PHY–396 K.   Solutions for problem set #13.

Problem **1**(a):

Change the integration variable from ξ to $C = \xi A + (1 - \xi)B$. Then $dC = (A - B)d\xi$ and hence

$$\int\limits_0^1 d\xi \frac{1}{[\xi A + (1 - \xi)B]^2} = \int\limits_B^A \frac{dC}{A - B} \times \frac{1}{C^2} = \frac{1}{A - B} \times \frac{-1}{C}\Big|_B^A = \frac{1}{A - B} \times \left(\frac{1}{B} - \frac{1}{A}\right) = \frac{1}{AB},$$
(S.1)

which proves (F.a).

Problem **1**(b):

Let's take $n - 1$ derivatives of both sides of eq. (F.a) with respect to A. On one hand,

$$\left(-\frac{\partial}{\partial A}\right)^{n-1} \frac{1}{AB} = \frac{(n - 1)!}{A^n B},$$
(S.2)

on the other hand

$$\left(-\frac{\partial}{\partial A}\right)^{n-1} \int\limits_0^1 d\xi \frac{1}{[\xi A + (1 - \xi)B]^2} = \int\limits_0^1 d\xi \left(-\frac{\partial}{\partial A}\right)^{n-1} \frac{1}{[\xi A + (1 - \xi)B]^2}$$

$$= \int\limits_0^1 d\xi \frac{\xi^{n-1} \times n!}{[\xi A + (1 - \xi)B]^{n+1}}.$$
(S.3)

Comparing these two formulae and dividing by $(n - 1)!$, we immediately obtain

$$\frac{1}{A^n B} = \int\limits_0^1 d\xi \frac{\xi^{n-1} \times n}{[\xi A + (1 - \xi)B]^{n+1}}.$$
(F.b)

Problem **1**(c):

Now let's take $m - 1$ derivatives of both sides of eq. (F.b) with respect to B. On the left hand side

$$\left(-\frac{\partial}{\partial B}\right)^{m-1} \frac{1}{A^n B} = \frac{(m-1)!}{A^n B^m}, \tag{S.4}$$

while on the right hand side

$$\left(-\frac{\partial}{\partial B}\right)^{m-1} \int_0^1 d\xi \frac{n\xi^{n-1}}{[\xi A + (1-\xi)B]^{n+1}} = \int_0^1 d\xi \left(-\frac{\partial}{\partial B}\right)^{m-1} \frac{n\xi^{n-1}}{[\xi A + (1-\xi)B]^{n+1}}$$

$$= \int_0^1 d\xi \frac{n\xi^{n-1} \times (1-\xi)^{m-1} \times (n+1) \cdots (n+m-1)}{[\xi A + (1-\xi)B]^{n+m}}$$

$$= \frac{(n+m-1)!}{(n-1)!} \times \int_0^1 d\xi \frac{\xi^{n-1}(1-\xi)^{m-1}}{[\xi A + (1-\xi)B]^{n+m}}. \tag{S.5}$$

Comparing these two formulae, we immediately obtain

$$\frac{1}{A^n B^m} = \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \int_0^1 d\xi \frac{\xi^{n-1}(1-\xi)^{m-1}}{[\xi A + (1-\xi)B]^{n+m}}. \tag{F.c}$$

Problem **1**(d):

To evaluate the product of 3 denominators, we may first combine two of them using eq. (F.a), and then use eq. (F.c) to combine with the third denominator. Thus, using integration variables z and w instead of ξ, we obtain

$$\frac{1}{ABC} = \frac{1}{C} \times \frac{1}{AB} = \frac{1}{C} \times \int_0^1 dz \frac{1}{[zA + (1-z)B]^2}$$

$$= \int_0^1 dz \frac{1}{[zA + (1-z)B]^2 \times C} = \int_0^1 dz \int_0^1 dw \frac{2w}{\{w[zA + (1-z)B] + (1-w)C\}^3}. \tag{S.6}$$

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Now let's change variables from w and z to

$$\xi \;=\; w \times z \quad \text{and} \quad \eta \;=\; w \times (1-z) \tag{S.7}$$

which span the triangle

$$\xi \;\geq\; 0, \quad \eta \;\geq\; 0, \quad \xi + \eta \;=\; w \;\leq\; 1. \tag{S.8}$$

