k = N$, which means

$$\kappa_1 \;=\; \cdots \;=\; \kappa_N \;=\; C^2 \;=\; \frac{-m^2}{\alpha + N\beta} \;>\; 0, \tag{12}$$

and hence $\langle\Phi\rangle = C \times$ a unitary matrix as in eq. (13). We shall focus on this regime through the rest of this problem.

- For $\alpha < 0$ or $\beta < 0$, the situation is more complicated:

  ∗ When $\alpha + \beta < 0$ or $\alpha + N\beta < 0$, the scalar potential (10) is unbounded from below and the theory is sick.

  ∗ When $\alpha > 0$ and $\beta < 0$ but $\alpha + N\beta > 0$, the solutions are similar to the $\beta > 0$ case: For $m^2 > 0$ all $\kappa_i = 0$, while for $m^2 < 0$ the κ_i are as in eq. (12).

  ∗ When $\beta > 0$ and $\alpha < 0$ but $\alpha + \beta > 0$: for $m^2 > 0$ the only solution is $k = 0$, meaning $\langle\Phi\rangle = 0$, but for $m^2 < 0$ all the solutions (S.32) with $k = 1, 2, \ldots, N$ are good local minima.

  To find the global minimum, we compare the potentials at the local minima,

$$\begin{aligned} V(\text{minimum}\#k) \;&=\; \frac{\alpha}{2} \times kC^4 \;+\; \frac{\beta}{2} \times (kC^2)^2 \;+\; m^2 \times kC^2 \\ &=\; \frac{k\alpha + k^2\beta}{2} \times \frac{m^4}{(\alpha + k\beta)^2} \;+\; km^2 \times \frac{-m^2}{(\alpha + k\beta)} \\ &=\; -\frac{m^4}{2} \times \frac{k}{k\beta + \alpha}\,. \end{aligned} \tag{S.35}$$

  Since $\alpha < 0$ but $\alpha + \beta > 0$, the deepest minimum obtains for $k = 1$, thus

$$\kappa_1 \;=\; \frac{-m^2}{\alpha + \beta}\,, \quad \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; 0. \tag{S.36}$$

Problem $\mathbf{2}$(c):

Let's act with some $SU(N)_L \times SU(N)_R \times U(1)$ symmetry (11) on the vacuum expectation values (14):

$$\langle\Phi\rangle \;=\; C \times \mathbf{1}_{N\times N} \;\to\; e^{i\theta} U_L \,\langle\Phi\rangle\, U_R^\dagger \;=\; C \times e^{i\theta} U_L U_R^\dagger\,. \tag{S.37}$$

Clearly, to keep the VEVs $\langle\Phi\rangle$ invariant, we need

$$e^{i\theta} U_L U_R^\dagger \;=\; \mathbf{1}_{N\times N} \tag{S.38}$$

and hence

$$U_R \;=\; e^{i\theta} \times U_L\,. \tag{S.39}$$

Moreover, since the U_L and U_R matrices have unit determinants, this requires

$$\det\left(e^{i\theta} \times \mathbf{1}_{N\times N}\right) \;=\; 1 \quad\Longrightarrow\quad N \times \theta \;=\; 0 \pmod{2\pi}. \tag{S.40}$$

Such a phase can be absorbed into the $U_L \in SU(N)$, so without loss of generality we need

$$e^{i\theta} \;=\; 1 \quad\text{and}\quad U_L \;=\; U_R \;\in\; SU(N). \tag{S.41}$$

In other words, the *unbroken* symmetry group is $SU(N)_V$ which acts on the scalar fields as

$$\Phi(x) \;\to\; U\Phi(x)\,U^\dagger\,, \quad U \in SU(N). \tag{S.42}$$

Problem $\mathbf{2}$(d):

