Problem **1**(a):

Feynman rules for the diagram (1) evaluate to

$$-i\Sigma(p^2) \;=\; \frac{(-i\lambda)^2}{3!}\int\!\frac{d^4q_1}{(2\pi)^4}\int\!\frac{d^4q_2}{(2\pi)^4}\,\frac{i}{q_1^2-m^2+i\epsilon}\times\frac{i}{q_2^2-m^2+i\epsilon}\times\frac{i}{q_3^2-m^2+i\epsilon} \qquad \text{(S.1)}$$

where $q_3 \equiv p - q_1 - q_2$ while the overall 1/3! factor comes from the permutation symmetry between the 3 propagators. Using Feynman's parameter tricks — specifically, eq. (F.d) from the homework set#13 — we may combine the denominators of the three propagators into a complete cube,

$$\prod_{i=1}^{3}\frac{1}{q_i^2-m^2+i\epsilon} \;=\; \int\limits_{\triangle}\! d(FP)\,\frac{2}{\mathcal{D}^3} \;\stackrel{\rm def}{=}\; \iiint\limits_{\xi,\eta,\zeta\ge0}\! d\xi\,d\eta\,d\zeta\,\delta(\xi+\eta+\zeta-1)\times\frac{2}{\mathcal{D}^3} \qquad \text{(13.F.d)}$$

where

$$\begin{aligned}\mathcal{D}(\xi,\eta,\zeta) \;&=\; \xi\times(q_1^2-m^2+i\epsilon) \;+\; \eta\times(q_2^2-m^2+i\epsilon) \;+\; \zeta\times(q_3^2-m^2+i\epsilon)\\ &=\; \xi\times q_1^2 \;+\; \eta\times q_2^2 \;+\; \zeta\times(q_3=p-q_1-q_2)^2 \;-\; m^2 \;+\; i\epsilon.\end{aligned} \qquad \text{(S.2)}$$

Consequently, we may rewrite eq. (S.1) as

$$\Sigma(p^2) \;=\; -\frac{\lambda^2}{3}\int\limits_{\triangle}\! d(FP)\int\!\frac{d^4q_1}{(2\pi)^4}\int\!\frac{d^4q_2}{(2\pi)^4}\,\frac{1}{\mathcal{D}^3}\,, \qquad \text{(S.3)}$$

cf. eq. (2).

Problem **1**(b):

As a function of the momenta q_1, q_2, and p, the $\mathcal{D}$ in eq. (S.2) is a quadratic polynomial. So let us shift the loop momentum variables from q_1 and q_2 to some k_1 and k_2 so that $\mathcal{D}$ takes the sum-of-squares form (3).

We start by expanding the $\zeta(q_3 = p - q_1 - q_2)^2$ term in eq. (S.2) and then collecting all the terms containing the q_1 momentum into a full square,

$$\begin{aligned}
\mathcal{D} \,+\, m^2 \;&=\; \xi \times q_1^2 \;+\; \eta \times q_2^2 \;+\; \zeta \times (p - q_1 - q_2)^2 \\
&=\; (\xi + \zeta) \times q_1^2 \;+\; 2\zeta \times q_1^\mu (q_2 - p)_\mu \;+\; \zeta \times (q_2 - p)^2 \;+\; \eta \times q_2^2 \\
&=\; (\xi + \zeta) \times \left(q_1 \;+\; \frac{\zeta}{\xi + \zeta}\,(q_2 - p) \right)^2 \;+\; \frac{\xi\zeta}{\xi + \zeta} \times (q_2 - p)^2 \;+\; \eta \times q_2^2 \,.
\end{aligned} \tag{S.4}$$

Naturally, we interpret the first term on the last line as $\alpha \times k_1^2$, thus

$$\alpha \;=\; (\xi + \zeta), \qquad k_1 \;=\; q_1 \,+\, \frac{\zeta}{\xi + \zeta} \times (q_2 - p). \tag{S.5}$$