In terms of the new variables,

$$w[zA + (1-z)B] \;+\; (1-w)C \;=\; \xi \times A \;+\; \eta \times B \;+\; (1 - \xi - \eta) \times C \tag{S.9}$$

while

$$d\xi \, d\eta \;=\; w \, dw \, dz \,. \tag{S.10}$$

Consequently, eq. (S.6) becomes

$$\frac{1}{ABC} \;=\; \iint\limits_{\text{triangle}} d\xi \, d\eta \; \frac{2}{[\xi A + \eta B + (1 - \xi - \eta)C]^3} \,. \tag{F.d}$$

Note: there are two ways to parametrize the triangle (S.8), hence two lines of eq (F.d) in the problem text. On the top line, the triangle is a part of the (ξ, η) plane delimited by the conditions (S.8), or equivalently $0 \leq \eta \leq 1 - \xi$ for $0 \leq \xi \leq 1$. On the bottom line, the same triangle is embedded into a 2D plane $\xi + \eta + \zeta = 1$ in the 3D space, hence formal integration over three variables $d\xi \, d\eta \, d\zeta$ accompanied by the $\delta(\xi + \eta + \zeta - 1)$ function that restricts us to the 2D plane in 3D. Within that plane, the triangle is delimited by the conditions $\xi \geq 0$, $\eta \geq 0$, and $\zeta = 1 - \xi - \eta \geq 0$.

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<u>Problem **1**(e):</u>

Note: the integral on the RHS of eq. (F.e) is over the $(k-1)$-dimensional simplex embedded into the $\xi_1 + \cdots \xi_k = 1$ hyperplane in the k-dimensional space $(\xi_1, \ldots, \xi_k)$.

The simplest way to prove eq. (F.e) is by induction in k. The induction base is provided by eqs. (F.a) for $k = 2$ and (F.d) for $k = 3$. Now, the induction step: assuming eq. (F.e) is valid for some k, let's prove it for $k + 1$. Similar to the previous sub-problem, we obtain

$$
\frac{1}{A_1 \cdots A_{k+1}} = \frac{1}{A_{k+1}} \times \frac{1}{A_1 \cdots A_k}
$$

$$
\langle\!\langle \text{ by induction assumption for } k \rangle\!\rangle
$$

$$
= \frac{1}{A_{k+1}} \times \int\limits_{\xi_1, \ldots, \xi_k \geq 0} \!\!\! \cdots \int d^k \xi \, \delta(\xi_1 + \cdots + \xi_k - 1) \frac{(k-1)!}{[\xi_1 A_1 + \cdots + \xi_k A_k]^k}
$$

$$
= \int\limits_{\xi_1, \ldots, \xi_k \geq 0} \!\!\! \cdots \int d^k \xi \, \delta(\xi_1 + \cdots + \xi_k - 1) \frac{(k-1)!}{[\xi_1 A_1 + \cdots + \xi_k A_k]^k \times A_{k+1}} \qquad \text{(S.11)}
$$

$$
= \int\limits_{\xi_1, \ldots, \xi_k \geq 0} \!\!\! \cdots \int d^k \xi \, \delta(\xi_1 + \cdots + \xi_k - 1) \times
$$

$$
\times \int\limits_0^1 dw \, \frac{k! \times w^{k-1}}{\{w[\xi_1 A_1 + \cdots + \xi_k A_k] + (1-w)A_{k+1}\}^{k+1}}
$$

where the last equality follows from eq. (F.b) where w plays the role of ξ, $\xi_1 A_1 + \cdots + \xi_k A_k$ plays the role of A, and A_{k+1} plays the role of B.

