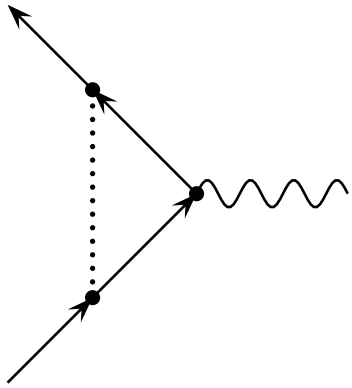
Problem **1**(b):

The form factor $F_2(q^2)$ of the muon — and hence the anomalous magnetic moment $a = F_2(0)$ — is a part of the 'dressed QED vertex' $ie\Gamma^\mu$, which is *net 1PI amplitude* for two on-shell muons and one off-shell photon; additional fields (besides the EM and the muon's Ψ) affect this amplitude through loop diagrams containing their propagators. For a neutral scalar S with a Yukawa coupling to the muon, there is a single one-loop diagram for the $ie\Gamma^\mu$(muon), namely



(S.1)

At the two-loop and higher-loop levels there are many more diagrams involving the scalar S, but they are beyond the scope of this exercise.

Evaluating the one-loop diagram (S.1), we obtain

$$\begin{aligned}
\Delta_S[ie\Gamma^\mu(p',p)] &= \int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-M_s^2+i0}\times(-ig)\frac{i}{\not p'+\not k-m+i0}\,(ie\gamma^\mu)\,\frac{i}{\not p+\not k-m+i0}(-ig)\\
&= -eg^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-M_s^2+i0}\times\frac{\not p'+\not k+m}{(p'+k)^2-m^2+i0}\,(ie\gamma^\mu)\,\frac{\not p+\not k+m}{(p+k)^2-m^2+i0}\\
&= -eg^2\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{\mathcal{D}}
\end{aligned}\tag{S.2}$$

where in the numerator

$$\mathcal{N}^\mu \;=\; (\not p'+\not k+m)\times\gamma^\mu\times(\not p+\not k+m)\tag{S.3}$$

and in the denominator

$$\frac{1}{\mathcal{D}} = \frac{1}{k^2 - M_s^2 + i0} \times \frac{1}{(k+p')^2 - m^2 + i0} \times \frac{1}{(k+p)^2 - m^2 + i0}\,. \tag{S.4}$$

As usual, we may combine the 3 denominator factors using the Feynman parameter trick. Proceeding similarly to the QED calculation of the Γ^μ in class — *cf.* eqs. (13) through (21) of my notes on the dressed QED vertex — we have

$$\begin{aligned}\frac{1}{\mathcal{D}} &= \frac{1}{k^2 - M_s^2 + i0} \times \frac{1}{(k+p')^2 - m^2 + i0} \times \frac{1}{(k+p)^2 - m^2 + i0}\\ &= \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{(\ell^2 - \Delta + i0)^3}\,.\end{aligned} \tag{S.5}$$

where

$$\ell^2 - \Delta = z(k^2 - M_s^2) + x((k+p)^2 - m^2) + y((k+p')^2 - m^2) \tag{S.6}$$

and hence

$$\ell = k + xp + yp', \tag{S.7}$$

$$\begin{aligned}\Delta &= zm_S^2 + (1-z)m^2 - xzp^2 - yzp'^2 - xyq^2\\ &\qquad \langle\!\langle \text{ for on-shell muon momenta } p^2 = p'^2 = m^2 \,\rangle\!\rangle\\ &= zM_s^2 + (1-z)^2 m^2 - xyq^2.\end{aligned} \tag{S.8}$$

As usual, we now substitute the denominator (S.5) into the bottom line of eq. (S.2), change the order of integration over momentum and over the Feynman parameters, and then change the momentum integration variable from k^μ to ℓ^μ, thus

$$\Delta_S\Gamma^\mu(p',p) = 2ig^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{(\ell^2 - \Delta + i0)^3}\,, \tag{S.9}$$

which has the desired form (2).