In terms of the shifted fields $\delta\Phi = \Phi - C \times \mathbf{1}_{N\times N}$ and $\delta\Phi^\dagger = \Phi^\dagger - C \times \mathbf{1}_{N\times N}$,

$$\mathrm{tr}(\Phi^\dagger\Phi) \;=\; C^2 \times N \;+\; C\,\mathrm{tr}(\delta\Phi^\dagger + \delta\Phi) \;+\; \mathrm{tr}(\delta\Phi^\dagger\delta\Phi), \tag{S.43}$$

$$\begin{aligned}\mathrm{tr}^2(\Phi^\dagger\Phi) \;=\;& N^2C^4 \;+\; 2NC^3\,\mathrm{tr}(\delta\Phi^\dagger + \delta\Phi)\\ &+\; 2NC^2\,\mathrm{tr}(\delta\Phi^\dagger\delta\Phi) \;+\; C^2\,\mathrm{tr}^2(\delta\Phi^\dagger + \delta\Phi)\\ &+\; 2C\,\mathrm{tr}(\delta\Phi^\dagger + \delta\Phi) \times \mathrm{tr}(\delta\Phi^\dagger\delta\Phi) \;+\; \mathrm{tr}^2(\delta\Phi^\dagger\delta\Phi),\end{aligned} \tag{S.44}$$

$$
\begin{aligned}
\mathrm{tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi) \;=\; & C^4\times N \;+\; 2C^3\,\mathrm{tr}(\delta\Phi^\dagger+\delta\Phi) \\
& +\; C^2\,\mathrm{tr}\big((\delta\Phi)^2+(\delta\Phi^\dagger)^2+4(\delta\Phi^\dagger)(\delta\Phi)\big) \\
& +\; 2C\,\mathrm{tr}\big((\delta\Phi^\dagger)^2(\delta\Phi)+(\delta\Phi^\dagger)(\delta\Phi)^2\big) \\
& +\; \mathrm{tr}\big((\delta\Phi^\dagger)(\delta\Phi)(\delta\Phi^\dagger)(\delta\Phi)\big).
\end{aligned}
\tag{S.45}
$$

Plugging these formulae into the potential (10) and arranging by the powers of $\delta\Phi$ and $\delta\Phi^\dagger$ according to eq. (14), we get

$$
V_1 \;=\; \big(\alpha C^3+\beta NC^3+m^2C\big)\times\mathrm{tr}(\delta\Phi^\dagger+\delta\Phi),
\tag{S.46}
$$

$$
\begin{aligned}
V_2 \;=\; & \big(2\alpha C^2+\beta NC^2+m^2\big)\times\mathrm{tr}(\delta\Phi^\dagger\delta\Phi) \\
& +\; \frac{\alpha C^2}{2}\times\mathrm{tr}\big((\delta\Phi)^2+(\delta\Phi^\dagger)^2\big)+\;\frac{\beta C^2}{2}\times\mathrm{tr}^2(\delta\Phi^\dagger+\delta\Phi),
\end{aligned}
\tag{S.47}
$$

$$
\begin{aligned}
V_3 \;=\; & \alpha C\times\mathrm{tr}\big((\delta\Phi^\dagger)^2(\delta\Phi)+(\delta\Phi^\dagger)(\delta\Phi)^2\big) \\
& +\; \beta C\times\mathrm{tr}(\delta\Phi^\dagger+\delta\Phi)\times\mathrm{tr}(\delta\Phi^\dagger\delta\Phi),
\end{aligned}
\tag{S.48}
$$

$$
\begin{aligned}
V_4 \;=\; & \frac{\alpha}{2}\times\mathrm{tr}\big((\delta\Phi^\dagger)(\delta\Phi)(\delta\Phi^\dagger)(\delta\Phi)\big) \\
& +\; \frac{\beta}{2}\times\mathrm{tr}^2(\delta\Phi^\dagger\delta\Phi).
\end{aligned}
\tag{S.49}
$$

Furthermore, the specific value of C^2 in eq. (13) leads to

$$
\alpha C^2 \;+\; \beta NC^2 \;+\; m^2 \;=\; 0
\tag{S.50}
$$

and hence $V_1=0$. Also, eq. (S.47) for the V_2 simplifies to

$$
\begin{aligned}
V_2 \;&=\; \alpha C^2\times\mathrm{tr}(\delta\Phi^\dagger\delta\Phi)\;+\;\frac{\alpha C^2}{2}\times\mathrm{tr}\big((\delta\Phi)^2+(\delta\Phi^\dagger)^2\big)\;+\;\frac{\beta C^2}{2}\times\mathrm{tr}^2(\delta\Phi^\dagger+\delta\Phi) \\
&=\; \frac{\alpha C^2}{2}\times\mathrm{tr}\big((\delta\Phi^\dagger+\delta\Phi)^2\big)\;+\;\frac{\beta C^2}{2}\times\mathrm{tr}^2(\delta\Phi^\dagger+\delta\Phi),
\end{aligned}
\tag{S.51}
$$

exactly as in eq. (16).