Now let's change variables from w and $\xi_1, \ldots, \xi_k$ to

$$
\zeta_1 = w \times \xi_1, \quad \zeta_2 = w \times \xi_2, \quad \ldots \quad \zeta_k = w \times \xi_k, \quad \text{and} \quad \zeta_{k+1} = 1 - w, \quad \text{(S.12)}
$$

which parametrize a k-dimensional simplex in the $\zeta_1 + \cdots + \zeta_{k+1} = 1$ hyperplane in the $(k+1)$ dimensional space of the $(\zeta_1, \ldots, \zeta_{k+1})$. Indeed,

$$
0 \leq w \leq 1 \;\&\; \xi_1, \ldots, \xi_k \geq 0 \;\&\; \xi_i + \cdots + \xi_k = 1 \quad \Longleftrightarrow \quad \zeta_1, \ldots, \zeta_{k+1} \geq 0 \;\&\; \zeta_1 + \cdots + \zeta_{k+1} = 1.
$$
$$
\text{(S.13)}
$$

Also, for the integral on the last 2 lines of eq. (S.11), we have

$$dw\, w^{k-1} d^k \xi\, \delta(\xi_1 + \cdots + \xi_k - 1) = d^{k+1}\zeta\, \delta(\zeta_1 + \cdots + \zeta_{k+1} - 1) \qquad (\text{S.14})$$

while

$$w[\xi_1 A_1 + \cdots + \xi_k A_k] + (1-w)A_{k+1} = \zeta_1 A_1 + \cdots + \zeta_k A_k + \zeta_{k+1} A_{k+1}, \qquad (\text{S.15})$$

hence

$$\frac{1}{A_1 \cdots A_{k+1}} = \int \cdots \int_{\zeta_1,\dots,\zeta_{k+1} \geq 0} d^{k+1}\zeta\, \delta(\zeta_1 + \cdots + \zeta_{k+1} - 1) \times \frac{k!}{[\zeta_1 A_1 + \cdots + \zeta_{k+1} A_{k+1}]^{k+1}}. \qquad (\text{S.16})$$

Altogether, assuming eq. (F.e) works for some k, we have proved it also works for $k+1$. By induction, the formula should work for all k.

Problem **1**(f):
Eq. (F.f) follows from eq. (F.e) by taking derivatives of both sides with respect to $A_1, \dots, A_k$. Indeed, on the left hand side,

$$\left(-\frac{\partial}{\partial A_1}\right)^{n_1-1} \cdots \left(-\frac{\partial}{\partial A_k}\right)^{n_k-1} \frac{1}{A_1 \times \cdots \times A_k} = \frac{(n_1-1)! \times \cdots \times (n_k-1)!}{A_1^{n_1} \times \cdots \times A_k^{n_k}} \qquad (\text{S.17})$$

while on the right hand side

$$\left(-\frac{\partial}{\partial A_1}\right)^{n_1-1} \cdots \left(-\frac{\partial}{\partial A_k}\right)^{n_k-1} \int \cdots \int_{\xi_1,\dots,\xi_k \geq 0} d^k\xi\, \delta(\xi_1 + \cdots + \xi_k - 1) \frac{(k-1)!}{[\xi_1 A_1 + \cdots + \xi_k A_k]^k} =$$

$$= \int \cdots \int_{\xi_1,\dots,\xi_k \geq 0} d^k\xi\, \delta(\xi_1 + \cdots + \xi_k - 1) \frac{(n_1 + \cdots + n_k - 1)! \times \xi_1^{n_1-1} \times \cdots \times \xi_k^{n_k-1}}{[\xi_1 A_1 + \cdots + \xi_k A_k]^{n_1 + \cdots n_k}}$$

$$(\text{S.18})$$

Comparing these two formulae, we immediately obtain

$$\frac{1}{A_1^{n_1} \times \cdots \times A_k^{n_k}} = \frac{(n_1 + \cdots + n_k - 1)!}{(n_1 - 1)! \cdots (n_k - 1)!} \times$$

$$\times \int\cdots\int_{\xi_1,\ldots,\xi_k \geq 0} d^k\xi \, \delta(\xi_1 + \cdots + \xi_k - 1) \frac{\xi_1^{n_1-1} \cdots \xi_k^{n_k-1}}{[\xi_1 A_1 + \cdots + \xi_k A_k]^{n_1+\cdots n_k}} . \tag{F.f}$$

Problem **2**:

The one-loop diagram (1) yields amplitude

$$\mathcal{M}(\text{diagram}) = F(q_{\text{net}}) = \frac{-i\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{(q_2 = q_{\text{net}} - q_1)^2 - m^2 + i0},$$
$$\tag{S.19}$$

but the momentum integral here diverges logarithmically as $q_1 \to \infty$. In class, we have regularized this diagram using a hard-edge cutoff and obtained

$$F_{\text{HE}}(q_{\text{net}}^2 = t) = \frac{\lambda_{\text{bare}}^2}{32\pi^2} \times \left(\log \frac{\Lambda_{\text{HE}}^2}{m^2} - 1 - J(t/m^2) + \text{negligible} \right) \tag{S.20}$$

where $J(t/m^2)$ is as in eq. (3) and 'negligible' means 'vanishes as a negative power of the cutoff scale $\Lambda \to \infty$. Clearly, the amplitude (S.20) has form (2) for

$$C_{\text{HE}} = -1. \tag{S.21}$$

Now lets re-calculate the amplitude using the Pauli–Villars regularization scheme, where one subtracts from (1) a similar diagram where internal lines belong to ghost fields of extremely large mass $\Lambda \gg m$. The subtraction is done before the momentum integration,

$$F_{\text{PV}}(\delta k) = \frac{-i\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{(q_2 = q_{\text{net}} - q_1)^2 - m^2 + i0} \right.$$
$$\left. - \frac{1}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{(q_2 = q_{\text{net}} - q_1)^2 - \Lambda^2 + i0} \right\}, \tag{S.22}$$

so for $q^2 \gg \Lambda^2$ the net integrand behaves as $O(\Lambda^2/q^6)$ instead of $1/q^4$, which makes the integral convergent.

Out task is to evaluate the integral (S.22), so let's start with the Feynman's trick for simplifying the propagator product. As discussed in class,

$$
\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{(q_2 = q_{\text{net}} - q_1)^2 - m^2 + i0} =
$$
$$
= \int_0^1 d\xi \, \frac{1}{[(1-\xi)(q_1^2 - m^2 + i0) + \xi(q_2^2 - m^2 + i0)]^2}
$$
$$
= \int_0^1 d\xi \, \frac{1}{[k^2 + \xi(1-\xi) \times q_{\text{net}}^2 - m^2 + i0]^2}
$$

$$(S.23)$$

where $k = q_1 - \xi \times q_{\text{net}}$.

Similarly,

$$
\frac{1}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{(q_2 = q_{\text{net}} - q_1)^2 - \Lambda^2 + i0} = \int_0^1 d\xi \, \frac{1}{[k^2 + \xi(1-\xi) \times q_{\text{net}}^2 - \Lambda^2 + i0]^2}
$$

$$(S.24)$$

for exactly same $k = q_1 - \xi q_{\text{net}}$. Hence, we plug both propagator products into eq. (S.22), change the order of integration, and than change the momentum variable from q_1 to k,

$$
F_{\text{PV}}(q_{\text{net}}^2 = t) = \int_0^1 d\xi \, \mathcal{F}_{\text{PV}}(t, \xi)
$$

$$(S.25)$$

where

$$
\mathcal{F}_{\text{PV}}(t, \xi) = \frac{-i\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \Big\{ \frac{1}{[(k = q_1 - \xi q_{\text{net}})^2 + t\xi(1-\xi) - m^2 + i0]^2}
$$
$$
- \frac{1}{[(k = q_1 - \xi q_{\text{net}})^2 + t\xi(1-\xi) - \Lambda^2 + i0]^2} \Big\}
$$
$$
= \frac{-i\lambda^2}{2} \int \frac{d^4 k}{(2\pi)^4} \Big\{ \frac{1}{[k^2 + t\xi(1-\xi) - m^2 + i0]^2} - \frac{1}{[k^2 + t\xi(1-\xi) - \Lambda^2 + i0]^2} \Big\}.
$$

$$(S.26)$$

Next, we analytically continue the momentum integral from the Minkowski momentum k^μ

to the Euclidean Momentum k_E^μ, thus

$$d^4k \;\rightarrow\; i d^4 k_E\,, \qquad k^2 \;\rightarrow\; -k_E^2\,, \tag{S.27}$$

and consequently

$$\mathcal{F}_{\mathrm{PV}}(t,\xi) \;=\; \frac{\lambda^2}{2} \int \frac{d^4 k_E}{(2\pi)^4} \left\{ \frac{1}{[k_E^2 + m^2 - \xi(1-\xi)t]^2} - \frac{1}{[k_E^2 + \Lambda^2 - \xi(1-\xi)t]^2} \right\}. \tag{S.28}$$