Problem $\mathbf{1}$(c):

Our next task is to simplify the numerator (S.3) in the present context. That is, we re-express $\mathcal{N}^\mu$ in terms of the shifted loop momentum ℓ, discard terms which integrate to zero (because they are odd with respect to $\ell \to -\ell$ or $x \leftrightarrow y$ symmetries), and also make use of the $\bar{u}(p')\,\Gamma^\mu\,u(p)$ context which allows us to substitute $\not{p} \to m$ in the rightmost factor and $\not{p}' \to m$ in the leftmost factor. Thus, proceeding similarly to eqs. (26) through (43) of my notes, we obtain

$$\begin{aligned}
\mathcal{N}^\mu &= (\not{k} + \not{p}' + m)\,\gamma^\mu\,(\not{k} + \not{p} + m) \\
&= \big((\not{\ell} - x\not{p} - y\not{p}') + \not{p}' + m\big)\,\gamma^\mu\,\big((\not{\ell} - x\not{p} - y\not{p}') + \not{p} + m\big) \\
&\quad \langle\!\langle \text{ disregarding odd powers of } \ell \,\rangle\!\rangle \\
&\cong \not{\ell}\gamma^\mu\not{\ell} \;+\; (\not{p}' - x\not{p} - y\not{p}' + m)\,\gamma^\mu(\not{p} - x\not{p} - y\not{p}' + m) \\
&= \not{\ell}\gamma^\mu\not{\ell} + \;(z\not{p}' + x\not{q} + m)\,\gamma^\mu\,(z\not{p} - y\not{q} + m) \\
&\quad \langle\!\langle \text{ between } \bar{u}(p') \text{ and } u(p) \,\rangle\!\rangle \\
&\cong \not{\ell}\gamma^\mu\not{\ell} \;+\; ((z+1)m + x\not{q})\,\gamma^\mu\,((z+1)m - y\not{q}) \\
&= \not{\ell}\gamma^\mu\not{\ell} \;+\; (z+1)^2 m^2\,\gamma^\mu \;-\; xy\,\not{q}\gamma^\mu\not{q} \\
&\qquad +\; (z+1)m\Big(x\not{q}\gamma^\mu \;-\; y\gamma^\mu\not{q} \;=\; (x-y)q^\mu \;+\; (x+y)i\sigma^{\mu\nu}q_\nu\Big) \\
&\cong \not{\ell}\gamma^\mu\not{\ell} \;+\; (z+1)^2 m^2\,\gamma^\mu \;+\; xyq^2\,\gamma^\mu + \; m(1-z^2)\times i\sigma^{\mu\nu}q_\nu \;+\; (x-y)\times(z+1)mq^\mu.
\end{aligned} \tag{S.10}$$

On the bottom line here, the last term $(x-y)\times\cdots$ integrates to zero thanks to the $x \leftrightarrow y$ symmetry of the integral over the Feynman parameters, thus in the present context

$$(x-y)\times(z+1)mq^\mu \;\cong\; 0. \tag{S.11}$$

Also, thanks to the Lorentz symmetry of the $\int d^4\ell$ and of the denominator,

$$\not{\ell}\gamma^\mu\not{\ell} \;=\; \gamma^\lambda\gamma^\mu\gamma^\nu\times\ell^\lambda\ell^\nu \;\cong\; \gamma^\lambda\gamma^\mu\gamma^\nu\times g^{\lambda\nu}\,\frac{\ell^2}{D} \;=\; (2-D)\gamma^\mu\times\frac{\ell^2}{D}. \tag{S.12}$$

Plugging these formulae into eq. (S.10), we arrive at

$$\begin{aligned}
\mathcal{N}^\mu &\cong (2-D)\gamma^\mu\times\frac{\ell^2}{D} \;+\; (z+1)^2 m^2\,\gamma^\mu \;+\; xyq^2\,\gamma^\mu + \; m(1-z^2)\times i\sigma^{\mu\nu}q_\nu \\
&\equiv \mathcal{N}_1\times\gamma^\mu \;+\; \mathcal{N}_2\times\frac{i\sigma^{\mu\nu}q_\nu}{2m}
\end{aligned} \tag{S.13}$$

where

$$
\begin{aligned}
\mathcal{N}_1 &= -\frac{D-2}{D}\,\ell^2 \,+\, (1+z)^2 m^2 \,+\, xyq^2 && \text{(S.14)}\\
\text{and}\quad \mathcal{N}_2 &= 2(1-z^2)m^2. && \text{(S.15)}
\end{aligned}
$$