Problem **2**(e):

First, let's check that the fields χ_1, χ_2, φ_1, and φ_2 have properly normalized kinetic terms. Since the φ_1 and φ_2 matrices are traceless, we immediately have

$$\begin{aligned}
\mathcal{L}_{\rm kin} \;&=\; \mathrm{tr}\bigl(\partial_\mu\Phi^\dagger \times \partial^\mu\Phi\bigr) \;=\; \mathrm{tr}\bigl(\partial_\mu\delta\Phi^\dagger \times \partial^\mu\delta\Phi\bigr) \\
&=\; \frac{1}{2N}\partial_\mu(\chi_1 - i\chi_2)\partial^\mu(\chi_1 + i\chi^2) \times \mathrm{tr}(\mathbf{1}_{N\times N}) \;+\; \frac{1}{2}\,\mathrm{tr}\bigl(\partial_\mu(\varphi_1 - i\varphi_2)\partial^\mu(\varphi_1 + i\varphi^2)\bigr) \\
&=\; \frac{1}{2}\bigl((\partial_\mu\chi_1)^2 \;+\; (\partial_\mu\chi_2)^2\bigr) \;+\; \frac{1}{2}\,\mathrm{tr}\bigl((\partial_\mu\varphi_1)^2 \;+\; (\partial_\mu\varphi_2)^2\bigr),
\end{aligned} \tag{S.52}$$

so all the kinetic terms are indeed properly normalized.

Now consider the mass terms V_2 in the scalar potential. Both terms in eq. (16) involve

$$\delta\Phi^\dagger \;+\; \delta\Phi \;=\; \sqrt{\frac{2}{N}}\,\chi_1 \times \mathbf{1}_{N\times N} \;+\; \sqrt{2}\varphi_1\,, \tag{S.53}$$

hence

$$\begin{aligned}
V_2 \;&=\; \alpha C^2 \times \mathrm{tr}\left(\left(\varphi_1 \;+\; \frac{\chi_1}{\sqrt{N}} \times \mathbf{1}_{N\times N}\right)^2\right) \;+\; \beta C^2 \times \mathrm{tr}^2\left(\varphi_1 \;+\; \frac{\chi_1}{\sqrt{N}} \times \mathbf{1}_{N\times N}\right) \\
&=\; \alpha C^2 \times \left(\mathrm{tr}\bigl(\varphi_1^2\bigr) \;+\; \chi_1^2\right) +\; \beta C^2 \times \left(\frac{\chi_1}{\sqrt{N}} \times N\right)^2 \\
&=\; \alpha C^2 \times \mathrm{tr}\bigl(\varphi_1^2\bigr) \;+\; (\alpha C^2 + \beta N C^2) \times \chi_1^2.
\end{aligned} \tag{S.54}$$

Therefore:

- The χ_1 field has mass $2(\alpha + N\beta)C^2$.
- All components of the traceless matrix φ_1 have mass $2\alpha C^2$.
- The χ_2 field and all components of the φ_2 matrix are massless.