At this point, we go to spherical coordinates in the 4D Euclidean space and focus on the radial coordinate $k_e = |k_E|$. This gives us

$$d^4 k_E \;=\; 2\pi^2\, k_e^3\, dk_e \;=\; \pi^2\, k_e^2\, dk_e^2 \tag{S.29}$$

and hence

$$\mathcal{F}_{\mathrm{PV}}(t,\xi) \;=\; \frac{\lambda^2}{32\pi^2} \int\limits_0^\infty dk_e^2 \left\{ \frac{k_e^2}{[k_e^2 + m^2 - \xi(1-\xi)t]^2} - \frac{k_e^2}{[k_e^2 + \Lambda^2 - \xi(1-\xi)t]^2} \right\}. \tag{S.30}$$

This last integral has form

$$\int\limits_0^\infty dy \left(\frac{y}{(y+A)^2} - \frac{y}{(y+B)^2} \right) \tag{S.31}$$

which evaluates to $\log(B/A)$. Indeed,

$$
\begin{aligned}
\int\limits_0^\infty dy \left(\frac{y}{(y+A)^2} - \frac{y}{(y+B)^2} \right) &= \int\limits_0^\infty dy \left(\frac{1}{y+A} - \frac{1}{y+B} - \frac{A}{(y+A)^2} + \frac{B}{(y+B)^2} \right) \\
&= \left(\log\frac{y+A}{y+B} + \frac{A}{y+A} - \frac{B}{y+B} \right)\Bigg|_0^\infty \\
&= \left(\log\frac{A+\infty}{B+\infty} - \log\frac{A}{B} \right) + \left(\frac{A}{\infty} - \frac{A}{A} \right) - \left(\frac{B}{\infty} - \frac{B}{B} \right) \\
&= \left(0 - \log\frac{A}{B} \right) + (0 - 1) - (0 - 1) \\
&= \log\frac{B}{A}\,.
\end{aligned}
\tag{S.32}
$$

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Therefore,

$$\mathcal{F}_{\mathrm{PV}}(t,\xi) = \frac{\lambda^2}{32\pi^2} \times \log \frac{\Lambda^2 - \xi(1-\xi)t}{m^2 - \xi(1-\xi)t} \approx \frac{\lambda^2}{32\pi^2} \times \log \frac{\Lambda^2}{m^2 - \xi(1-\xi)t} \tag{S.33}$$

since we assume not only $\Lambda \gg m$ but also $\Lambda^2 \gg |t|, |u|, |s|$.

Integrating this formula over ξ, we arrive at the Pauli–Villars regularized amplitude,

$$
\begin{aligned}
F_{\mathrm{PV}}(t) &= \frac{\lambda^2}{32\pi^2} \int\limits_0^1 d\xi \, \log \frac{\Lambda^2}{m^2 - \xi(1-\xi)t} \\
&= \frac{\lambda^2}{32\pi^2} \int\limits_0^1 d\xi \left(\log \frac{\Lambda^2}{m^2} - \log \frac{m^2 - \xi(1-\xi)t}{m^2} \right) \\
&= \frac{\lambda^2}{32\pi^2} \left(\log \frac{\Lambda^2}{m^2} - J\left(\frac{t}{m^2}\right) \right)
\end{aligned} \tag{S.34}
$$

where $J(t/m^2)$ is as in eq. (3). Clearly, the amplitude (S.34) has the requisite form (2), but this time