In light of the Dirac-matrix structure of the last line of eq. (S.13), the $\mathcal{N}_1$ contributes to the F_1 form-factor of the muon while the $\mathcal{N}_2$ contributes to the F_2 form factor,

$$
\begin{aligned}
\Delta_S F_1(q^2) &= 2ig^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_1}{(\ell^2-\Delta+i0)^3}\\
&\qquad - \text{similar integral for } q^2=0 \text{ because of } \Delta_S\delta_1\,, && \text{(S.16)}\\
\Delta_S F_2(q^2) &= 2ig^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2}{(\ell^2-\Delta+i0)^3}\,. && \text{(S.17)}
\end{aligned}
$$

Problem **1**(d):

In this exercise, we are interested in the anomalous magnetic moment of the muon, so all we need is the F_2 for $q^2=0$, and we do not need to worry about the counterterm δ_1 because it affects only the other form factor F_1. In eq. (S.17) for the F_2, the numerator $\mathcal{N}_2$ does not depend on the loop momentum ℓ, so the $\int d^4\ell$ integral converges without any regularization, UV or IR,

$$
i\int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^3} = \int \frac{d^4\ell_E}{(2\pi)^4}\,\frac{1}{(\ell_E^2+\Delta)^3} = \frac{1}{16\pi^2}\int_0^2 d\ell_E^2\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3} = \frac{1}{16\pi^2}\times\frac{1}{2\Delta}\,. \tag{S.18}
$$

Consequently,

$$
\Delta_S F_2(q^2) = \frac{g^2}{16\pi^2}\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{\mathcal{N}_2 = 2m^2(1-z^2)}{\Delta} \tag{S.19}
$$

where Δ is as in eq. (S.8). In particular, for $q^2 = 0$ we have $\Delta = zM_s^2 + (1-z)^2 m^2$, hence

$$\begin{aligned}
\Delta_S\left(\frac{g_\mu - 2}{2}\right) &= \Delta_S F_2(q^2 = 0) \\
&= \frac{g^2}{16\pi^2}\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z^2)}{zM_s^2 + (1-z)^2 m^2} \\
&= \frac{g^2}{16\pi^2}\int_0^1 dz\,(1-z)\times\frac{2m^2(1-z^2)}{zM_s^2 + (1-z)^2 m^2}\,.
\end{aligned}\tag{S.20}$$

The last integral here is a complicated function of the muon-to-scalar mass ratio m/M_s, but for the problem at hand, the scalar is much heavier than the muon. Hence, we approximate the denominator as

$$\begin{aligned}
zM_s^2 \;+\; (1-z)^2 m^2 &\approx \begin{cases} zM_s^2 \;+\; 0 & \text{except when } z \approx 0 \\ zM_s^2 \;+\; m^2 & \text{for } z \approx 0 \end{cases} \\
&\approx zM_s^2 \;+\; m^2 \quad \text{for all } z,
\end{aligned}\tag{S.21}$$

and consequently evaluate

$$\begin{aligned}
\int_0^1 dz\,\frac{2m^2(1-z^2)(1-z)}{zM_s^2 + m^2} &\approx \frac{2m^2}{M_s^2}\int_0^1 dz\,\frac{1 - z - z^2 + z^3}{z + (m^2/M_S^2)} \\
&\approx \frac{2m^2}{M_s^2}\int_0^1 dz\left(\frac{1}{z + (m^2/M_S^2)} \;-\; 1 \;-\; z \;+\; z^2 \;+\; O(m^2/M_s^2)\right) \\
&= \frac{2m^2}{M_s^2}\left(\log\frac{M_s^2}{m^2} \;-\; \frac{7}{6}\right) \;+\; O\left(\frac{m^4}{M_s^4}\right).
\end{aligned}\tag{S.22}$$

Thus, to the leading orders in the Yukawa coupling g and in the m/M_s mass ratio, the scalar's effect on the anomalous magnetic moment of the muon amounts to

$$\Delta_S a_\mu \;\approx\; \frac{g^2}{8\pi^2}\,\frac{m^2}{M_s^2}\left(\log\frac{M_s^2}{m^2} \;-\; \frac{7}{6}\right).\tag{S.23}$$

Problem **1**(e):