Problem **2**(f):

The unbroken $SU(N)_V$ symmetry acts on the scalar fields according to

$$\Phi(x) \;\to\; U \times \Phi(x) \times U^\dagger. \tag{S.42}$$

and since the VEV (14) is invariant, the shifted fields $\delta\Phi(x) \,=\, \Phi(x) - \langle\Phi\rangle$ also transform according to

$$\delta\Phi(x) \;\to\; U\times\delta\Phi(x)\times U^\dagger. \tag{S.55}$$

Moreover, the unitarity transforms like this preserve the decomposition (17) of $\delta\Phi$ into its hermitian and antihermitian parts and also into the traces $\chi_{1,2}$ and the traceless parts $\varphi_{1,2}$. In particular, the trace parts $\chi_1(x)$ and $\chi_2(x)$ remain invariant — which makes them singlets of the $SU(N)_V$ symmetry, — while the traceless matrices $\varphi_1(x)$ and $\varphi_2(x)$ transform as

$$\varphi_1(x) \;\to\; U\varphi_1(x)\,U^\dagger, \qquad \varphi_2(x) \;\to\; U\varphi_2(x)\,U^\dagger, \tag{S.56}$$

which makes them *adjoint multiplets* of the $SU(N)_V$.

Clearly, this multiplet structure agrees with the masses we have obtained in part (e): All N^2-1 members of the adjoint multiplet φ_1 have the same mass $M=2\alpha C^2$, while all N^2-1 members of the adjoint multiplet φ_2 have $M=0$. On the other hand, the singlet χ_1 has a different mass from the adjoint fields φ_2, but that's OK since they belong to different multiplets.

However, the second singlet χ_2 has the same zero mass as the second adjoint multiplet φ_2, but that's because they are both Goldstone bosons of the spontaneously broken continuous symmetries

$$G/H \;=\; \Big(SU(N)_L\times SU(N)_R\times U(1)\Big)\Big/\,SU(N)\,. \tag{S.57}$$

Specifically, the singlet χ_2 is the Goldstone boson of the broken $U(1)$ symmetry. Indeed, the $U(1)$'s generator commutes with all the other generators, so it belongs in its own singlet of the symmetry, and the corresponding Goldstone particle should also be a singlet.

Now consider the non-abelian generators. Generators T^a_L of the $SU(N)_L$ form an adjoint multiplet of the $SU(N)_L$, but are invariant under the $SU(N)_R$. Likewise, generators T^a_R of the $SU(N)_R$ form an adjoint multiplet of the $SU(N)_R$, but are invariant under the $SU(N)_L$. In other words, under an $(U_L,U_R)\in SU(N)_L\times SU(N)_R$ they transform as

$$T^a_L \;\to\; U_L T^a_L U^\dagger_L\,, \qquad T^a_R \;\to\; U_R T^a_R U^\dagger_R\,. \tag{S.58}$$

When the $SU(N)_L\times SU(N)_R$ is broken down to a single $SU(N)$ spanning $U_L=U_R=U$, both T_L^a and T_R^a transform as

$$T_L^a \;\to\; UT_L^a U^\dagger, \qquad T_R^a \;\to\; UT_R^a U^\dagger, \tag{S.59}$$

which puts them into *two adjoint multiplets* of the unbroken $SU(N)$. Equivalently, we may form two adjoint multiplets out of

$$T_V^a \;=\; T_L^a + T_R^a \qquad \text{and} \qquad T_A^a \;=\; T_L^a - T_R^a, \tag{S.60}$$

which act on the scalar fields according to

$$T_V^a\Phi \;=\; \tfrac{i}{2}[\lambda^a,\Phi], \qquad T_A^a\Phi \;=\; \tfrac{i}{2}\{\lambda^a,\Phi\}. \tag{S.61}$$

The T_V^a generate the unbroken $SU(N)_V$ symmetry, *cf.* eq. (S.42). The T_A^a generators are spontaneously broken, hence there should be an adjoint multiplet of massless Goldstone bosons. And indeed there is — the φ_2.