$$C_{\mathrm{PV}} = 0. \tag{S.35}$$

At first glance, having different constant terms $C_{\mathrm{PV}} \neq C_{\mathrm{HE}}$ looks like the two regulators yield different amplitudes. However, this difference can be canceled by having slightly different cutoff scale parameters $\Lambda_{\mathrm{PV}} \neq \Lambda_{\mathrm{HE}}$ for the two regulators. Indeed, the cutoff scale Λ_{HE} of the hard-edge regulator — the maximal value of the Euclidean momenta allowed in that scheme — does not have to be exactly the same as the mass Λ_{PV} of the ghost fields in the Pauli–Villars regularization scheme. To produce a similar physical effect, the two scales should have similar orders of magnitude, but this generally means

$$\Lambda_{\mathrm{PV}}^2 = \Lambda_{\mathrm{HE}}^2 \times \text{an } O(1) \text{ constant} \tag{S.36}$$

rather than naive identification $\Lambda_{\mathrm{PV}} = \Lambda_{\mathrm{HE}}$. In particular, we may set

$$\Lambda_{\mathrm{PV}}^2 = \Lambda_{\mathrm{HE}}^2 \times \exp\left(C_{\mathrm{HE}} - C_{\mathrm{PV}} = -1\right) \tag{S.37}$$

to obtain

$$\log \Lambda^2_{\mathrm{PV}} + C_{\mathrm{PV}} = \log \Lambda^2_{\mathrm{HE}} + C_{\mathrm{HE}} \tag{S.38}$$

and hence perfect agreement between the amplitudes (S.34) and (S.20).

Now consider the higher-derivative regularization scheme. In this scheme, the scalar field ϕ has very small higher-derivative terms in its Lagrangian,

$$\mathcal{L}_{\mathrm{HD}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{24}\phi^4 - \frac{1}{2\Lambda^2}(\partial^2 \phi)^2, \tag{S.39}$$

which softens the scalar's propagator for very high momenta $q^2 \gtrsim \Lambda^2$:

$$\frac{i}{q^2 - m^2 + i0} \rightarrow \frac{i}{q^2 - m^2 + i0 - \Lambda^{-2}q^4} \approx \frac{i}{q^2 - m^2 + i0} \times \frac{-\Lambda^2}{q^2 - \Lambda^2 + i0}. \tag{S.40}$$

Consequently, in the higher-derivative regularization scheme, the one-loop amplitude (1) becomes

$$F_{\mathrm{HD}}(q^2_{\mathrm{net}} = t) = \frac{-i\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{q_1^2 - m^2 + i0} \frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} \frac{-\Lambda^2}{q_2^2 - \Lambda^2 + i0} \tag{S.41}$$

where $q_2 \equiv q_{\mathrm{net}} - q_1$. For all but extremely large momenta $q^2 \ll \Lambda^2$, the integrand here is indistinguishable from the un-regularized loop integral (S.19), but for $q^2 \gtrsim \Lambda^2$ it becomes softer — behaves like Λ^4/q^8 for $q^2 \to \infty$ instead of $1/q^4$ — so the integral (S.41) converges.

Our task is to evaluate this integral, so let's start by simplifying the propagator product by using the Feynman's trick. Specifically, let's apply the trick to the un-regulated factors $1/(q_1^2 - m^2 + i0)$ and $1/(q_2^2 - m^2 + i0)$ only,

$$\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{(q_2 = q_{\mathrm{net}} - q_1)^2 - m^2 + i0} = \int_0^1 d\xi \frac{1}{[(q_1 - \xi q_{\mathrm{net}})^2 + \xi(1 - \xi)q^2_{\mathrm{net}} - m^2 + i0]^2}. \tag{S.42}$$

As to the regulating factors $-\Lambda^2/(q_1^2 - \Lambda^2 + i\epsilon)$ and $-\Lambda^2/(q_2^2 - \Lambda^2 + i\epsilon)$, they are indistinguishable from 1 unless the momenta q_1 and q_2 are of order Λ or larger. But for so

large momenta, the $O(q_{\text{net}}) \ll \Lambda$ differences between the q_1, the $-q_2 = q_1 - q_{\text{net}}$ and the $k = q_1 - \xi q_{\text{net}}$ become unimportant, so we may approximate

$$\frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i\epsilon} \approx \frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i\epsilon} \approx \frac{-\Lambda^2}{(k = q_1 - \xi q_{\text{net}})^2 - \Lambda^2 + i\epsilon} . \tag{9}$$