Experimentally, muon's anomalous magnetic moment agrees with the MSM (Minimal Standard Model) to 8 decimal places; beyond that, we have eq. (1). Interpreting that equation as a limit on contributions from outside the MSM — *i.e.*, as limit on the $\Delta_S g_\mu$, — we have

$$\Delta_S a_\mu \;<\; 1.83\cdot 10^{-9} \tag{S.24}$$

at 95% confidence level, where the RHS here is the central value plus two sigmas from eq. (1). At the same time, for $m_\mu = 106$ MeV and $M_S = 500$ GeV, we have

$$\frac{m^2}{M_s^2}\times\left(\log\frac{M_s^2}{m^2}\;-\;\frac{7}{6}\right)\;=\;0.71\cdot 10^{-6}, \tag{S.25}$$

so plugging this mass-dependent factor into eq. (S.23) we immediately translate the limit (S.24) to

$$\frac{g^2}{8\pi^2}\;<\;2.6\cdot 10^{-3} \tag{S.26}$$

and hence

$$g\;<\;0.45. \tag{S.27}$$

More generally, for a scalar of mass $M_S =$ few hundreds GEV, its Yukawa coupling to the muon field should not exceed

$$g_{\max}\;\approx\;0.45\times\left(\frac{M_S}{500\,\mathrm{GeV}}\right). \tag{S.28}$$

Problem **2**:

The δ_2 and δ_m counterterms of QED are related to the electron's self-energy correction

$$\Sigma_e^{\rm net}(\not{p}) \;=\; \Sigma_e^{\rm loops}(\not{p}) \;+\; \delta_m \;-\; \delta_2\times\not{p} \tag{S.29}$$

which satisfies renormalization conditions

$$\textbf{for}\quad \not{p}=m,\quad \textbf{both}\quad \Sigma_e^{\rm net}=0 \quad\textbf{and}\quad \frac{d\Sigma_e^{\rm net}}{d\not{p}}=0. \tag{S.30}$$

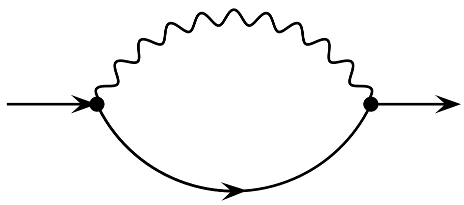
Thanks to these conditions,

$$\delta_2 \;=\; \frac{d\Sigma_e^{\rm loops}}{d\not{p}}\bigg|_{\not{p}\,=\,m} \tag{S.31}$$

and also

$$\delta_m \;=\; m\times\delta_2 \;-\; \Sigma_e^{\rm loops}\Big|_{\not{p}\,=\,m}. \tag{S.32}$$

At the one loop level of analysis, there is only one diagram for the electron's self energy, namely



(S.33)

For consistency with the calculation of the δ_1 counterterm in my notes on the dressed QED vertex, we should use the Feynman gauge for the photon propagator here. Thus, the diagram (S.33) evaluates to

$$-i\Sigma^{1\,\rm loop}(\not{p}) \;=\; \int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\lambda\nu}}{k^2+i0}\times ie\gamma_\lambda\,\frac{i}{\not{k}+\not{p}-m_e+i0}\times ie\gamma_\nu \tag{S.34}$$

where the the momentum integral should be regulated exactly as in my notes for the calculation of the δ_1 counterterm: spacetime dimension $D=4-2\epsilon$ to regulate the UV divergence, and a tiny photon's mass m_γ to regulate the infrared divergence. Thus,

$$\begin{aligned}\Sigma^{1\,\mathrm{loop}}(\not{p}) &= i\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{-ig^{\lambda\nu}}{k^2-m_\gamma^2+i0}\times ie\gamma_\lambda\,\frac{i}{\not{k}+\not{p}-m_e+i0}\times ie\gamma_\nu \\ &= -ie^2\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{\mathcal{N}}{\mathcal{D}}\end{aligned}\tag{S.35}$$

where the numerator is

$$\mathcal{N} = \gamma^\nu(\not{k}+\not{p}+m_e)\gamma_\nu = Dm_e - (D-2)(\not{p}+\not{k})\tag{S.36}$$

and the denominator is

$$\frac{1}{\mathcal{D}} = \frac{1}{k^2-m_\gamma^2+i0}\times\frac{1}{(k+p)^2-m_e^2+i0} = \int\limits_0^1 dx\,\frac{1}{(\ell^2-\Delta+i0)^2}\tag{S.37}$$