Problem **3**(a):

When a gauge symmetry is spontaneously broken, the gauge fields acquire masses — which come from the gauge-covariant kinetic terms for the scalar fields with non-zero VEVs (vacuum expectation values). The simplest way to separate the vectors' mass terms from the shifted scalars' kinetic energies and from the scalar-vector interactions is to freeze the scalar fields to their VEVs. Indeed, let's freeze $\Phi(x) \equiv \langle\Phi\rangle = C \times \mathbf{1}_{N\times N}$. Then according to eqs. (19),

$$\begin{aligned} D_\mu\langle\Phi\rangle \;&=\; ig'B_\mu\langle\Phi\rangle \;+\; igL_\mu\langle\Phi\rangle \;-\; ig\langle\Phi\rangle R_\mu \\ &=\; ig'C\,B_\mu \times \mathbf{1}_{N\times N} \;+\; igC \times (L_\mu \,-\, R_\mu) \\ &=\; ig'C\,B_\mu \times \mathbf{1}_{N\times N} \;+\; \frac{igC}{2}\sum_a (L_\mu^a - R_\mu^a) \times \lambda^a, \end{aligned} \tag{S.62}$$

and consequently

$$\mathrm{tr}\Big(\big(D_\mu\langle\Phi\rangle^\dagger\big)\big(D^\mu\langle\Phi\rangle\big)\Big) \;=\; Ng'^2C^2 \times B_\mu B^\mu \;+\; \frac{g^2C^2}{2}\sum_a (L_\mu^a - R_\mu^a)(L^{a\mu} - R^{a\mu}). \tag{S.63}$$

Thus, the abelian B_μ field has mass $M_B^2 = 2Ng'^2C^2$ while the non-abelian fields L_μ^a and R_μ^a

have non-diagonal mass terms. To diagonalize those terms, let's mix the fields according to

$$V^a_\mu \;=\; \frac{1}{\sqrt{2}}\left(L^a_\mu \,+\, R^a_\mu\right), \qquad X^a_\mu \;=\; \frac{1}{\sqrt{2}}\left(L^a_\mu \,-\, R^a_\mu\right), \tag{S.64}$$

where the $1/\sqrt{2}$ coefficients make the V^a_μ and X^a_μ canonically normalized, *i.e.*

$$\mathcal{L}^{\rm kin}_{L,R} \;=\; -\frac{1}{4}\sum_a\left(\left(\partial_{[\mu}L^a_{\nu]}\right)^2 \,+\, \left(\partial_{[\mu}R^a_{\nu]}\right)^2\right) \;=\; -\frac{1}{4}\sum_a\left(\left(\partial_{[\mu}X^a_{\nu]}\right)^2 \,+\, \left(\partial_{[\mu}V^a_{\nu]}\right)^2\right). \tag{S.65}$$

In terms of the V^a_μ and X^a_μ, the mass terms for L^a_μ and R^a_μ in eq. (S.63) become

$$\mathcal{L}^{\rm masses}_{L,R} \;=\; g^2C^2\times X^a_\mu X^{a\mu}. \tag{S.66}$$

Thus, the V^a_μ fields remain massless while the X^a_μ acquire common mass $M^2_X = 2g^2C^2$.

Problem **3**(b):

To write down an effective theory for the massless fields, we simply freeze all the massive vector fields B_μ and X^a_μ as well as all the scalar fields comprising the $\delta\Phi = \Phi - \langle\Phi\rangle$; only the massless vector fields V^a_μ remain un-frozen. In other words, we let

$$\Phi(x)\;\equiv\;\langle\Phi\rangle\;=\;C\times\mathbf{1}_{N\times N}\,, \qquad B_\mu(x)\;\equiv\;0, \qquad L^a_\mu(x)\;=\;R^a_\mu(x)\;=\;\frac{1}{\sqrt{2}}\,V^a_\mu(x), \tag{S.67}$$

and then substitute these values into the Lagrangian (15). According to eq. (S.62), for fields as in eq. (S.67) $D_\mu\Phi = 0$, so the only un-frozen terms in the Lagrangian are

$$\begin{aligned}
\mathcal{L}^{\rm unfrozen} \;&=\; -\tfrac{1}{2}\,{\rm tr}(L_{\mu\nu}L^{\mu\nu}) \;-\; \tfrac{1}{2}\,{\rm tr}(R_{\mu\nu}R^{\mu\nu}) \qquad \langle\!\langle\,\text{for } L^a_{\mu\nu} = R^a_{\mu\nu}\,\rangle\!\rangle\\
&=\; -\,{\rm tr}(L_{\mu\nu}L^{\mu\nu}) \;=\; -\frac{1}{2}\sum_a\left(L^a_{\mu\nu}\right)^2\\
&=\; -\frac{1}{4}\sum_a\left(V^a_{\mu\nu}\right)^2
\end{aligned} \tag{S.68}$$