The ξ on the RHS here is as in eq. (S.42). That is, we first put the regulating factors inside the Feynman-trick integral (S.42) and then apply the approximation (9). Altogether, this gives us

$$\text{product of HD regulated propagators} = \int\limits_0^1 d\xi \, \frac{1}{[k^2 + \xi(1 - \xi)t - m^2 + i0]^2} \times \frac{\Lambda^4}{[k^2 - \Lambda^2 + i\epsilon]^2} \tag{S.43}$$

where $k = q_1 - \xi q_{\text{net}}$ and $t = q_{\text{net}}^2$.

To integrate this product of regulated propagators over the loop momentum q_1^μ, we integrate over the momentum before integrating over the Feynman parameter ξ, and for each ξ we change the momentum variable from q_1^μ to $k^\mu = q_1^\mu - \xi q_{\text{net}}^\mu$, thus

$$F_{\text{HD}}(q_{\text{net}}^2 = t) = \int\limits_0^1 d\xi \, \mathcal{F}_{\text{HD}}(t, \xi) \tag{S.44}$$

where

$$\mathcal{F}_{\text{HD}}(t, \xi) = \frac{-i\lambda^2}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 + \xi(1 - \xi)t - m^2 + i0]^2} \times \frac{\Lambda^4}{[k^2 - \Lambda^2 + i\epsilon]^2} . \tag{S.45}$$

Next, we analytically continue the momentum integral from the Minkowski momentum k^μ to the Euclidean momentum k_E^μ: d^4k becomes id^4k_E, k^2 becomes $-k_E^2$, and the integral (S.45) becomes

$$\mathcal{F}_{\text{HD}}(q_{\text{net}}^2 = t, \xi) = \frac{\lambda^2}{2} \int \frac{d^4 k_E}{(2\pi)^2} \frac{1}{[k_E^2 + m^2 - \xi(1 - \xi)t]^2} \times \frac{\Lambda^4}{[k_E^2 + \Lambda^2]^2}$$

$$\langle\!\langle \text{ integrating over the directions of the } k_E^\mu \text{ in the 4D Euclidean space} \rangle\!\rangle$$

$$= \frac{\lambda^2}{32\pi^2} \int\limits_0^\infty \frac{dk_e^2 \, k_e^2 \times \Lambda^4}{[k_e^2 + m^2 - \xi(1 - \xi)t]^2 \times [k_e^2 + \Lambda^2]^2} . \tag{S.46}$$

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The remaining radial integral has form

$$\int\limits_0^\infty dy \, \frac{yB^2}{(y+A)^2(y+B)^2} \tag{S.47}$$

where $A = m^2 - \xi(1-\xi)t$ is much less than $B = \Lambda^2$. The simplest way to evaluate this integral is to split it at some point C which is much bigger than A but much smaller that B, thus

$$\int\limits_0^\infty dy \, \frac{yB^2}{(y+A)^2(y+B)^2} = \int\limits_0^C dy \, \frac{yB^2}{(y+A)^2(y+B)^2} + \int\limits_C^\infty dy \, \frac{yB^2}{(y+A)^2(y+B)^2} \tag{S.48}$$

for $A \ll C \ll B$. In the first integral on the RHS we have $y \le C \ll B$, which allows us to approximate $y + B \approx B$ and hence

$$\int\limits_0^C dy \, \frac{yB^2}{(y+A)^2(y+B)^2} \approx \int\limits_0^C dy \, \frac{y}{(y+A)^2} = \log\frac{A+C}{A} - \frac{C}{C+A} \approx \log\frac{C}{A} - 1. \tag{S.49}$$

In the second integral on the RHS of (S.48) we have $y \ge C \gg A$, which allows for a different approximation $y + A \approx y$ and hence

$$
\begin{aligned}
\int\limits_C^\infty dy \, \frac{yB^2}{(y+A)^2(y+B)^2} &\approx \int\limits_C^\infty dy \, \frac{B^2}{y(y+B)^2} \\
&= \int\limits_C^\infty dy \left(\frac{1}{y} - \frac{1}{y+B} - \frac{B}{(y+B)^2} \right) \\
&= \log\frac{B+C}{C} - \frac{B}{B+C} \\
&\approx \log\frac{B}{C} - 1.
\end{aligned}
\tag{S.50}
$$