For the other two terms on the last line of (S.4), we expand the $(q_2 - p)^2$ and collect all terms containing the q_2 momentum into another full square, thus

$$\begin{aligned}
\frac{\xi\zeta}{\xi + \zeta} \times (q_2 - p)^2 \;+\; \eta \times q_2^2 \;&=\; \left(\frac{\xi\zeta}{\xi + \zeta} + \eta \right) q_2^2 \;-\; \frac{2\xi\zeta}{\xi + \zeta}(q_2 p) \;+\; \frac{\xi\zeta}{\xi + \zeta} p^2 \\
&=\; \frac{\xi\zeta + \eta(\xi + \zeta)}{\xi + \zeta} \times \left(q_2 \;-\; \frac{\xi\zeta}{\xi\zeta + \eta(\xi + \zeta)}\, p \right)^2 \\
&\qquad +\; \left(\frac{\xi\zeta}{\xi + \zeta} - \frac{(\xi\zeta)^2}{(\xi + \zeta)(\xi\zeta + \eta(\xi + \zeta))} \right) \times p^2 \\
&=\; \beta \times k_2^2 \;+\; \gamma \times p^2
\end{aligned} \tag{S.6}$$

for

$$\beta \;=\; \frac{\xi\zeta + \eta(\xi + \zeta) \;=\; \xi\eta + \xi\zeta + \eta\zeta}{\xi + \zeta}\,, \tag{S.7}$$

$$k_2^\mu \;=\; q_2^\mu \;-\; \frac{\xi\zeta}{\xi\eta + \xi\zeta + \eta\zeta}\, p^\mu, \tag{S.8}$$

$$\gamma \;=\; \frac{\xi\zeta}{\xi + \zeta} \;-\; \frac{(\xi\zeta)^2}{(\xi + \zeta)(\xi\eta + \xi\zeta + \eta\zeta)} \;=\; \frac{\xi\eta\zeta}{\xi\eta + \xi\zeta + \eta\zeta}\,. \tag{S.9}$$

Altogether, we end up with

$$\xi \times q_1^2 \;+\; \eta \times q_2^2 \;+\; \zeta \times q_3^2 \;=\; \alpha \times k_1^2 \;+\; \beta \times k_2^2 \;+\; \gamma \times p^2 \tag{S.10}$$

and hence

$$\mathcal{D} \;=\; \alpha \times k_1^2 \;+\; \beta \times k_2^2 \;+\; \gamma \times p^2 \;-\; m^2 \;+\; i\epsilon \tag{3}$$

for the Feynman-parameter-dependent coefficients α, β, and γ being precisely as in eq. (4).

Finally, we need to check the Jacobian of replacing the original independent loop momenta q_1 and q_2 with k_1 and k_2. In light of eqs. (S.5) and (S.8), it is easy to see that

$$\frac{\partial(k_1, k_2)}{\partial(q_1, q_2)} \;=\; \det\begin{pmatrix} 1 & \frac{\zeta}{\xi+\zeta} \\ 0 & 1 \end{pmatrix} \;=\; 1, \tag{S.11}$$

and therefore $dk_1\,dk_2 = dq_1\,dq_2$, dimension by dimension. In other words, for fixed Feynman parameters

$$\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4} \;=\; \int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\,, \tag{S.12}$$

and therefore

$$\Sigma^{2\,\text{loop}}(p^2) \;=\; -\frac{\lambda^2}{3}\int\limits_{\triangle} d(FP)\iint\frac{d^4k_1\,d^4k_2}{(2\pi)^8}\,\frac{1}{[\mathcal{D} \;=\; \alpha k_1^2+\beta k_2^2+\gamma p^2-m^2+i\epsilon]^3}\,. \tag{S.13}$$

Problem **1**(c):

The momentum integral in eq. (S.13) has form

$$\int\frac{d^8k}{[k^2+\cdots]^3}\,, \tag{S.14}$$

which is quadratically divergent for $k\to\infty$. However, the quadratic divergence here is a p–independent constant, so it does not affect the derivative $d\Sigma/dp^2$ and hence the field strength renormalization factor Z. Instead, the derivative is only logarithmically divergent.

To see how this works, let's take d/dp^2 derivatives of both sides of eq. (S.13). On the right hand side, the only p-dependent thing is the γp^2 term in $\mathcal{D}$, hence

$$\frac{\partial\mathcal{D}}{\partial p^2} = \gamma \quad \Longrightarrow \quad \frac{\partial}{\partial p^2}\frac{1}{\mathcal{D}^3} = \frac{-3\gamma}{\mathcal{D}^4} \tag{S.15}$$

and therefore

$$\frac{d\Sigma}{dp^2} = +\lambda^2 \int_{\triangle} d(FP)\,\gamma \times \iint \frac{d^4k_1\,d^4k_2}{(2\pi)^8}\,\frac{1}{\mathcal{D}^4}\,. \tag{S.16}$$

Here, the momentum integral has form

$$\int \frac{d^8k}{[k^2+\cdots]^4}\,, \tag{S.17}$$

so its UV divergence for $k\to\infty$ is logarithmic rather than quadratic.