— which is precisely the Yang-Mills Lagrangian for the canonically normalized tension fields

$$V^a_{\mu\nu} \;=\; \frac{L^a_{\mu\nu}+R^a_{\mu\nu}}{\sqrt{2}} \;\to\; \sqrt{2}L^a_{\mu\nu} \quad \text{when } L^a_{\mu\nu} = R^a_{\mu\nu}\,. \tag{S.69}$$

of the un-broken $SU(N)_V$ gauge theory. Indeed, in terms of the canonically normalized

$SU(N)_V$ potential fields V^a_μ,

$$\begin{aligned} V^a_{\mu\nu} &= \sqrt{2}\Big(\partial_\mu L^a_\nu - \partial_\nu L^a_\mu - gf^{abc}L^b_\mu L^c_\nu\Big) \\ &= \sqrt{2}\left(\partial_\mu \frac{V^a_\nu}{\sqrt{2}} - \partial_\nu \frac{V^a_\mu}{\sqrt{2}} - gf^{abc}\,\frac{V^b_\mu}{\sqrt{2}}\,\frac{V^c_\nu}{\sqrt{2}}\right) \\ &= \partial_\mu V^a_\nu - \partial_\nu V^a_\mu - \frac{g}{\sqrt{2}}\,f^{abc}V^b_\mu V^c_\nu\,. \end{aligned} \tag{S.70}$$

The coefficient of the non-abelian last term on the bottom line is the gauge coupling of the unbroken $SU(N)_V$ gauge group

$$g_v = \frac{g}{\sqrt{2}}\,. \tag{S.71}$$

Problem **3**($\star$):

For $g_L \neq g_R$, the covariant derivatives of the scalar fields become

$$D_\mu\Phi = \partial_\mu\Phi + ig'B_\mu\Phi + ig_L L_\mu\Phi - ig_R\Phi R_\mu\,. \tag{S.72}$$

As in part (a), the mass terms for the vector fields obtain from plugging $\langle\Phi\rangle$ into these covariant derivatives and then expanding the kinetic terms for the scalars:

$$D_\mu\langle\Phi\rangle = ig'CB_\mu \times \mathbf{1}_{N\times N} + iC(g_L L^a_\mu - g_R R^a_\mu) \times \frac{\lambda^a}{2} \tag{S.73}$$

and hence

$$\mathcal{L} \supset \mathrm{tr}(D_\mu\Phi^\dagger D^\mu\Phi) \supset Ng'^2C^2 \times B_\mu B^\mu + \frac{C^2}{2} \times (g_L L^a_\mu - g_R R^a_\mu)(g_L L^{a\mu} - g_R R^{a\mu}). \tag{S.74}$$

As in part (a), the abelian gauge fields gets mass $M_B^2 = 2Ng'^2C^2$, while the non-abelian vector mass is more tricky.

Let's define the coupling $\tilde{g}$ and the *mixing angle* θ according to

$$g_L = \tilde{g}\times\cos\theta, \quad g_R = \tilde{g}\times\sin\theta \quad\Longrightarrow\quad \tilde{g}^2 = g_L^2 + g_R^2, \quad \tan\theta = \frac{g_R}{g_L}. \tag{S.75}$$

Then the non-abelian mass term in eq. (S.74) becomes

$$\frac{C^2\tilde{g}^2}{2}\times\left(L_\mu^a\cos\theta - R_\mu^a\sin\theta\right)^2, \tag{S.76}$$

which tells us which particular combination of the non-abelian gauge fields become massive. Indeed, if we let

$$\begin{aligned} X_\mu^a &= \cos\theta\times L_\mu^a - \sin\theta\times R_\mu^a, \\ Y_\mu^a &= \sin\theta\times L_\mu^a + \cos\theta\times R_\mu^a, \end{aligned} \tag{S.77}$$

then both combinations of vector fields are canonically normalized — indeed,

$$\mathcal{L}_{L,R}^{\mathrm{kin}} = -\frac{1}{4}\sum_a\left(\left(\partial_{[\mu}L_{\nu]}^a\right)^2 + \left(\partial_{[\mu}R_{\nu]}^a\right)^2\right) = -\frac{1}{4}\sum_a\left(\left(\partial_{[\mu}X_{\nu]}^a\right)^2 + \left(\partial_{[\mu}Y_{\nu]}^a\right)^2\right), \tag{S.78}$$

