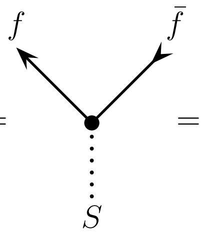
Problem **2**(a):

At the tree level, the scalar decay amplitude is simply

$$i\mathcal{M}(S \to f + \bar{f}) \;=$$



$$= \;\bar{u}_f(-ig)v_{\bar{f}}\,. \tag{S.1}$$

Summing over spins of the outgoing fermions, we get

$$\sum|\mathcal{M}|^2 \;=\; g^2 \times \mathrm{tr}\left[(\not{p}_1 + m_f)(\not{p}_2 - m_f)\right] \;=\; g^2 \times (4p_1p_2 - 4m_f^2) \;=\; 2g^2 \times (M_s^2 - 4m_f^2), \tag{S.2}$$

where the last equality follows from

$$2p_1p_2 \;-\; 2m_f^2 \;=\; (p_1 + p_2)^2 \;-\; p_1^2 \;-\; p_2^2 \;-\; 2m_f^2 \;=\; M_s^2 \;-\; 4m_f^2. \tag{S.3}$$

The phase space factor for one particle decaying into two — in the frame of the initial particle where the momenta of the final particles are $\pm\mathbf{p}$ — is

$$\begin{aligned}
\mathcal{P} \;&=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{(2M_s)(2E_1)(2E_2)} \times (2\pi)\delta(E_1 + E_2 - M_s) \\
&=\; \int\limits_0^\infty \frac{4\pi|\mathbf{p}|^2\,d|\mathbf{p}|}{32\pi^2\,M_sE_1E_2}\,\delta(E_1 + E_2 - M_s) \\
&=\; \frac{4\pi|\mathbf{p}|^2}{32\pi^2\,M_sE_1E_2} \times \left(\frac{d(E_1 + E_2)}{d|\mathbf{p}|} \;=\; \frac{|\mathbf{p}|M_s}{E_1E_2}\right)^{-1} \\
&=\; \frac{|\mathbf{p}|}{8\pi M_s^2}\,.
\end{aligned} \tag{S.4}$$

Consequently, the net tree-level decay rate is

$$\Gamma(S \to f + \bar{f}) \;=\; \mathcal{P} \times \sum|\mathcal{M}|^2 \;=\; \frac{g^2}{4\pi} \times \frac{M_s^2 - 4m_f^2}{M_s^2} \times |\mathbf{p}|. \tag{S.5}$$

By energy conservation,

$$|\mathbf{p}| \;=\; \sqrt{(\tfrac{1}{2}M_s)^2 - m_f^2} \;=\; \frac{\beta M_s}{2} \quad\text{where}\quad \beta \;=\; \sqrt{1 - \frac{4m_f^2}{M_s^2}}\,, \tag{S.6}$$

so in terms of the fermions' speed β,

$$\Gamma^{\rm tree}(S\to f+\bar f) \;=\; \frac{g^2}{8\pi}\times\beta^3 M_s\,. \tag{S.7}$$

Note that for weak Yukawa coupling $g^2 \ll 8\pi$, the decay rate is small compared to the scalar's mass, $\Gamma \ll M_s$, so the resonance in the scalar's correlation function should be narrow.

Problem **2**(b):

For real p^2, everything under the integral in eq. (2) is real — except for the logarithm when $\Delta(\xi)$ happens to be negative, in which case log $=$ real $\pm\,\pi i$. To determine the sign, we let $p^2 = \text{real} + i\epsilon$, hence

$$\Delta \;=\; m_f^2 \;-\; \xi(1-\xi)\times p^2 \;=\; \text{real} \;-\; i\epsilon \tag{S.8}$$

and therefore

$$\operatorname{Im}\log\frac{4\pi m^2}{\Delta} \;=\; -\operatorname{Im}\log(\Delta - i\epsilon) \;=\; +\pi\times\Theta(\Delta<0) \;\stackrel{\rm def}{=}\; +\pi\times\begin{cases}1 & \text{when } \Delta<0,\\ 0 & \text{when } \Delta>0.\end{cases} \tag{S.9}$$

Consequently, the imaginary part of Σ_ϕ is given by

$$\operatorname{Im}\Sigma_\phi^{1\,\rm loop}(p^2+i\epsilon) \;=\; \frac{12g^2}{16\pi}\times\int_0^1 d\xi\,(m_f^2 - \xi(1-\xi)p^2)\times\Theta(m_f^2 - \xi(1-\xi)p^2 < 0). \tag{S.10}$$

Technically, the m_f here is the *bare* fermion mass, but at the $O(g^2)$ level of accuracy we may neglect the difference between $m_f^{\rm bare}$ and $m_f^{\rm phys}$. Consequently, the threshold for the imaginary part (S.9) lies at $p^2_{\rm min} = (2m_f^{\rm phys})^2$ — which is precisely the lowest scalar mass $(M_s^{\rm phys})^2$ that allows for decay $S\to f+\bar f$.