<u>Problem **1**(d–e):</u>
Rotating both loop momenta k_1 and k_2 into the Euclidean momentum space, we have $d^4k_1 \to id^4k_1^E$, $d^4k_2 \to id^4k_2^E$, and

$$\mathcal{D} \to -\alpha\times(k_1^E)^2 - \beta\times(k_2^E)^2 + \gamma\times p^2 - m^2 \tag{S.18}$$

hence

$$\frac{d\Sigma}{dp^2} = -\lambda^2 \int_{\triangle} d(FP)\,\gamma \times \int\frac{d^4k_1^E}{(2\pi)^4}\int\frac{d^4k_2^E}{(2\pi)^4}\,\frac{1}{[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]^4}\,. \tag{S.19}$$

Next, we need dimensional regularization to actually perform the momentum integrals.

Changing

$$\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \ \to\ \mu^{2(4-D)} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \tag{S.20}$$

(Euclidean signature for all dimensions), we have

$$\begin{aligned}
&\mu^{8-2D} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \frac{1}{[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2]^4} = \\
&\quad \langle\langle \text{using eq. (6)} \rangle\rangle \\
&\quad = \frac{\mu^{8-2D}}{6} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} \int_0^\infty dt\, t^3 \exp\Big(-t \times \big[\alpha(k_1^E)^2 + \beta(k_2^E)^2 + m^2 - \gamma p^2\big]\Big) \\
&\quad = \frac{\mu^{8-2D}}{6} \int_0^\infty dt\, t^3 e^{-t(m^2-\gamma p^2)} \int \frac{d^Dk_1}{(2\pi)^D} \int \frac{d^Dk_2}{(2\pi)^D} e^{-t\alpha k_1^2} e^{-t\beta k_2^2} \\
&\quad \langle\langle \text{using eq. (7)} \rangle\rangle \\
&\quad = \frac{\mu^{8-2D}}{6} \int_0^\infty dt\, t^3 e^{-t(m^2-\gamma p^2)} \times (4\pi\alpha t)^{-D/2} (4\pi\beta t)^{-D/2} \\
&\quad = \frac{\mu^{8-2D}}{6(4\pi)^D(\alpha\beta)^{D/2}} \times \int_0^\infty dt\, t^{3-D} e^{-t(m^2-\gamma p^2)} \\
&\quad = \frac{\mu^{8-2D}}{6(4\pi)^D(\alpha\beta)^{D/2}} \times \Gamma(4-D)(m^2 - \gamma p^2)^{D-4}.
\end{aligned} \tag{S.21}$$

Note the $\Gamma(4-D)$ factor: It has a pole at $D = 4$ but no poles at $D < 4$. This is dimensional regularization's way to show that the momentum integrals diverge, but only logarithmically.

At this point, we may take $D = 4 - 2\epsilon$ for an infinitesimally small ϵ. Hence, the last line of eq. (S.21) becomes

$$\frac{1}{6(4\pi)^4(\alpha\beta)^2} \times \Gamma(2\epsilon) \times \left(\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2 - \gamma p^2}\right)^{2\epsilon} \ \xrightarrow[\epsilon\to 0]{}\ \frac{1}{6(4\pi)^4(\alpha\beta)^2} \times \left(\frac{1}{2\epsilon} - \gamma_E + \log\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2 - \gamma p^2}\right), \tag{S.22}$$

where the limit is taken according to eq. (8). Plugging this formula back into eq. (S.19) and assembling all the factors, we finally arrive at

$$\frac{d\Sigma}{dp^2} = -\frac{\lambda^2}{12(4\pi)^4}\int\limits_{\triangle} d(FP)\,\frac{\gamma}{(\alpha\beta)^2}\times\left\{\frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} + \log\frac{\alpha\beta}{[1-(p^2/m^2)\gamma]^2}\right\} \quad \text{(S.23)}$$

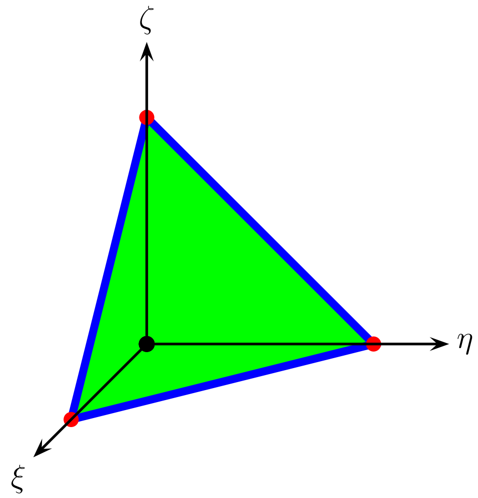
where α, β, and γ depend on the Feynman parameters ξ,η,ζ according to eq. (4). Plugging in their explicit form — and also the explicit form of the Feynman parameter integral — we immediately obtain eq. (9). *Quod erat demonstrandum.*