— while the mass term (S.76) becomes

$$\mathcal{L}_{L,R}^{\mathrm{mass}} = \frac{C^2\tilde{g}^2}{2}\times X_\mu^a X^{a\mu}. \tag{S.79}$$

Thus, the X_μ^a fields have mass $M_X = \tilde{g}C$, while the Y_μ^a fields remain massless.

Now let's derive the effective Lagrangian for just the massless vector fields $Y_\mu^a(x)$ while freezing all the other fields, *i.e.* setting $\Phi(x)\equiv\langle\Phi\rangle$, $B_\mu(x)\equiv 0$, and $X_\mu^a(x)\equiv 0$. In terms of the L_μ^a and R_μ^a fields, this means

$$L_\mu^a = Y_\mu^a\times\sin\theta, \quad R_\mu^a = Y_\mu^a\times\cos\theta, \tag{S.80}$$

or in terms of the group-normalized gauge fields

$$\mathcal{L}_\mu = g_L\times L_\mu = g_L\sin\theta\times Y_\mu, \quad \mathcal{R}_\mu = g_R\times R_\mu = g_R\cos\theta\times Y_\mu. \tag{S.81}$$

However, for the mixing angle θ related to the couplings as in eq. (S.75), we have

$$g_L \sin\theta \;=\; g_R \cos\theta \;=\; \frac{g_L g_R}{\tilde{g}} \;\stackrel{\rm def}{=}\; g_Y\,, \tag{S.82}$$

and therefore

$$\mathcal{L}_\mu(x) \;=\; \mathcal{R}_\mu(x) \;=\; g_Y \times Y_\mu(x) \;\stackrel{\rm def}{=}\; \mathcal{Y}_\mu(x). \tag{S.83}$$

In terms of the $\mathcal{Y}_\mu(x)$ gauge field, the non-abelian tension fields are

$$\mathcal{L}_{\mu\nu}(x) \;=\; \mathcal{R}_{\mu\nu}(x) \;=\; \mathcal{Y}_{\mu\nu}(x) \;=\; \partial_\mu\mathcal{Y}_\nu(x) \;-\; \partial_\nu\mathcal{Y}_\mu(x) \;+\; i[\mathcal{Y}_\mu,\mathcal{Y}_\nu] \tag{S.84}$$

— precisely as for a group-normalized $SU(N)$ connection $\mathcal{Y}_\mu(x)$ — while the net YM Lagrangian is

$$\mathcal{L}_{YM} \;=\; -\frac{1}{2g_L^2}\,\mathrm{tr}\big(\mathcal{L}_{\mu\nu}\mathcal{L}^{\mu\nu}\big) \;-\; \frac{1}{2g_R^2}\,\mathrm{tr}\big(\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\big) \;=\; -\frac{1}{2}\left(\frac{1}{g_L^2}+\frac{1}{g_R^2}\right)\times\mathrm{tr}\big(\mathcal{Y}_{\mu\nu}\mathcal{Y}^{\mu\nu}\big) \tag{S.85}$$

— precisely as for the $SU(N)$ YM theory with inverse gauge coupling

$$\frac{1}{g_Y^2} \;=\; \frac{1}{g_L^2} \;+\; \frac{1}{g_R^2}\,. \tag{S.86}$$

After a bit of algebra, this formula becomes

$$g_Y \;=\; \frac{g_L g_R}{\sqrt{g_L^2+g_R^2}}\,, \tag{24}$$

in perfect agreement with $\mathcal{Y}^a_\mu(x) = g_Y \times Y^a_\mu(x)$ for the canonically normalized gauge fields $Y^a_\mu(x)$.