Altogether, for $A \ll C \ll B$,

$$\int\limits_0^\infty dy \, \frac{yB^2}{(y+A)^2(y+B)^2} \approx \log\frac{C}{A} - 1 + \log\frac{B}{C} - 1 = \log\frac{B}{A} - 2. \tag{S.51}$$

Note that C drops out of net result; if it did not, our approximations would be inconsistent.

Alternatively, we may evaluate the integral (S.47) without using any approximations by expanding the integrand — which is a rational function of y — into its poles:

$$\frac{yB^2}{(y+A)^2(y+B)^2} = \frac{\alpha}{(y+A)^2} + \frac{\beta}{(y+B)^2} + \frac{\gamma}{y+A} + \frac{\delta}{y+B} \qquad (S.52)$$

for some constants $\alpha, \beta, \gamma, \delta$. The values of α and β follow by matching the coefficients of the double poles at $y = -A$ and $y = -B$ at both sides, thus

$$\alpha = \frac{-AB^2}{(B-A)^2}, \qquad \beta = \frac{-B^3}{(B-A)^2}. \qquad (S.53)$$

Subtracting the double poles from both sides of eq. (S.52) and matching the residues of the remaining single poles, we obtain

$$\gamma = -\delta = \frac{+B^2(B+A)}{(B-A)^3}. \qquad (S.54)$$

Consequently,

$$\int_0^\infty dy \, \frac{yB^2}{(y+A)^2(y+B)^2} =$$

$$= \frac{B^2}{(B-A)^2} \times \int_0^\infty dy \left[\frac{B+A}{B-A} \left(\frac{1}{y+A} - \frac{1}{y+B} \right) - \frac{A}{(y+A)^2} - \frac{B}{(y+B)^2} \right] \quad (S.55)$$

$$= \frac{B^2}{(B-A)^2} \times \left[\frac{B+A}{B-A} \times \log\frac{B}{A} - 1 - 1 \right] \qquad \langle\!\langle \text{ for any } B > A > 0 \,\rangle\!\rangle$$

$$\approx \log\frac{B}{A} - 2 \qquad\qquad\qquad\qquad\qquad \langle\!\langle \text{ for } B \gg A \,\rangle\!\rangle,$$

in perfect agreement with eq. (S.51).

Plugging this formula into the momentum integral (S.46), we obtain

$$\mathcal{F}_{\text{HD}}(t,\xi) = \frac{\lambda^2}{32\pi^2} \times \left(\log\frac{\Lambda^2}{m^2 - \xi(1-\xi)t} - 2 \right) \qquad (S.56)$$

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and consequently

$$
\begin{aligned}
F_{\mathrm{HD}}(t) &= \frac{\lambda^2}{32\pi^2} \int\limits_0^1 d\xi \left(\log \frac{\Lambda^2}{m^2 - \xi(1-\xi)t} - 2 \right) \\
&= \frac{\lambda^2}{32\pi^2} \left(\log \frac{\Lambda^2}{m^2} - 2 - J\left(\frac{t}{m^2}\right) \right)
\end{aligned}
\tag{S.57}
$$

where $J(t/m^2)$ is as in eq. (3). As promised the higher-derivative regularization also yields the amplitude of the form (2), but for a different constant C that the other two regulators, namely

$$
C_{\mathrm{HD}} = -2.
\tag{S.58}
$$

Therefore, to identify the amplitude (S.57) with the amplitudes produced by the other two regularization schemes, we should set the HD cutoff scale Λ_{HD} to

$$
\Lambda_{\mathrm{HD}}^2 = \Lambda_{\mathrm{PV}}^2 \times \exp\left(C_{\mathrm{PV}} - C_{\mathrm{HD}} = +2 \right) = \Lambda_{\mathrm{HE}}^2 \times \exp\left(C_{\mathrm{HE}} - C_{\mathrm{HD}} = +1 \right).
\tag{S.59}
$$