Problem **1**(f):

When a divergent diagram is regularized using DR (dimensional regularization), the $1/\epsilon$ poles could come from several places. Most commonly, they appear as $\Gamma(\epsilon)$ or $\Gamma(2\epsilon)$ factors from integrals over t–like parameters introduced to make the momentum integral Gaussian, for example see the last couple of lines of eq. (S.21). But for some diagrams — especially with nested or overlapping divergences, see §10.5 of the textbook for an example — there are additional singularities for $\epsilon\to 0$ coming from divergent integrals over the Feynman parameters.

Fortunately, this does not happen for the two-loop amplitude in question, and that's what we need to verify in this part of the problem.

We have 3 Feynman parameters ξ,η,ζ satisfying $\xi+\eta+\zeta=1$ and $\xi,\eta,\zeta\geq 0$; together, they span a 2D area (since only 2 are independent) in the shape of an equilateral triangle



(S.24)

We are to verify that the functions

$$F(\xi,\eta,\zeta) \;=\; \frac{\xi\eta\zeta}{[\xi\eta+\xi\zeta+\eta\zeta]^3} \tag{S.25}$$

and

$$\begin{aligned} H(\xi,\eta,\zeta) \;&=\; F(\xi,\eta,\zeta)\times\log G(\xi,\eta,\zeta) \\ \text{where } G \;&=\; \frac{[\xi\eta+\xi\zeta+\eta\zeta]^3}{[\xi\eta+\xi\zeta+\eta\zeta\;-\;\xi\eta\zeta(p^2/m^2)]^2} \end{aligned} \tag{S.26}$$

may be safely integrated over the triangle (S.24), so let's start with the $F(\xi,\eta,\zeta)$ and check it for singularities. The denominator $[\xi\eta+\xi\zeta+\eta\zeta]^3$ stays positive in the interior of the triangle (green area in fig. (S.24) where all three of ξ,η,ζ are positive) and also along the edges (blue lines where precisely one of the ξ,η,ζ becomes zero), but it vanishes in the vertices (red dots where two variables go to zero at the same time). So as far as the first integral (14) is concerned, the only potentially dangerous parts of the triangle are the vertices, all other places are completely safe.

Let's take a closer look at any one vertex (they are related by symmetry), say $\xi,\eta\to 0$ while $\zeta\approx 1$. Near this vertex

$$F \;\approx\; \frac{\xi\eta}{(\xi+\eta)^3}\,, \tag{S.27}$$

and if we approach this vertex along a line $\eta=\xi\times$ a constant, then

$$F \;\propto\; \frac{1}{\xi} \;\to\; \infty \quad \text{as } \xi\to 0. \tag{S.28}$$

This behavior would create a divergence in one-dimensional integral $\int d\xi$, but not for the 2D integral we are interested in. Indeed, let's change our coordinates according to eq. (15) and consider what happens for $w\to 0$. In this limit — which corresponds to the corner $\xi,\zeta\to 0$ — we have

$$F \;\approx\; \frac{x(1-x)}{w} \tag{S.29}$$

but the differential

$$F\,d\xi\,d\eta \;=\; F\times w\,dw\,dx \;\approx\; x(1-x)\,dx\times dw \tag{S.30}$$

remains perfectly finite for $w\to 0$, so the integral converges just fine.

Now consider the second integral (14) where we have an extra $\log G(\xi,\eta,\zeta)$ factor in the integrand. Since G is a rational function, $\log G$ does not have any singularities worse than logarithmic, and log singularities may be safely integrated over. The only potential danger comes from singularities of the $\log G$ coinciding with singularities of the F factor, so the net singularity becomes worse.

Since F's singularities lie at the 3 corner of the triangle, let's see how the G function and its log behave hear the corners. Going back to the $\xi,\eta\to 0$, $\zeta\approx 1$ corner, we have

$$G \;\approx\; \frac{(\xi+\eta)^3}{[\xi+\eta\,-\,\xi\eta(p^2/m^2)]^2} \approx\; (\xi+\eta) \tag{S.31}$$

so $\log G$ has a logarithmic singularity on top of the "pole" of F. However, in terms of the w,x coordinates, the differential

$$F\times\log G\times d\xi\,d\eta \;\approx\; x(1-x)\,dx\times\log(w)\,dw \tag{S.32}$$

has only a mild logarithmic singularity at $w\to 0$ and the integral converges.