Letting $p^2 = M_s^2 > 4m_f^2$, we have

$$\frac{m_f^2}{p^2} = \frac{1-\beta^2}{4} \quad\Longrightarrow\quad \Delta(\xi) = \frac{M_s^2}{4}\times\bigl(1-\beta^2-4\xi(1-\xi)\bigr) = \frac{M_s^2}{4}\times\bigl((1-2\xi)^2-\beta^2\bigr). \tag{S.11}$$

This expression becomes negative for $\frac{1-\beta}{2} < \xi < \frac{1+\beta}{2}$. Consequently, the integral in eq. (S.10) evaluates to

$$\frac{M_s^2}{4}\times\int\limits_{\frac{1}{2}(1-\beta)}^{\frac{1}{2}(1+\beta)} d\xi\bigl[(1-2\xi)^2-\beta^2\bigr] = -\frac{M_s^2}{8}\times\int\limits_{-\beta}^{+\beta} d(1-2\xi)\,\bigl[\beta^2-(1-2\xi)^2\bigr] = -\frac{M_s^2}{8}\times\frac{4\beta^3}{3} \tag{S.12}$$

and therefore

$$\operatorname{Im}\Sigma_\phi^{1\,\text{loop}}(M_s^2+i\epsilon) = -\frac{g^2}{8\pi}\times\beta^3 M_s^2\,. \tag{S.13}$$

<u>Problem **2**(c)</u>:

By inspection of eqs. (S.7) and (S.13), eq. (4) holds true:

$$\operatorname{Im}\Sigma_\phi^{1\,\text{loop}}(p^2=M_s^2+i\epsilon) = -\frac{g^2}{8\pi}\times\beta^3 M_s^2 = -M_s\times\Gamma^{\text{tree}}(S\to f+\bar f). \tag{1}$$

Higher-loop imaginary parts are similarly related to the decay rates calculated to higher orders. In the bare perturbation theory (using the bare coupling and mass parameters and Z factors instead of the counterterms),

$$\operatorname{Im}\Sigma_\phi^{\text{bare pert. theory}}(p^2=(M_s^{\text{phys}})^2+i\epsilon) = -M_s^{\text{phys}}\times\Gamma_{\text{total}}(S\to\text{anything})\times Z_\phi; \tag{S.14}$$

in the perturbation theory using counterterms, the $\Sigma_\phi(p^2)$ amplitude has a different normalization by a $1/Z_\phi$ factor, so we have simply

$$\operatorname{Im}\Sigma_\phi^{\text{counterterm pert. theory}}(p^2=(M_s^{\text{phys}})^2+i\epsilon) = -M_s^{\text{phys}}\times\Gamma_{\text{total}}(S\to\text{anything}). \tag{S.15}$$

Eqs. (S.14) and (S.15) work in all quantum field theories. For any field $\hat\phi(x)$ which can create an unstable particle U of physical mass M_U and lifetime $1/\Gamma_U \gg 1/M_U$, the imaginary part of

Σ_ϕ for that field satisfies

$$\begin{aligned}\operatorname{Im}\Sigma_\phi^{\text{bare pert. theory}}(p^2=(M_U^{\text{phys}})^2+i\epsilon) &= -M_U^{\text{phys}}\times\Gamma_{\text{total}}(U\to\text{anything})\times Z_\phi\,,\\ \operatorname{Im}\Sigma_\phi^{\text{counterterm pert. theory}}(p^2=(M_U^{\text{phys}})^2+i\epsilon) &= -M_U^{\text{phys}}\times\Gamma_{\text{total}}(U\to\text{anything}).\end{aligned} \tag{S.16}$$

The relation (S.16) follows from the optical theorem, which makes a narrow resonance out of any slowly-decaying particle. Consequently, the propagator of the field creating such particles should have form

$$\mathcal{F}_{\phi\phi}(p^2+i\epsilon) = \frac{iZ}{p^2-(M_U^{\text{phys}})^2+iM_U^{\text{phys}}\times\Gamma_{\text{tot}}(U\to\text{anything})} + \text{smooth} \tag{S.17}$$

for p^2 near $(M_U^{\text{phys}})^2$. The bare perturbation theory gives this propagator as

$$\mathcal{F}_{\phi\phi}(p^2) = \frac{i}{p^2-m_{\text{bare}}^2-\Sigma_\phi(p^2)}\,, \tag{S.18}$$

so to make a Breit–Wigner resonance (S.17) out of this formula, we need

$$(M_U^{\text{phys}})^2-(m_\phi^{\text{bare}})^2 = \operatorname{Re}\Sigma_\phi(p^2=(M_U^{\text{phys}})^2+i\epsilon), \tag{S.19}$$

$$\frac{1}{Z_\phi} = 1-\operatorname{Re}\left.\frac{d\Sigma_\phi}{dp^2}\right|_{p^2=(M_U^{\text{phys}})^2+i\epsilon}, \tag{S.20}$$

$$\operatorname{Im}\Sigma_\phi(p^2=(M_U^{\text{phys}})^2+i\epsilon) < 0 \quad \text{(this is essential!)}, \tag{S.21}$$

$$M_U^{\text{phys}}\times\Gamma_{\text{tot}}(U\to\text{anything})\times Z_\phi = -\operatorname{Im}\Sigma_\phi(p^2=(M_U^{\text{phys}})^2+i\epsilon). \tag{S.22}$$

In addition, we also assume that $\Gamma_{\text{tot}}(U)\ll M_U^{\text{phys}}$ and that the imaginary part $\operatorname{Im}\Sigma_\phi(p^2+i\epsilon)$ does not change much for $p^2=(M_U^{\text{phys}})^2\pm O(M_U^{\text{phys}}\times\Gamma_{\text{tot}}(U))$. If these assumptions fail, the resonance looks wide and/or deformed rather than a nice Breit–Wigner peak (S.17).