for

$$\ell^2 - \Delta = (1-x)(k^2-m_\gamma^2) + x((k+p)^2-m_e^2)\tag{S.38}$$

and hence

$$\ell = k + xp,\tag{S.39}$$

$$\Delta = xm_e^2 - x(1-x)p^2 + (1-x)m_\gamma^2\,.\tag{S.40}$$

As usual, we re-express the numerator (S.36) in terms of the shifted loop momentum ℓ and then discard the odd powers of ℓ, thus

$$\mathcal{N} = Dm_e - (D-2)(\not{\ell}-x\not{p}+\not{p}) \cong Dm_e - (D-2)(1-x)\not{p},\tag{S.41}$$

and therefore

$$\Sigma^{1\,\mathrm{loop}}(\not{p}) = e^2\int\limits_0^1 dx\left(Dm_e - (D-2)(1-x)\not{p}\right)\times\int\frac{d^D\ell}{(2\pi)^D}\,\frac{-i\mu^{4-D}}{(\ell^2-\Delta+i0)^2}\,.\tag{S.42}$$

The momentum integral here should be familiar to you by now, so let me simply write it down,

$$\int\frac{d^D\ell}{(2\pi)^D}\,\frac{-i\mu^{4-D}}{(\ell^2-\Delta+i0)^2}\;=\;\frac{\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(2-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-2}\;=\;\frac{1}{16\pi^2}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}. \tag{S.43}$$

Consequently,

$$\Sigma^{1\,\text{loop}}(\not p)\;=\;\frac{\alpha}{2\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^{\epsilon}\int\limits_0^1\!dx\,\frac{(2-\epsilon)m_e\,-\,(1-\epsilon)(1-x)\,\not p}{\Delta^{\epsilon}(z)}\,. \tag{S.44}$$

By itself, the Feynman parameter integral (S.44) does not need the IR regulator: for $m_\gamma^2=0$ it converges for any $\epsilon<\frac{1}{2}$ (*i.e.*, $D>3$) for the on-shell momentum $p^2=m_e^2$, and for any $\epsilon<1$ (*i.e.*, $D>2$) for off-shell momenta $p^2<m_e^2$. However, while the resulting $\Sigma(\not p)$ remains finite for $\not p=m_e$, its derivative $d\Sigma(\not p)/d\not p$ develops a mild singularity of IR origin. Regulating *that* singularity is why we need a tiny photon mass $m_\gamma>0$.

Indeed, taking the derivative of eq. (S.44) with respect to $\not p$, we have

$$\frac{d\Sigma^{1\,\text{loop}}}{d\not p}\;=\;\frac{\alpha}{2\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^{\epsilon}\int\limits_0^1\!dx\left(\begin{array}{c}\dfrac{-(1-\epsilon)(1-x)}{\Delta^{\epsilon}}\;-\;\epsilon\dfrac{(2-\epsilon)m_e-(1-\epsilon)(1-x)\not p}{\Delta^{1+\epsilon}}\times\\ \times\left[\dfrac{\partial\Delta}{\partial\not p}\;=\;-2x(1-x)\,\not p\right]\end{array}\right). \tag{S.45}$$

Let us neglect the IR regulator for a moment and take $m_\gamma^2=0$, hence $\Delta=x\times(m^2-p^2+xp^2)$. Then for $x\to 0$, the integrand of eq. (S.45) behaves as

$$\frac{1}{x^{\epsilon}}\times\left[-\frac{1-\epsilon}{(m^2-p^2+xp^2)^{\epsilon}}\;+\;\frac{2\epsilon\,\not p((2-\epsilon)m-(1-\epsilon)\,\not p)}{(m^2-p^2+xp^2)^{1+\epsilon}}\right]. \tag{S.46}$$

For off-shell momenta $p^2<m^2$, the expression in the square brackets here is finite and the $\int dx\,x^{-\epsilon}$ is perfectly finite as long as $\epsilon<1$ *i.e.*, $D>2$. But for the on-shell momentum $p^2=m^2$, the second term in the square brackets blows up at $x\to 0$ and we end up with

$$\int\limits_0^1\!dx\,\frac{\text{finite}}{x^{1+2\epsilon}}\,, \tag{S.47}$$

which diverges for any $\epsilon\geq 0$ *i.e.*, $D\leq 4$. And that's why we need the IR regulator $m_\gamma^2>0$.