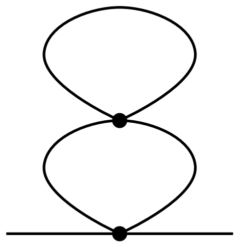
Optional problem $\mathbf{1}(\star)$: my Mathematica code; my numeric code.

Problem $\mathbf{1}$(g):

Having verified that the integral (9) over the Feynman parameters converges, we now face the daunting task of actually evaluating the integral. Fortunately, we do not need to evaluate its as an analytic function of the external momentum p^2 — for the purpose of calculating the field strength renormalization factor Z we are interested in only one value of p^2, namely p^2 = physical mass^2. Moreover, since we are working at the leading order of perturbation theory which contributes to the $d\Sigma/dp^2$, we may neglect the difference between the physical and the bare masses as higher-order correction and set $p^2=m^2$. Consequently, the integral (9) reduces to a combination of the integrals (14), thus

$$\left.\frac{d\Sigma^{2\,\mathrm{loops}}}{dp^2}\right|_{p^2=m^2} \;=\; -\frac{\lambda^2}{24(4\pi)^4}\times\left\{\frac{1}{\epsilon}\,-\,2\gamma_E\,+\,2\log\frac{4\pi\mu^2}{m^2}\,-\,\frac{3}{2}\right\}. \tag{S.33}$$

Note: there are two two-loop 1PI diagrams for the $\Sigma(p^2)$, namely (1) and also



(S.34)

However, the diagram (S.32) produces a p–independent Σ, so it does not contribute to the $d\Sigma/dp^2$. This means that eq. (S.33) is the entire two-loop contribution to the derivative. Also, this two-loop contribution is leading (in the power series in λ) because the one-loop contribution happens to vanish in the $\lambda\phi^4$ theory, thus

$$\frac{d\Sigma^{\rm net}}{dp^2}\bigg|_{p^2=M^2} = -\frac{\lambda^2}{24(4\pi)^4}\times\left\{\frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} - \frac{3}{2}\right\} + O(\lambda^3). \tag{S.35}$$

Consequently, the field strength renormalization factor is

$$Z = \frac{1}{1-\frac{d\Sigma}{dp^2}}\Bigg|_{p^2=M^2} = 1 + \frac{\lambda^2}{6144\pi^4}\left\{\frac{1}{\epsilon} + 2\log\frac{\mu^2}{m^2} + C\right\} + O(\lambda^3) \tag{S.36}$$

where C is a numeric constant, specifically

$$C = 2\log(4\pi) - 2\gamma_E - \frac{3}{2} \approx 2.41. \tag{S.37}$$

<u>Problem **2**(a):</u>

Consider a connected Feynman diagram with L loops, P_B bosonic propagators, and P_F fermionic propagators. At large momenta, bosonic propagators behave as $1/q^2$ while fermionic propagators behave as $1/q$, hence in 4 dimensions the diagram has the superficial degree of UV divergence

$$\mathcal{D} = 4L - 2P_B - P_F\,. \tag{S.38}$$

As in the $\lambda\phi^4$ theory, we can relate this expression to the numbers of external legs using the vertex valences. The Feynman rules of the theory has two vertex types — Yukawa and 4-scalar,

— so let V_Y and V_λ be the respective numbers such vertioces in the diagram. Counting the line ends connected tho these vertices, we have

$$\begin{aligned} 2P_F \;+\; E_F \;&=\; 2V_Y\,, \\ 2P_B \;+\; E_B \;&=\; V_Y \;+\; 4V_\lambda\,, \end{aligned} \tag{S.39}$$

while the Euler formula says

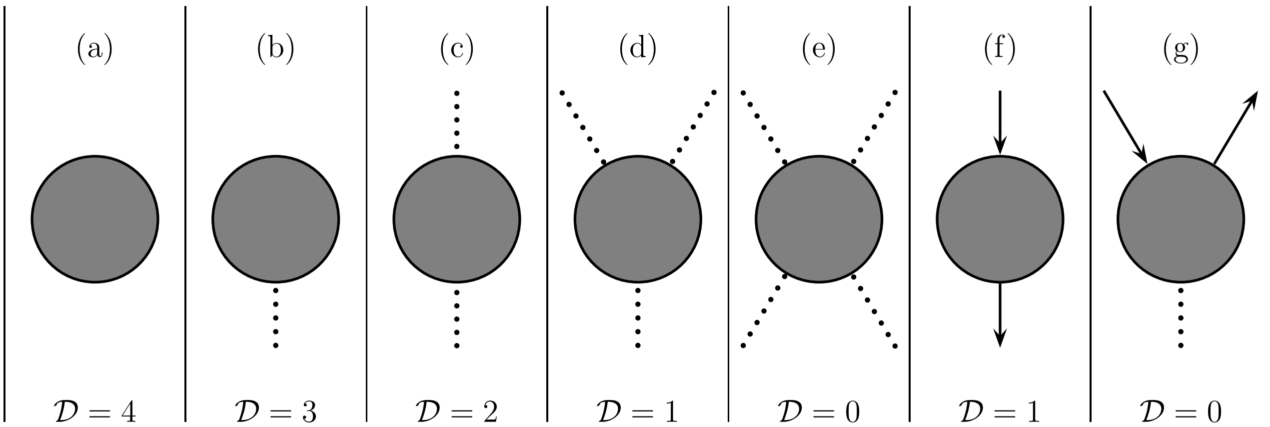
$$L \;-\; P \;+\; V \;\equiv\; L \;-\; P_B \;-\; P_F \;+\; V_Y \;+\; V_\lambda \;=\; 1, \tag{S.40}$$

Combining these three equations, we obtain

$$\begin{aligned} \mathcal{D} \;&=\; 4L - 2P_B - P_F \;=\; 4(L - P_B - P_F) \;+\; 3P_F \;+\; 2P_B \\ &=\; 4(1 - V_Y - V_\lambda) \;+\; \tfrac{3}{2}(2V_Y - E_F) \;+\; (V_Y + 4V_\lambda - E_B) \\ &=\; 4 \;-\; \tfrac{3}{2}E_F \;-\; E_B\,. \end{aligned} \tag{S.41}$$

Thus, the external legs of a diagram completely determine its superficial degree of divergence.

Consequently, for any number of loops, there are only seven superficially divergent amplitudes, namely



Furthermore, the amplitude (a) here is the vacuum energy while the amplitudes (b) and (d) vanish because of the parity symmetry. Indeed, the *pseudo*-scalar field Φ is parity-odd, hence the amplitudes involving odd number of pseudoscalar particles and no fermions must have parity-odd dependence on the particles' momenta. But to construct a parity-odd Lorentz-invariant combination of the Lorentz vectors $p_1^\alpha, p_2^\beta, \ldots$, one needs ϵ tensors, *e.g.* $\epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta$, which requires at least 4 linearly independent momenta (in $d = 4$ spacetime) and hence $n \geq 5$ external legs. For the amplitudes (b) and (d) involving one or three pseudoscalars only and no fermions, such construction is not available and the amplitudes vanish identically.

Altogether, the Yukawa theory has only 4 UV-divergent amplitudes, namely

$$\begin{aligned} &(\mathrm{c}) \quad \Sigma_\phi(p^2), \quad \mathcal{D} = 2, \\ &(\mathrm{e}) \quad V(p_1, p_2, p_3, p_4), \quad \mathcal{D} = 0, \\ &(\mathrm{f}) \quad \Sigma_\psi(\not{p}), \quad \mathcal{D} = 1, \\ &(\mathrm{g}) \quad \Gamma^5(p', p), \quad \mathcal{D} = 0. \end{aligned} \tag{S.42}$$

<u>Problem **2**(b)</u>:

Now consider the divergent parts of the amplitudes (S.42). Similar to the $\lambda\Phi^4$ theory, for the (c) amplitude

$$\mathcal{D}[\Sigma_\phi] \;=\; +2, \quad \mathcal{D}[d\Sigma_\phi/dp^2] \;=\; 0, \quad \mathcal{D}[d^2\Sigma_\phi/(dp^2)^2] \;=\; -2, \tag{S.43}$$

hence

$$\Sigma_\phi(p^2) \;=\; O(\Lambda^2) \times 1 + \; O(\log\Lambda) \times p^2 \;+\; \text{finite}(p^2). \tag{S.44}$$