So let us put the IR regulator back where it belongs and calculate the derivative (S.45) for the on-shell momentum. For $\not p = m_e$, the integrand on the RHS of eq. (S.45) simplifies to

$$-\frac{(1-\epsilon)(1-x)}{\Delta^\epsilon} + \frac{2\epsilon x(1-x)[1+(1-\epsilon)x]\times m_e^2}{\Delta^{1+\epsilon}} \tag{S.48}$$

where

$$\Delta = x^2 m_e^2 + (1-x)m_\gamma^2 \approx x^2 m_e^2 + m_\gamma^2 . \tag{S.49}$$

The approximation here follows from the IR regulator being important only for $x \to 0$, and it allows us to extract a total derivative from the integrand:

$$(\text{S.48}) = -\frac{d}{dx}\left(\frac{(1-x)[1+(1-\epsilon)x]}{\Delta^\epsilon}\right) - \frac{1+(1-\epsilon)x}{\Delta^\epsilon} . \tag{S.50}$$

For $\epsilon < \frac{1}{2}$ — *i.e.*, for $D > 3$ — the second term here can be integrated without the photon's mass,

$$\int_0^1 dx\, \frac{1+(1-\epsilon)x}{[\Delta = x^2 m_e^2]^\epsilon} = \frac{1}{m_e^{2\epsilon}} \times \left(\frac{1}{1-2\epsilon} + \frac{1-\epsilon}{2-2\epsilon}\right), \tag{S.51}$$

while the first term yields

$$-\left.\frac{(1-x)[1+(1-\epsilon)x]}{\Delta^\epsilon}\right|_0^1 = \frac{+1}{\Delta^\epsilon(x=0)} = \frac{+1}{m_\gamma^{2\epsilon}} . \tag{S.52}$$

Altogether

$$\int_0^1 dx\,(\text{S.48}) = \frac{1}{m_\gamma^{2\epsilon}} - \frac{1}{m_e^{2\epsilon}} \times \left(\frac{1}{1-2\epsilon} + \frac{1}{2}\right) \tag{S.53}$$

and therefore

$$\begin{aligned}
\delta_2^{\text{order}\,\alpha} &= \left.\frac{d\Sigma^{1\,\text{loop}}}{d\not p}\right|_{\not p = m} \\
&= \frac{\alpha}{2\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \times \left[\left(\frac{m_e^2}{m_\gamma^2}\right)^\epsilon - \frac{1}{1-2\epsilon} - \frac{1}{2}\right] \\
&= -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \times \left[1 + \frac{2}{1-2\epsilon} - 2\left(\frac{m_e^2}{m_\gamma^2}\right)^\epsilon\right]
\end{aligned} \tag{S.54}$$

In the $D \to 4$ limit, this counterterm becomes

$$\delta_2^{\text{order}\,\alpha} \;=\; -\frac{\alpha}{4\pi}\left[\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{m_e^2}\,+\,4\,-\,2\log\frac{m_e^2}{m_\gamma^2}\right]. \tag{S.55}$$

By comparison, in class I have calculated

$$\begin{aligned}\delta_1^{\text{order}\,\alpha} \;&=\; -\frac{\alpha}{4\pi}\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon}\times\left[1\,+\,\frac{2}{1-2\epsilon}\,-\,2\left(\frac{m_e^2}{m_\gamma^2}\right)^{\epsilon}\right]\\ &\xrightarrow[D\to 4]{}\;-\frac{\alpha}{4\pi}\left[\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{m_e^2}\,+\,4\,-\,2\log\frac{m_e^2}{m_\gamma^2}\right],\end{aligned} \tag{S.56}$$

cf. eqs. (95–96) of my notes on the dressed QED vertex (page 21). Thus, $\delta_2=\delta_1$ (to order α), and this equality holds for any dimension between 3 and 4. *Quod erat demonstrandum.*

In fact, the identity $\delta_1=\delta_2$ holds in any spacetime dimension and even for a finite photon mass $m_\gamma \lll m_e$, but proving *that* goes beyond the scope of this exercise.