Likewise, for the (e) amplitude

$$\mathcal{D}[V] \;=\; 0, \quad \mathcal{D}[\text{any } \partial V/\partial p_i] \;=\; -1, \tag{S.45}$$

hence

$$V(p_1, p_2, p_3, p_4) \;=\; O(\log\Lambda) \times 1 \;+\; \text{finite}(p_1, p_2, p_3, p_4). \tag{S.46}$$

Next, the (f) amplitude behaves similar to (e), except in terms of $\not{p}$ instead of p^2, thus

$$\mathcal{D}[\Sigma_\psi] \;=\; +1, \quad \mathcal{D}[d\Sigma_\phi/d\not{p}] \;=\; 0, \quad \mathcal{D}[d^2\Sigma_\phi/(d\not{p})^2] \;=\; -1, \tag{S.47}$$

hence

$$\Sigma_\psi(\not{p}) \;=\; O(\Lambda^1)\times 1 \;+\; O(\log\Lambda)\times \not{p} \;+\; \text{finite}(\not{p}). \tag{S.48}$$

Finally, for the (g) amplitude we have

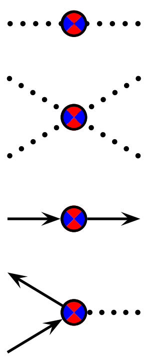
$$\mathcal{D}[\Gamma^5] \;=\; 0, \quad \mathcal{D}[\partial\Gamma^5/\partial p] \;=\; \mathcal{D}[\partial\Gamma^5/\partial p'] \;=\; -1, \tag{S.49}$$

hence

$$\Gamma^5(p',p) \;=\; O(\log\Lambda)\times\gamma^5 \;+\; \text{finite}(p',p). \tag{S.50}$$

where the γ^5 factor in the divergent term follows from the negative parity of the Φ field.

Altogether, all the divergences are fixed-degree polynomials of the momenta, so they may be canceled *in situ* by just 4 types of counterterm vertices, namely



$$\begin{aligned} &= -i\delta^\phi_m \;+\; ip^2\,\delta^\phi_Z\,, \\ &= -i\delta_\lambda\,, \\ &= -i\delta^\psi_m \;+\; i\not{p}\,\delta^\psi_Z\,, \\ &= -\delta_g\gamma^5 \end{aligned} \tag{S.51}$$

parametrized by 6 divergent counterterm coefficients

$$\begin{aligned} \delta^\phi_m &= O(\Lambda^2), \\ \delta^\phi_Z &= O(\log\Lambda), \\ \delta_\lambda &= O(\log\Lambda), \\ \delta^\psi_m &= O(\Lambda^1), \\ \delta^\psi_Z &= O(\log\Lambda), \\ \delta_g &= O(\log\Lambda). \end{aligned} \tag{S.52}$$

In terms of the Feynman rules of the theory, the counterterm vertices (S.51) stems from the local(in x) counterterm Lagrangian

$$\mathcal{L}^{\rm counter}_{\rm terms} \;=\; \tfrac{1}{2}\delta^{\phi}_{Z}\,(\partial\Phi)^2 \;-\; \tfrac{1}{2}\delta^{\phi}_{m}\,\Phi^2 -\; \tfrac{1}{4!}\delta_\lambda\,\Phi^4 \;+\; i\delta^{\psi}_{Z}\,\overline{\Psi}\,\partial\!\!\!/\Psi \;-\; \delta^{\psi}_{m}\,\overline{\Psi}\Psi \;-\; i\delta_g\,\Phi\overline{\Psi}\gamma^5\Psi. \quad \text{(S.53)}$$

Physically, this counterterm Lagrangian is the difference between the bare Lagrangian of the quantum theory and the physical Lagrangian (16). To see how this works, we start with the bare Lagrangian

$$\mathcal{L}_{\rm bare} \;=\; \tfrac{1}{2}(\partial\Phi_b)^2 \;-\; \tfrac{1}{2}m_b^2\Phi_b^2 -\; \tfrac{1}{4!}\lambda_b\Phi_b^4 +\; \overline{\Psi}_b(i\,\partial\!\!\!/ - M_b)\Psi_b \;-\; ig_b\,\Phi_b\overline{\Psi}_b\gamma^5\Psi_b\,, \qquad \text{(S.54)}$$

relate the bare fields to the renormalized fields as

$$\Phi_b(x) \;=\; \sqrt{Z_\phi}\Phi_r(x), \quad \Psi_b(x) \;=\; \sqrt{Z_\psi}\Psi_r(x), \qquad \text{(S.55)}$$

hence

$$\mathcal{L}_{\rm bare} \;=\; \frac{Z_\phi}{2}(\partial\Phi_r)^2 \;-\; \frac{Z_\phi m_b^2}{2}\Phi_r^2 \;-\; \frac{Z_\phi^2\lambda_b}{24}\Phi_r^4 \;+\; Z_\psi\overline{\Psi}_r(i\,\partial\!\!\!/ - M_b)\Psi_r \;-\; ig_bZ_\psi\sqrt{Z_\phi}\times\Phi_r\overline{\Psi}_r\gamma^5\Psi_r\,, \qquad \text{(S.56)}$$

and then spit this bare Lagrangian into the Physical Lagrangian and the counterterms:

$$\mathcal{L}_{\rm bare} \;=\; \mathcal{L}_{\rm phys} \;+\; \mathcal{L}^{\rm counter}_{\rm terms}\,,$$

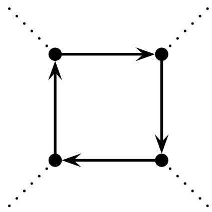
where $\mathcal{L}_{\rm phys}$ and $\mathcal{L}^{\rm counter}_{\rm terms}$ are exactly as in eqs. (16) and (S.53) (in terms of $\Phi = \Phi_r$ and $\Psi = \Psi_r$), while the counterterm coefficients (S.52) are

$$\delta^{\phi}_{Z} = Z_\phi - 1, \qquad \delta^{\psi}_{Z} = Z_\psi - 1, \qquad \delta^{\phi}_{m} = Z_\phi m_b^2 - m_{\rm ph}^2\,, \qquad \delta^{\psi}_{m} = Z_\psi M_b - M_{\rm ph},$$
$$\delta_\lambda = Z_\phi^2\lambda_b - \lambda_{\rm ph}\,, \qquad \text{and} \quad \delta_g = Z_\psi Z_\phi^{1/2} g_b - g_{\rm ph}\,.$$

Thus, we see that all the counterterms (S.52) we need to cancel the divergences of the Yukawa theory are already included in the bare Lagrangian of the theory.

Problem **2**(c):

In the next homework set#16 we are going to calculate the divergences of the YUkawa theory at the one-loop level, and the counterterms we need to cancel these divergences. That calculation will show that we do need all 6 of the counterterms (S.52) to cancel all the divergences. In particular, we shall see that even for $\lambda_{\text{ph}} = 0$ we still need an infinite δ_λ counterterm to cancel the divergences of the fermionic loop diagrams



$+$ five similar. (S.57)

For the moment, we do not need to evaluate such diagrams, all we need is the fact that such diagrams exist, diverge logarithmically (since they have $\mathcal{D} = 0$), and do not vanich (since they have no reason to). Consequently, to cancel the divergences of such diagrams, we do need an infinite $\delta_\lambda \neq 0$.

Thus, from the bare Lagrangian point of view, $\lambda_{\text{phys}} = 0$ has no special meaning: the bare coupling λ_b would be infinite in any case, so vanishing of a particular scattering amplitude we use to define the physical coupling λ_{ph} would be just an accident. In other words, we may *fine tune* λ_b to achieve $\lambda_{\text{ph}} = 0$ just as we can fine tune λ_b to achieve any other experimental value of the physical coupling, but it would not have any special meaning for the theory itself.

This is an example of the general rule: *barring fine tuning of the coupling parameters, a renormalizable quantum field theory has all the renormalizable couplings consistent with the theory's symmetries.* For the theory at hand, we have a Dirac field Ψ, a real pseudoscalar field Φ, and all the Lagrangian terms involving these fields should be invariant under Lorentz and parity transformations and have canonical dimensions ≤ 4 (for renormalizability's sake). There is only a finite number of such terms, and it is easy to see that the Lagrangian (16) comprises all such terms and no others. Consequently, the renormalized theory would not have any additional interactions.

Sometimes, in absence of some coupling the theory has an additional symmetry that would not be present otherwise. In such case, the extra symmetry would prevent such coupling from being restored by the renormalization procedure. For example, consider the Lagrangian (16) for $g = 0$ (but $\lambda \neq 0$): In the absence of the Yukawa coupling, the theory would have an extra symmetry $\Phi(x) \to -\Phi(x)$ (without parity), and this extra symmetry would prevent the renormalization procedure from restoring the Yukawa coupling. On the other hand, when $\lambda = 0$ but $g \neq 0$, the theory does not has any additional symmetries it wouldn't have for $\lambda \neq 0$, and that's why the renormalization gives rise to the $\lambda\Phi^4$ coupling even if it wasn't there to begin with.